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Nonlinearity Role in Long-Term Interaction of the Ocean Gravity Waves

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LONG TERM GOALS

Development of accurate and fast advanced statistical and dynamical nonlinear models of ocean surface waves, based on first physical principles, which will improve and accelerate both long term ocean surface waves forecasts and prediction of strongly coherent events, such as freak waves, tsunami and wave-breaking.

OBJECTIVES

Finding of physically correct wind input term for Hasselmann equation, understanding of the balance of source terms in Hasselmann equation, investigation of the problem of interaction of different scales on the ocean surface (sea and swell), development of new water surface analytic models and methods of their solution.

APPROACH

Advanced analytical techniques: Hamiltonian formalism, self-similar solutions, analytic solutions of integral equations; numerical methods for solution of integral and pseudo-differential equations; comparison of analytic and numerical results with experimental data

WORK COMPLETED

- Confirmation of the new wind input term through the limited fetch numerical simulation
- Analytical solution and its numerical confirmation for non-stationary Hasselmann equation without non-linear interaction term
- Detection of swell feedback by sea background through theory and experimental data
- Analytical and numerical proof of non-integrability of 2D free-surface hydrodynamics
- New findings in mechanisms of the energy dissipation during wave breaking event

RESULTS

1. Confirmation of the new wind input term through the limited fetch numerical simulation

The Hasselmann equation for energy spectral density $\varepsilon(\vec{r},\vec{k},t)$

$$\frac{\partial\varepsilon}{\partial t}+\frac{\partial\omega_k}{\partial\vec{k}}\frac{\partial\varepsilon}{\partial\vec{r}}=S_{nl}+S_{in}+S_{diss} \tag{1}$$

in the limited fetch situation can be reduced to

$$\frac{g}{2\omega}\cos\theta\frac{\partial\varepsilon}{\partial x}=S_{nl}+S_{wind}+S_{diss} \tag{2}$$

where $\boldsymbol{x}$ is the coordinate orthogonal to the shore and θ is the angle between individual wavenumber $\boldsymbol{k}$ and the axis $\boldsymbol{x}$.

Eq.(2) is somewhat difficult for numerical simulation, as it contains the singularity in the form of $(\cos\theta)^{-1}$ on the right-hand side. We overcame this problem of division by zero through zeroing one half of the Fourier space of the system for the waves propagating toward the shore. Since it is well–known that the energy in such waves is small with respect to waves propagating in the offshore direction, such approximation is quite reasonable for our purposes.

We are looking for wind input function in he form $S_{wind}=\gamma(\omega,\theta)\varepsilon(\omega,\theta)$ where $\gamma(\omega,\theta)=\alpha\omega^{1+s}f(\theta)$.

The same sort of self-similar analysis performed for the time domain situation (see [1]) can be repeated for the limited fetch one. The result of inserting the self-similar substitution

$$\varepsilon=x^{p+q}F(\omega x^q)$$

into Eq.(2) gives the values of the indices

$$p=\frac{10q-1}{2},\quad q=\frac{1}{s+2}$$

and

$$q=\frac{3}{10},\quad p=1,\ s=\frac{4}{3}$$

We found that in the fetch-limited case the wind forcing index $\boldsymbol{s}$ is similar to the time domain situation, and the wind forcing is given by (see [1]):

$$S_{wind}=0.2\frac{\rho_{air}}{\rho_{water}}\omega\left(\frac{\omega}{\omega_0}\right)^{4/3}f(\theta),\quad f(\theta)=\begin{cases}\cos\theta & \text{for } -\pi/2\le\theta\le\pi/2\\ 0 & \text{otherwise}\end{cases} \tag{3}$$

$$\omega_0=\frac{g}{u_{10}},\quad \frac{\rho_{air}}{\rho_{water}}=1.3\cdot10^{-3}$$

To check the theoretical finding (3), the numerical simulation of the Eq.(2) has been performed for S_{wind} input function (3), dissipation term S_{diss} similar to [3], where white-capping dissipation term was introduced implicitly through f^{-5} ($f=\frac{\omega}{2\pi}$) energy spectral tail stretching in frequency range from $f_d=1.1$ to $f_{\max}=2.0$.

Fig.1 presents characteristic spectral energy distribution for this limited fetch simulation.

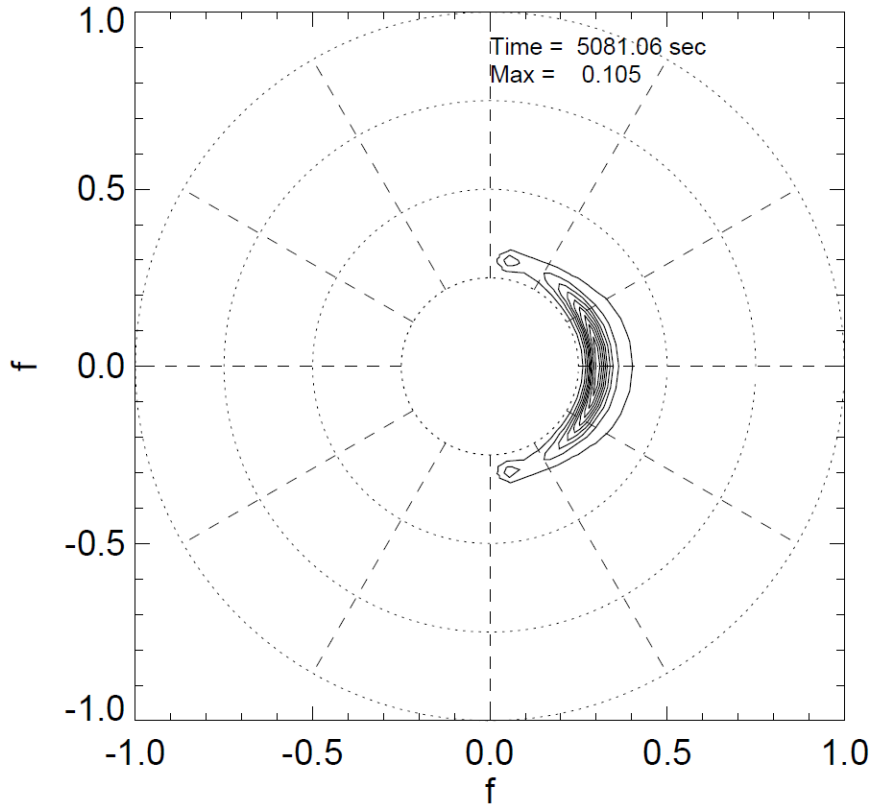

Fig.1: Energy spectral density $\varepsilon(f,\theta)$ as a function of f and angle θ in polar coordinates.

Fig.2 presents the plot of the function $\beta = F(k)\cdot k^{5/2}$ in terms of the parameter $(u_\lambda^2 C_p)^{1/3}/g^{1/2}$ for two different runs, corresponding to the same wind speed u_{10} *=10.0 m/sec*: one for time domain simulation (see [1]), and another for limited fetch simulation. Both simulations show good correspondence with Resio et al. (2004) [2] regression line.

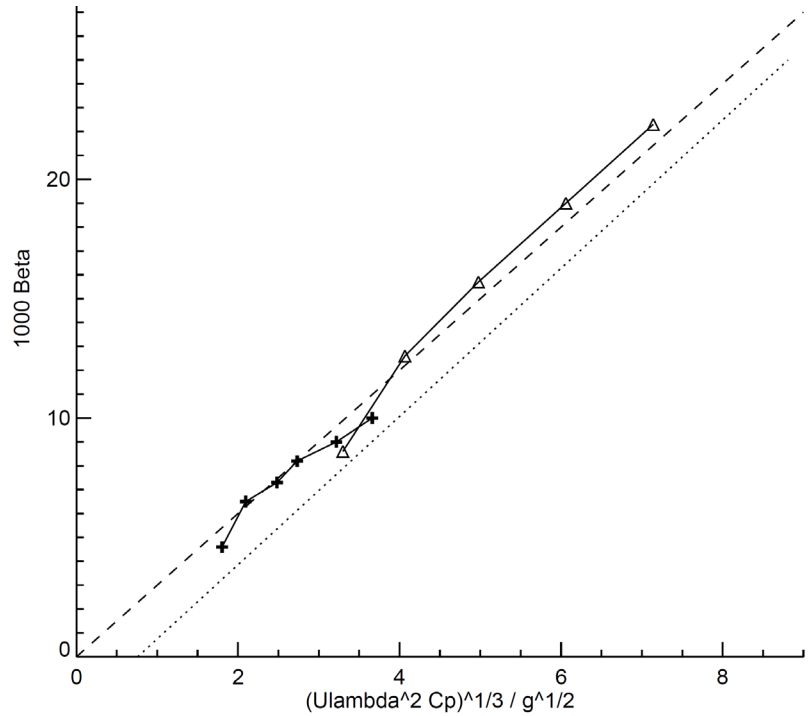

Fig.2: Two simulations for wind input u_{10}=10.0 m/sec: time-limited domain (triangles) and limited fetch (crosses). Dotted line – correlation of the equilibrium range coefficient β with $(u_\lambda^2 C_p)^{1/3} / g^{1/2}$ based on data from six disparate sources adopted from Resio at al, 2004 [2] . Dashed line – theoretical value of equilibrium range coefficient β.

2. *Analytical solution and its numerical tests for Hasselmann equation without nonlinear interaction*

The Hasselmann Eq.(1) includes essentially different physical processes' -- advection, nonlinear interaction and external forcing/damping. It is important to understand during numerical implementation of Hasselmann equation simulation that individual terms, responsible for particular physical process, work well. In this connection it's necessary to test each of them individually to be sure that discretization implementation errors do not overlap, creating wrongly working numerical code.

As far as concerns nonlinear interaction term S_{nl}, we can be sure that discrete implementation works quite well. Those tests include, in particular, numerical reproduction of analytical Komogorov-Zakharov and self-similar solutions for time-limited formulation of Hasselmann equation without advection term [1], [4].

The question of testing of the advection part of Hasselmann equation (1) remained open so far. We managed to find specific analytic solution and show numerically that it is reproduced by the corresponding numerical code.

We are starting with linear version of Hasselmann Eq.(1) stripped off the nonlinear interaction term S_{nl}:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial \omega_k}{\partial k}\frac{\partial \varepsilon}{\partial x} = \gamma\varepsilon$$

For one-dimensional situation and deep water dispersion relation $\omega_k = \sqrt{gk}$ it transforms into

$$\frac{\partial \varepsilon}{\partial t} + \frac{g}{2\omega}\cos\theta\frac{\partial \varepsilon}{\partial x} = \gamma\varepsilon \tag{4}$$

which after coordinate transformation $X = \frac{2\omega}{g\cos\theta}x$ becomes

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial \varepsilon}{\partial X} = \gamma\varepsilon \tag{5}$$

Eq.(5) has the following analytical solution:

$$\varepsilon(X_0, t) = \begin{cases} e^{\gamma t}, t < X_0 \\ e^{\gamma X_0}, t \geq X_0 \end{cases}$$

which means that original Eq.(4) has the exact analytical solution

$$\varepsilon(x_0, t) = \begin{cases} e^{\gamma t}, t < \frac{2\omega}{g\cos\theta}x_0 \\ e^{\gamma\frac{2\omega}{g\cos\theta}x_0}, t \geq \frac{2\omega}{g\cos\theta}x_0 \end{cases} \tag{6}$$

i.e. any initial condition in the spatial fetch point x_0, frequency ω and angle θ is growing exponentially until it reaches critical value $e^{\gamma\frac{2\omega}{g\cos\theta}x_0}$, and stays equal to that value thereafter.

We performed numerical tests of the Eq.(4) with the help of "rectangle" numerical scheme, having second order approximation in space and time. Such scheme is unconditionally stable, and the only condition of its correspondence to the physical reality is temporal resolution of the exponential growth with increment γ, i.e time step of the numerical integration $\tau << \frac{1}{\gamma}$. This condition has to be taken into account for any full-blown numerical simulation of Hasselmann Eq.(1) to get physically sensible results.

Fig.3 shows the results of numerical simulation of the Eq.(4): the initial condition is growing exponentially until it reaches critical value, and stays constant afterwards.

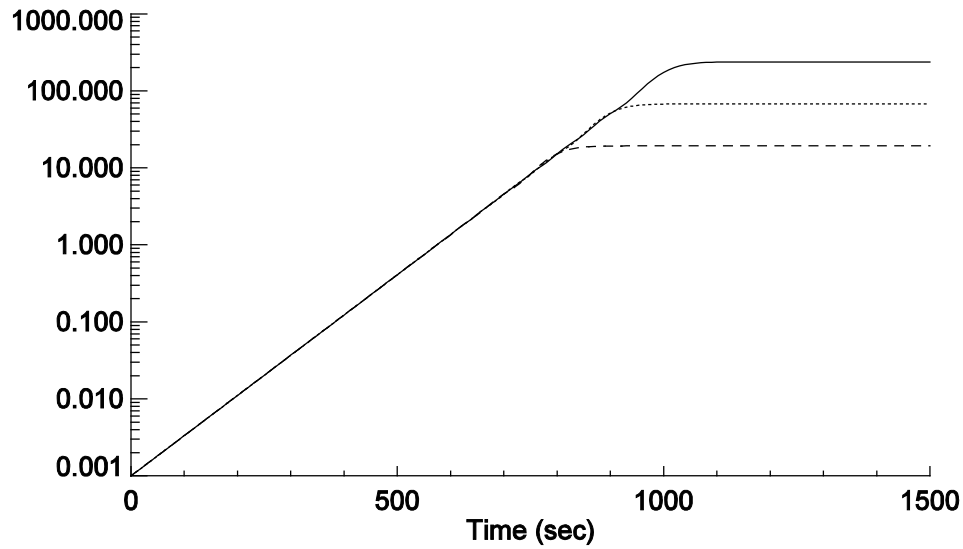

Fig.3: Logarithm of 3 spectral energy modes, located in different parts of the fetch. In accordance with analytical solution (6), they grow exponentially until reach the critical value.

Fig.4 shows total energy evolution as a function of the fetch and time and has important interpretation: negligence of the nonlinear interaction term leads to asymptotical stationary distribution of the wave amplitudes along the fetch. However, this distribution is exponential, and such distribution of energy has never been observed in the reality (the observed is the power one). Therefore, nonlinear term S_{nl} is at least absolutely necessary component of proper physical picture of the ocean wave surface. In fact, S_{nl} is the dominating source term in Hasselmann equation with respect to γ, as it was shown in [1].

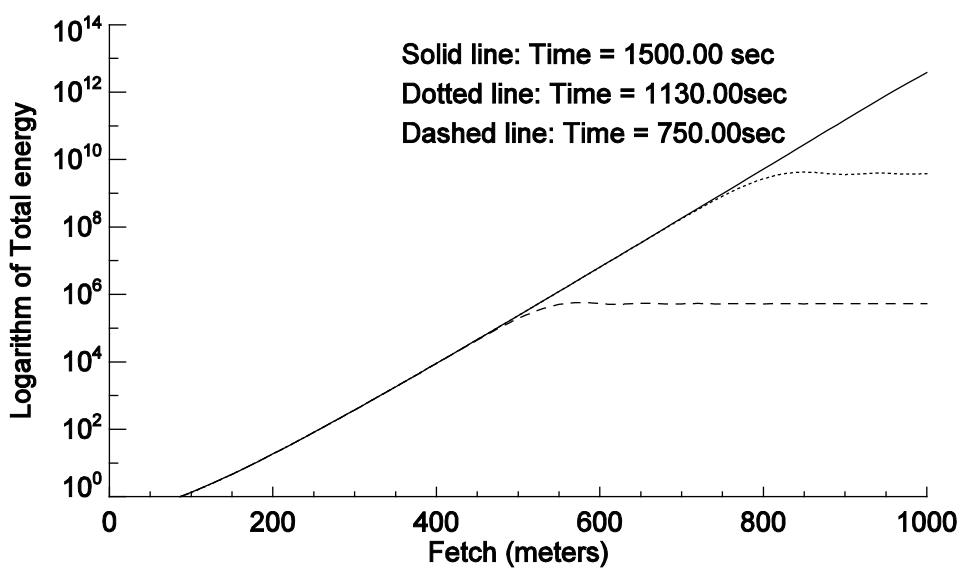

Fig.4: Total energy as a function of fetch and time

3. Detection of swell feedback by sea background through theory and experimental data

The global visual wave observations are reanalyzed within the theoretical concept of self-similar wind-driven seas. The core of the analysis is one-parameter dependencies of wave height on wave period. Theoretically, wind-driven seas are governed by power-like laws with exponents close to Toba's one 3/2, while the corresponding swell exponent (-1/2) has an opposite signature. This simple criterion was used and appeared to be adequate to the problem of swell and wind-driven waves discrimination. This theoretically based discrimination does not follow exactly the Voluntary Observing Ship (VOS) data. This important issue is considered both in the context of methodology of obtaining VOS data and within the physics of wind waves. The results are detailed for global estimates and for analysis of particular areas of the Pacific Ocean. Prospects of further studies are discussed. In particular, satellite data are seen to be promising for tracking ocean swell and for studies of physical mechanisms of its evolution.

Fig.5 gives a graphical summary of four reference cases of self-similar evolution of wind-driven waves. These cases are shown as different *R*, tangents of one-parametric dependencies

$$H \sim T^R$$

height-to-period in logarithmic axes. Reference cases of growing wind sea are shown as the young sea growth at permanent wave momentum production (exponent $R = 5/3$ by Hasselmann et al. 1976 [5]), growing Toba's sea ($R = 3/2$) and old premature sea by Zakharov and Zaslavsky 1983 [6] with $R = 4/3$.

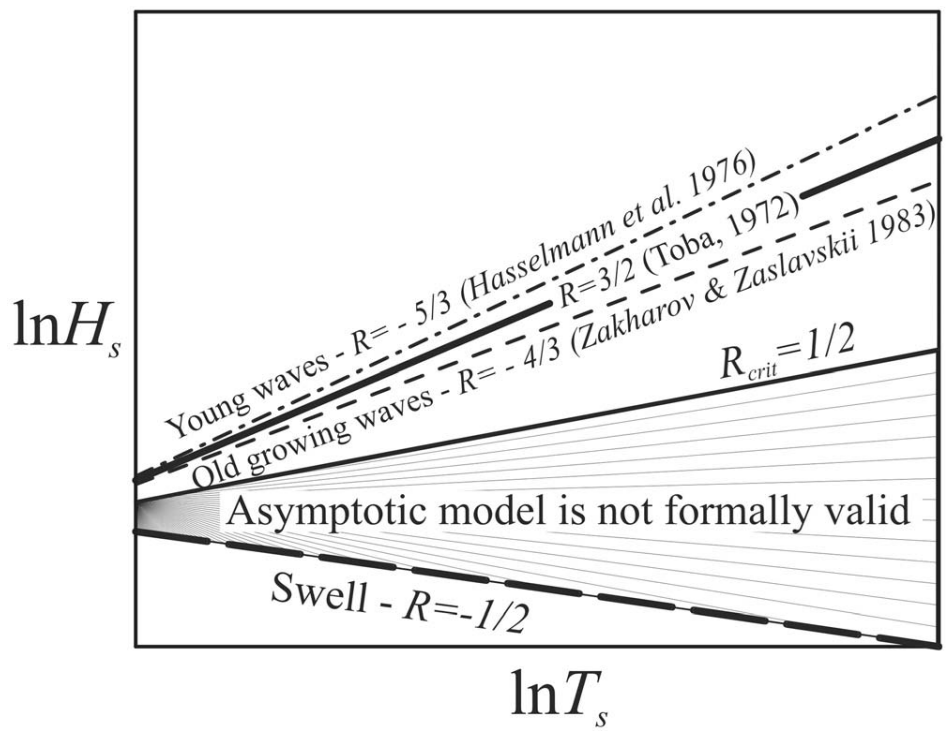

Fig. 5 Reference cases of wave growth as one-parametric dependencies $H_s(T_s)$. Cases of Toba [1972] R = 3/2 law and swell with R = -1/2 are shown by bold lines. Domain where the asymptotic scheme is formally invalid (Badulin et al., 2007 [7]) is shaded.

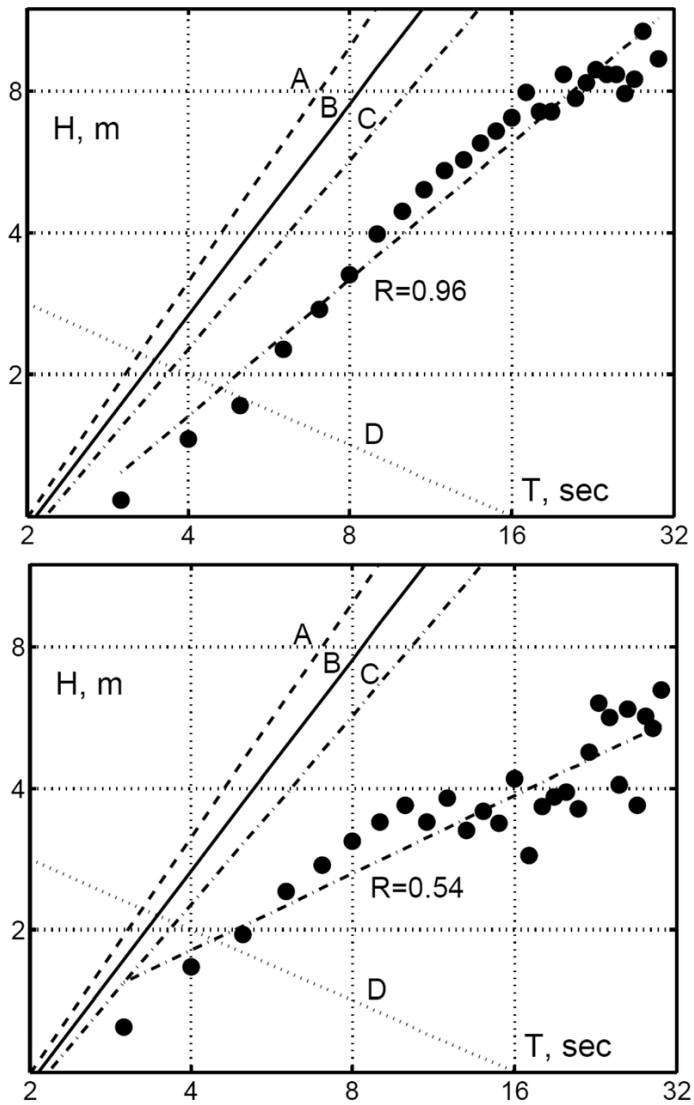

Fig.6 Dependencies H(T) and their power law fits for the whole World Ocean, 1970–2007 for (top) wind waves and (bottom) swell. Lines marked as A, B, C, D show reference power laws of Fig.5. The exponents of the experimental fits R = 0.96 for wind sea and R = 0.54 for swell are found to be quite far from the reference cases. Totally, 36,356,695 reports have been used for wind waves and 31,041,169 for swell observations.

The height-to-period dependencies *H(T)* derived from the VOS data are shown in Fig.6 for visually delineated wind waves (upper panel) and swell (bottom panel). The difference of these two cases is clearly seen in terms of exponents *R*. Thorough analysis of the experimental data uncovers more physically significant difference of two sea waves extremes. Selecting waves in wave ages and, what is more important, in wave periods we found definite indications on pumping of swell. Results of such selection is illustrated by Fig. 7 where histograms of exponents *R* estimated for 20° by 20° coordinate boxes are given. While for the wind waves the histogram is localized near a value more than 1, for swell the corresponding distribution is quite large that implies a variety of physical mechanisms responsible for the swell dynamics.

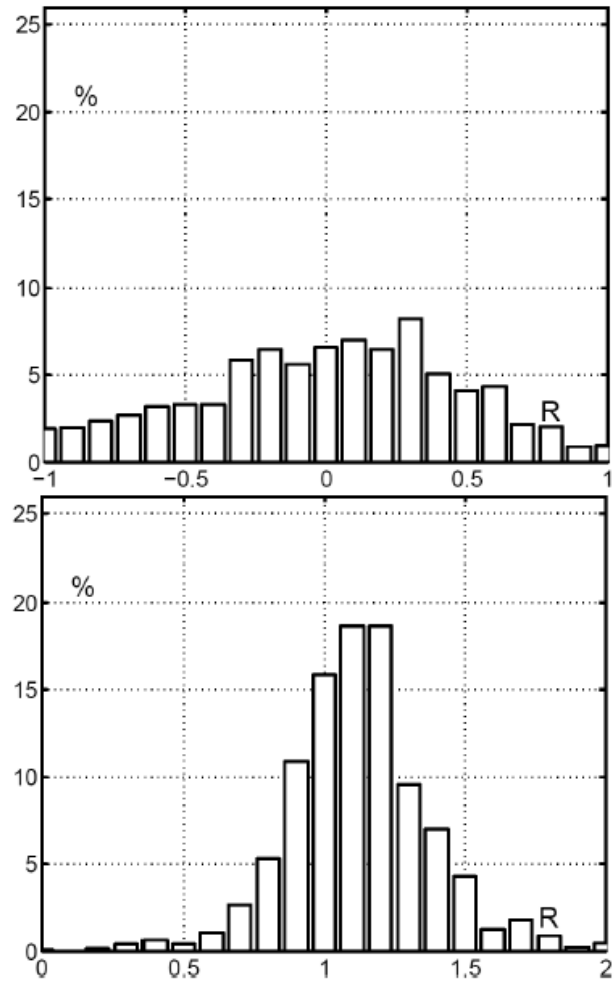

Fig. 7 Histograms of exponents R of power law fits of H(T) dependencies calculated monthly for 20x 20 boxes of the World Ocean (1481 dependencies of total 1782 = 8 x18x12 coordinate boxes) for special ranges of wave periods: (bottom) wind sea with T = 5–10 s and (top) swell with periods T = 10–20 s.

With the exponent *R* as indicator of sea wave dynamics we make a conceptual step: we study a link of wave heights *H* and periods *T* rather than features of the independent data sets. The separate analysis basing on VOS (Gulev et al., 2004 [8]) or satellite data (e.g., Zieger, 2010 [9]) gives valuable information on ranges of wave parameters and their geographical variability, but propose quite primitive vision of wave dynamics. Recent attempts to combine satellite altimeter observations of wave heights and mathematical modeling of wave dynamics (Laugel et al., 2012 [10]) propose reconstructions of full spatiotemporal structure of wind wavefield. This study is based on extensive simulations and requires thorough theoretical analysis. The interpretation of its results in the context of burning problems of sea wave physics shows a good prospect for further study.

4. Analytical and numerical proof of non-integrability of 2D free-surface hydrodynamics

We studied the problem of integrabily of 2D hydrodynamics of fluid with free surface in a gravity field. This conjecture was formulated in 1994. Here we studied the integrability for potential motion in the framework of Hamiltonian truncated equation up to the fourth order:

$$H=\frac{1}{2}\int g\eta^2+\psi\hat{k}\psi dx-\frac{1}{2}\int\{(\hat{k}\psi)^2-(\psi_x)^2\}\eta dx+\frac{1}{2}\int\{\psi_{xx}\eta^2\hat{k}\psi+\psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx+\ldots \qquad (7)$$

here $\eta(x,t)$ is the shape of a surface, $\psi(x,t)$ is a potential function of the flow given at the surface and g is gravitational acceleration. To simplify the Hamiltonian we applied canonical transformation which excludes all nonresonant temrs and instead of (7) we deal with the equivalent Hamiltonian

$$H=\int b^*\hat{\omega}_k b dx+\frac{1}{2}\int\left|\frac{\partial b}{\partial x}\right|^2\left[\frac{i}{2}\left(b\frac{\partial b^*}{\partial x}-b^*\frac{\partial b}{\partial x}\right)-\hat{K}\,|\,b\,|^2\right]dx. \qquad (8)$$

here $\hat{k}$ is the operator multiplying Fourier harmonics by mudulus of wave nambers. Corresponding equation of motion is:

$$i\frac{\partial b}{\partial t}=\hat{\omega}_k b+\frac{i}{4}\hat{P}^+\left[b^*\frac{\partial}{\partial x}(b'^2)-\frac{\partial}{\partial x}(b^*{}'\frac{\partial}{\partial x}b^2)\right]-\frac{1}{2}\hat{P}^+\left[b\cdot\hat{K}(|\,b'\,|^2)-\frac{\partial}{\partial x}(b'\hat{K}(|\,b\,|^2))\right], \qquad (9)$$

This equation has localized breather-type solution

$$b(x,t)=B(x-Vt)e^{i(k_0x-\omega_0t)}, \qquad (10)$$

where k_0 is the wavenumber of the carrier wave, V is the group velocity and ω_0 is the frequency close to ω_{k_0}. In the Fourier space breather can be written as follow:

$$b_k(t)=e^{-i(\Omega t+Vk)}\phi_k, \qquad (11)$$

where Ω is close to $\omega_{k_0}/2$. This solution is stable and does not radiate. In the integrable systems collisions of such beaters must be ellastic. We performed numerical simulation of collisions of two breathers and have found that it is not pure ellastic, see Fig.8.

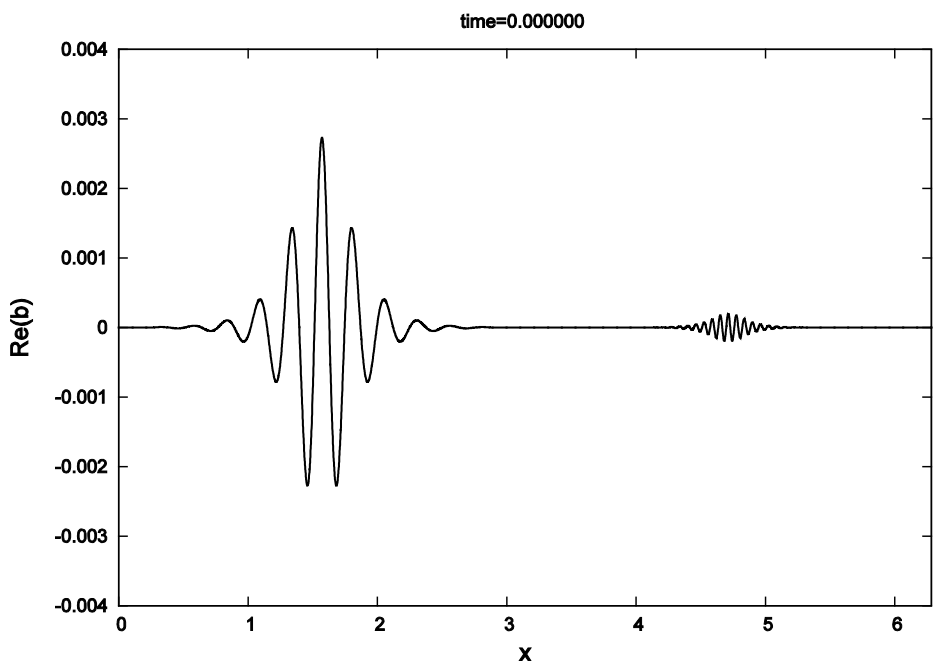

Fig.8 Surface profile of two breathers.

Also we studied analytically coefficient of 6-waves interaction as a superposition of 4-waves interaction. For integrable system it must be equal to zero on the resonance manifold. However it was found that it does not vanish. So, both numerical and anaylical study allow us to conclude that 2-D free surface hydrodynamics is not integrable system.

5. Dissipation of energy during the wave-breaking event

One of the most challenging tasks in improving the wave forecasting models is to develop an appropriate dissipation source term. The wave breaking phenomenon is the most important source of dissipation of energy for waves. Recently if has been found that the modulational instability is in principle a good candidate to explain the wave breaking mechanism in deep and intermediate water depth. The main idea is that a wave packet, if sufficiently steep, may go through a modulational instability process by which a single wave grows at the expense of the other waves. If the limiting steepness is reached, then a wave breaking is observed. Normally, the modulational instability is studied, at the leading order, by the Nonlinear Schrodinger equation and its exact solutions. Numerical simulations of the fully nonlinear Euler equation have also been performed in the past; however, the major limitation of these approaches is that they fail to capture the final stages of the wave breaking mechanism, therefore they are not capable of furnishing any result which concerns the dissipation of energy.

In the present work we have performed for the first time a direct numerical computation of the full Navier-Stokes equation for two-phase flow (air above and water below the surface) in order to study the wave breaking that results from the modulational instability mechanism. If the steepness of the initial wave is large enough, we observe a wave breaking, as shown in Fig.9, and the formation of large scale dipole structures in the air, see Fig.10. Because of the multiple steepening and breaking of the waves under unstable wave packets, a train of dipoles is released and propagate in the atmosphere at a height comparable with the wave length. The amount of energy dissipated by the breaker in water and air is considered and, contrary to expectations, we observe that the energy dissipation in the air is larger than the one in the water, see Fig.11.

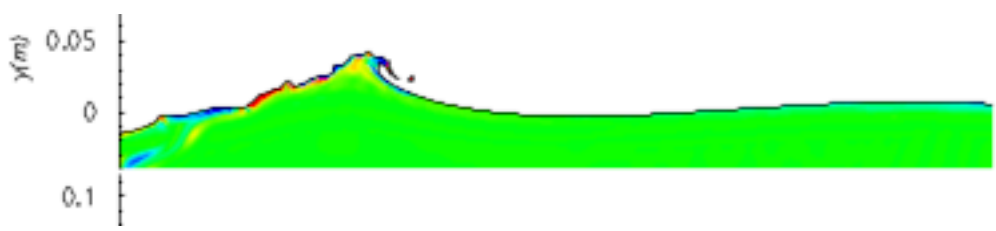

Fig.9 wave breaking during the modulational instability process

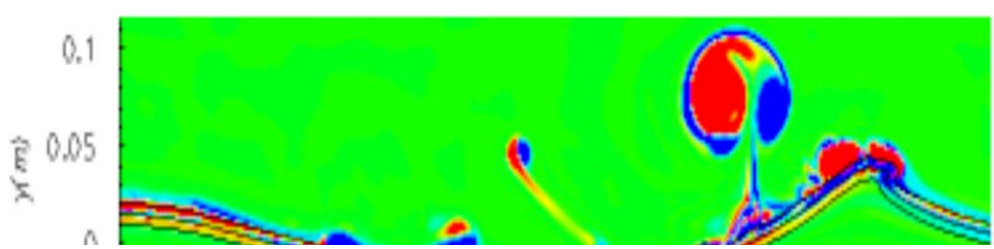

Fig.10 Dipole formation during wave breaking

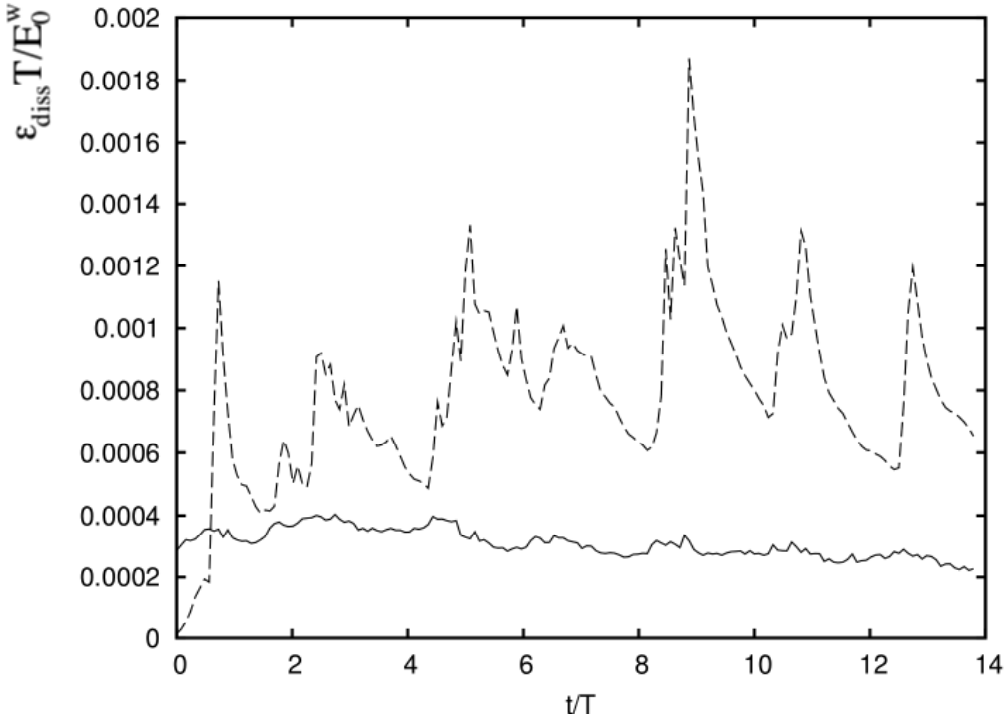

Fig.11 Dissipation of energy during multiple breaking events. Dashed line corresponds to dissipation in air and solid line to dissipation in water.

This result is somehow counter intuitive: most of the experimental research so far have measured dissipation due to wave breaking by looking at the dissipation of kinetic energy in the water. Possible consequences on the wave modeling of dissipation source term and on the exchange of aerosols and gases with the atmosphere are under investigation.

IMPACT/APPLICATIONS

- Confirmation of the correctness of proposed new wind input term for limited case situation should improve operational forecasting models
- New analytic solution for Hasselmann equation without nonlinear term will help to test operational models against correctness of the physics of described phenomena
- Detection of swell feedback by sea background through theory and experimental data helps to understand complex nonlinear mechanisms of different scales surface waves interaction in the ocean
- Analytical and numerical proof of non-integrability of 2D free-surface hydrodynamics will help to build simplified and fast nonlinear models of surface waves
- Discovery of large-scale dipole structures in the air appearing due to wave-breaking might alter current presentation of dissipation source terms in operational models

RELATED PROJECTS

None

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