CAVITY FLOWS OF VISCOUS LIQUIDS IN NARROW SPACES

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This communication is concerned with the motion of the interface between two immiscible viscous fluids contained in the narrow space between two closely spaced surfaces. One of the fluids may be of negligible viscosity and density, corresponding to the case of a cavity. When the surfaces are fixed, parallel, and plane, the configuration is known as a Hele-Shaw apparatus.

The penetration of a viscous fluid into a more viscous fluid contained in a straight-sided channel in a Hele-Shaw apparatus is discussed under certain simplifying assumptions about the physical conditions at the interface. These are that the pressure change on crossing the interface is constant, and that the sheets of the more viscous fluid left behind after the interface has passed are of constant thickness. The problem is not unique, possessing a singly infinite family of solutions, and some analytical features of the solution which single out a particular flow are described.

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INTRODUCTION

Taylor (1) has shown, and Lewis (2) has verified experimentally, that the surface between two superposed fluids of different densities and negligible viscosities which are accelerated in a direction perpendicular to their interface is stable or unstable to small deviations according as the acceleration is directed from the more dense to the less dense fluid or vice versa. If the fluids are confined within a long tube or channel, a final steady state appears to be attained in which a long bubble or finger of the less dense fluid advances along the tube whilst the more dense fluid drains away round the sides. This situation occurs when a vertical tube filled with fluid is closed at the top and opened at the bottom to the atmosphere. The steady motion of large bubbles or cavities rising through inviscid liquids in tubes and channels has been studied experimentally and theoretically; see, for example, Davies and Taylor (3); Garabedian (4), where other references may be found.

Saffman and Taylor (5) have shown that analogous phenomena occur when two superposed immiscible viscous fluids are forced with uniform velocity through a Hele-Shaw cell, that is, between two closely spaced, fixed parallel sheets. When the interface between the two fluids is straight and perpendicular to the direction of motion, a steady uniform motion is theoretically possible; but it can be shown that the interface is unstable to small deviations when the direction of motion is from the less viscous to the more viscous fluid (i.e., when the less viscous drives the more viscous) and stable when the more viscous pushes the less viscous. Actually this statement is strictly true only when the plates bounding the cell are horizontal, or the fluids have the same densities, so that the stabilizing or destabilizing effects of gravity are not important. When gravity effects have to be considered, the statement holds if the uniform velocity with which the fluids move through the cell is greater than a certain critical velocity (which may be negative) depending upon the density difference. However, when the plates are very close together or the fluids are very viscous, the pressures required to drive the fluids will be large compared with the hydrostatic pressure so that gravity may effectively be neglected even if the plane of the cell is vertical.

The instability was demonstrated experimentally and the motion into which the instability develops was observed. This consists of the penetration of the more viscous fluid by "fingers" of the less viscous one in a way analogous to the later stages of the instability of accelerated interfaces as reported by Lewis. A characteristic feature of the later stages of the growth of "instability" into "fingers" is the tendency of the fingers to space themselves so that the width of the fingers and the columns of more viscous fluid between them are of approximately the same breadth. Another characteristic feature was the inhibiting effect on the growth of its neighbours by any finger which gets ahead of them. These features are shown in the three photographs of Fig. 1 in which air was used to drive glycerine downwards through a Hele-Shaw cell consisting of two vertical rectangular sheets of flat glass 0.09 cm apart. The first photograph (a) was taken shortly after the beginning of the motion and shows the instability in its initial stages. The second photograph (b) was taken

(a) (b) (c)

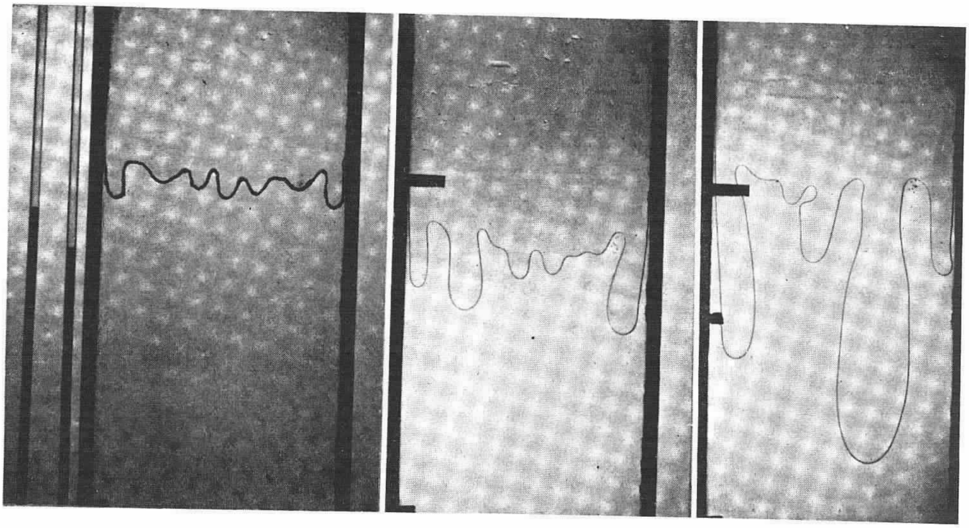

Fig. 1 - The development of the instability of an air-glycerine interface. The air is above and the glycerine below; the direction of motion is downwards. (a) An early stage, (b) a later stage, (c) a later stage than (b).

at a later stage of the instability (not the same experiment) and shows the tendency of the fingers to be separated by a distance approximately equal to their width. The third photograph (c), taken at a still later stage with another experiment, shows the inhibiting effect, because the three fingers on the right-hand side of this photograph all started to grow at the same time but the middle finger, which was slightly larger than its neighbours, has almost completely inhibited the growth of the others and has spread laterally.

In attempting to understand the mechanics of the formation and propagation of fingers from an unstable interface, the authors were led to the consideration of a single finger of fluid, or a cavity, propagating steadily through a channel of fixed width in a Hele-Shaw cell containing a more viscous fluid. Actually, it appears that if the superposed fluids with an unstable interface are confined within a sufficiently long parallel-sided channel in a Hele-Shaw cell, then "instability" finally develops into such a motion with the penetrating finger of less viscous fluid in the middle of the channel and a certain fraction of the channel width. Thus this problem will be closely related to the rise of an air bubble through a long vertical channel, closed at the top and open to the air at the bottom, in a Hele-Shaw cell, i.e., between two closely spaced sheets.

THE ANALYTICAL DESCRIPTION OF THE MOTION OF TWO IMMISCIBLE FLUIDS IN A HELE-SHAW CELL

For a viscous fluid moving between two closely spaced, fixed, parallel plates, the components of the mean velocity across the stratum are given by (see, for example, Lamb (6))

$$u = \frac{-b^2}{12\mu}\left(\frac{\partial p}{\partial x} + \rho g\right), \qquad v = \frac{-b^2}{12\mu}\frac{\partial p}{\partial y}, \tag{1}$$

where b denotes the (small) distance between the plates, p the pressure (assumed constant across the stratum), ρ the density and μ the viscosity of the fluid, and g the acceleration due to gravity. The plane of the plates is here taken as vertical, x and y are coordinates measured in the plane of the plates with the x-axis taken vertically upwards (the y-axis is therefore horizontal), and u and v are the components of the mean velocity averaged across the stratum in the x- and y-directions, respectively. The case in which the plates bounding the cell are horizontal is obtained by putting $g = 0$.

The mean velocity field is two dimensional and can be derived from a velocity potential

$$\varphi = \frac{-b^2}{12\mu}(p + \rho g x).$$

By virtue of the equation of continuity, $\nabla^2\varphi = 0$ and there exists a stream function ψ which is the harmonic conjugate of φ, so that the complex potential $\omega = \varphi + i\psi$ is an analytic function of $x + iy$. (This result was used by Hele-Shaw (7) for the purpose of demonstrating the streamlines in the irrotational flow of an inviscid liquid in two dimensions.)

Consider now the motion of the interface between two immiscible fluids of viscosities μ_1, μ_2 and densities ρ_1, ρ_2, and suppose the direction of motion of the interface is away from fluid 2 towards fluid 1; i.e., fluid 2 drives fluid 1. For a cavity flow, μ_2 and ρ_2 will be zero, but for the moment we shall suppose this is not necessarily so. Now fluid 1 is not necessarily completely expelled or replaced by fluid 2; a film of fluid 1 may wet the plates and adhere to them whilst a tongue of fluid 2 advances along the middle of the gap between the plates. The thickness of this tongue is a fraction, t say, of the gap between the plates and it may be expected that t will depend upon the viscosities of the fluids, the velocity of advance of the tongue, and the interfacial tension.

That is, the motion may be, in fact, a combination of two different types of cavity flow. There is that flow viewed by an observer standing in front of the cell looking in a direction perpendicular to the plates; he sees the motion of the two-dimensional interface between two regions, one occupied by fluid 1 and the other by fluids 2 and (possibly) 1, and is observing in fact the projection, on the plates bounding the cell, of the boundary of the region containing fluid 2. The velocity of the interface he sees is actually the velocity of the tip of the tongue between the plates. On the other hand, an observer standing at the side and looking between the plates (assuming this is possible) sees the penetration of a viscous fluid between two closely spaced parallel plates by a tongue of another viscous fluid, and this cavity flow, although schematically similar to the first, is in fact governed by different equations, namely those of slow creeping motion which reduce to the biharmonic equation for two-dimensional flow, whereas the first type of cavity flow may be described (under assumptions formulated below) by solutions of Laplace's equation. The two flows are not independent, but, by making plausible assumptions about the thickness of the tongue, it is possible to analyse the motion as seen by an observer standing in front, and this communication is concerned with cavity flows regarded from this viewpoint.

Suppose first that one fluid completely expels the other so that $t = 1$. In this case the tip of the meniscus viewed by an observer in front represents an interface completely separating the two fluids. Also, the streamlines which would be seen by this observer if the fluids were marked are those of the mean velocity field. The mean velocity across the stratum of fluid 1 ahead of the interface can be derived from the velocity potential

$$\varphi_1 = -(b^2/12\mu_1)(p + \rho_1 g x)$$

and that of fluid 2 from

$$\varphi_2 = -(b^2/12\mu_2)(p + \rho_2 g x),$$

where φ_1 and φ_2 both satisfy Laplace's equation. Since b is small, the width of the projection of the meniscus onto the plates is small compared with the length scale of the mean motion, and the interface may be regarded as a sharp line. It follows from continuity that the component of mean velocity normal to the interface is continuous across it and is equal to the velocity of the interface normal to itself; i.e.,

$$\partial\varphi_1/\partial n = \partial\varphi_2/\partial n, \qquad \text{or} \quad \psi_1 = \psi_2 \tag{2}$$

at the interface, where $\partial/\partial n$ is differentiation in the normal direction.

The authors (5) have given some evidence to show that, in cases likely to be met with in cavity flows of this type, t may be within 1-1/2 percent of unity. However, the conditions determining the value of t and the mechanics of the second type of cavity flow are far from being fully understood, and the assumption $t = 1$ will not be reasonable in all the cases likely to be encountered. On the other hand, it appears that the thickness of the tongue may be sensibly constant for some cases, and if this is so it transpires that it is possible to analyse the motion in a way similar to the case $t = 1$.

Suppose now that the thickness of the tongue is constant. The interface can now be identified with the tip of the tongue. Ahead of the tongue, the mean velocity is given by Eq. (1); behind the interface, the velocity distribution across the gap between the plates can be worked out making the same approximation used in calculating Eq. (1) that velocity gradients in directions parallel to the plates are small compared with those in the perpendicular direction and that the pressure is constant across the gap. The mean velocities in the tongue and the layers adhering to the plates can be shown to be derivable from velocity potentials which satisfy Laplace's equation, and continuity considerations at the interface give a boundary condition between the velocity potentials on each side. It can then be shown that the interface moves as if it were an interface completely separating two imaginary fluids of different viscosities and densities, one of which completely expels the other, i.e., for which $t = 1$. For the case in which a viscous fluid is penetrated by a cavity (i.e., $\mu_2 = \rho_2 = 0$), the motion of the interface is the same as if a fluid of viscosity $t\mu_1$ and density $\rho_1\{1 + (1 - t)(1 - t^2)\}$ were completely expelled, the pressures in the two flows being the same.

In other words, the case t = constant is equivalent, for the purposes of mathematical analysis, to $t = 1$. The analysis will henceforward be based on the assumption that t = constant, since otherwise a simple description of the motion is not possible; so that it is sufficient in fact to consider only the case $t = 1$ without loss of generality. The available evidence indicates that this is a reasonable assumption, but further work remains to be done.

To complete the description of the equations governing the flow, it is necessary to know what is the pressure drop across the interface. If the fluid wets the plane surfaces, it might be expected that under static conditions the pressure drop would be $T(2/b + 1/R)$, where R is the radius of curvature of the projection, on the plane of the plates bounding the cell, of the meniscus. When the fluids are moving, the pressure drop across the interface may depend upon a variety of physical circumstances which are not yet understood. The simplest assumption we can make is that the pressure drop has the same value as in the static case and hence that

$$p_2 - p_1 = \frac{T}{R} + \text{constant} \tag{3}$$

on the interface. Further, the pressure gradients in the viscous fluid are of order $\mu u/12b^2$, where u is the velocity of the interface, so that if $T/R \ll \mu uR/12b^2$, i.e., $\mu uR^2/(12Tb^2) \gg 1$ (which is likely to be the case if the motion is not very slow), it seems reasonable to neglect T/R and take the boundary condition at the interface as

$$p_2 - p_1 = \text{constant} = 0, \text{ say.} \tag{4}$$

The pressure inside a cavity is effectively zero, so that for cavity flows the boundary condition in this case is $P = 0$ at the surface of the cavity.

THE PENETRATION OF A CAVITY INTO A CHANNEL

The authors (5) have investigated the propagation of a finger of fluid through a straight parallel-sided channel in a Hele-Shaw cell containing a more viscous liquid. They showed that when the fluid in the finger has negligible viscosity, and hydrostatic pressures can be ignored, and it is assumed that the viscous fluid is completely expelled, i.e., $t = 1$ (or, as is mathematically equivalent, t = const.), and the pressure is assumed constant on the interface, i.e., Eq. (4) is used, then the potential and stream function of the mean velocity (across the stratum) of the viscous liquid satisfy the boundary conditions:

$$\varphi \sim Vx \text{ as } x \to +\infty$$

where V denotes the uniform mean velocity of the fluid at infinity ahead of the finger;

$$\psi = \pm V \quad \text{on} \quad y = \pm 1 ,$$

since the walls of the channel which are taken as $y = \pm 1$ are streamlines of the mean velocity (edge effects which occur at walls in the Hele-Shaw cell and invalidate Eq. (1) within distance b of them are neglected); $\varphi = 0$ on the interface, by virtue of Eq. (4); and (assuming the finger is symmetrical about the centre line of the channel)

$$\psi = Vy \quad \text{on} \quad \varphi = 0 ,$$

where V is the velocity of propagation of the finger. The solution of this free boundary problem was shown to be

$$x + iy = \frac{\varphi + i\psi}{V} + \frac{2}{\pi}(1 - \lambda) \log \frac{1}{2}\left[1 + \exp \frac{-\pi(\varphi + i\psi)}{V}\right], \tag{5}$$

and the velocity of the finger is related to the mean velocity at infinity by

$$V = \lambda U , \tag{6}$$

where λ is the fraction of the channel width occupied by the finger after the nose has passed; i.e., 2λ is the distance between the straight sides of the finger. The equation of the interface is the image of $\varphi = 0$ and is easily seen to be

$$x = \frac{1 - \lambda}{\pi} \log \frac{1}{2}\left(1 + \cos \frac{\pi y}{\lambda}\right) . \tag{7}$$

It was pointed out further that the case in which the fluid in the finger has a non-zero viscosity and hydrostatic pressures are not neglected can be reduced to this case, provided the same simplifying assumptions, namely those embodied in Eqs. (2) and (4), are made about the physical conditions at the interface. In particular, the equation of the interface is still given by (7). For the case of a cavity rising through a vertical channel in a vertical cell, the relation between the velocities is

$$\lambda U = V + (12\mu/b^2)(1 - \lambda)\rho g.$$

$V = 0$ corresponds to the case in which the channel is closed at the top, and the velocity of rise is then

$$U = \frac{1 - \lambda}{\lambda} \frac{12\mu}{b^2}\rho g .$$

Now there is nothing in the analysis which leads up to Eqs. (5), (6), and (7) which specifies the value of λ, the fraction of the channel occupied by the finger after the nose has passed. That is, if only the conditions at infinity in front of the cavity are specified, the mathematical problem does not appear to possess a unique solution and there is an infinity of possible shapes for the interface, each with a different velocity of penetration. (It is worth noting that, according to the analysis, the family of possible shapes is independent of the physical properties of the fluids.) The calculated shapes for $\lambda = 0.2$, 0.5, and 0.8 are shown in Fig. 2.

It was noted recently by Garabedian (4) that the analogous free-boundary problem of an air bubble rising through a vertical tube or channel containing an inviscid

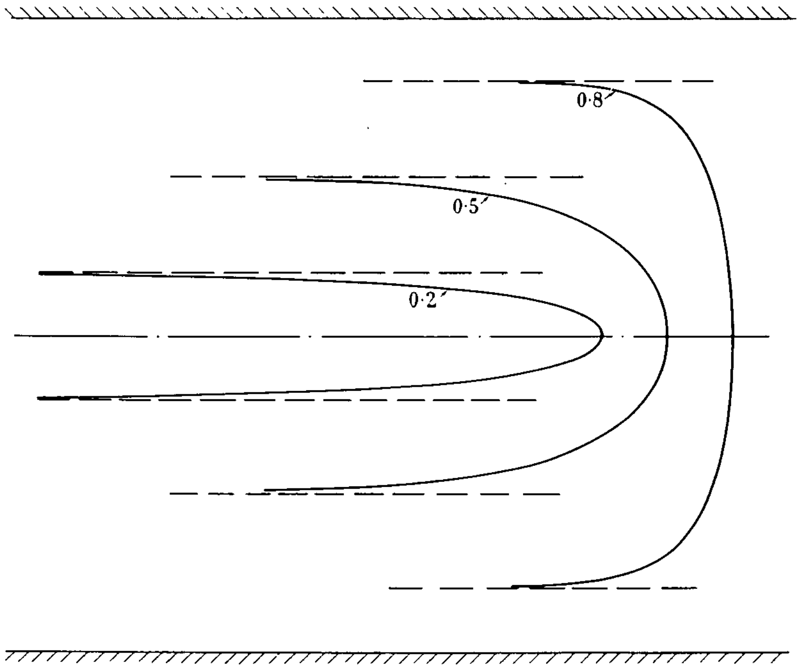

Fig. 2 - Calculated profiles with $\lambda = 0.2$, 0.5, 0.8 for a finger propagating through a channel

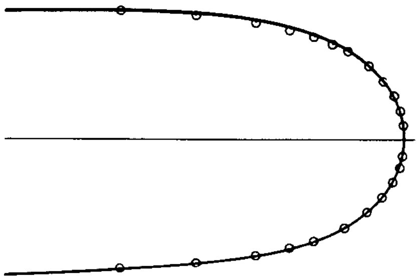

Fig. 3 - Comparison between an observed profile and the interface calculated from Eq. (7) with $\lambda = 1/2$

liquid also does not possess a unique solution. It is nice to be able, in the present, much easier, problem, to demonstrate explicitly a singly infinite family of solutions.

This motion was investigated experimentally using a variety of fluids. In the first series of experiments, a finger of air was blown through a channel containing glycerine and photographs were taken of the interface. It was found that the value of λ for several experiments was within 2 percent of 0.50. The observed shapes of the interface were compared with the calculated shape with $\lambda = 0.50$, and the agreement was found to be very good. In Fig. 3 the line is the photographed interface and the circles are surrounding points calculated from Eq. (7) with $\lambda = 1/2$. This agreement seems to justify the assumed boundary conditions.

The question now arises why in practice the finger with $\lambda = 1/2$ is singled out from the infinity of theoretically possible solutions. It might be thought that a hypothesis of a maximum or minimum rate of energy dissipation might single out a unique value of λ, but this in fact is not so, since it is easily seen that for given conditions ahead of the finger the rate of energy dissipation is independent of λ.

Further experiments were carried out in order to see whether λ depended upon the ratio of the viscosities of the fluids inside and outside the finger or upon any other of the physical properties of the fluids. These experiments showed that it was often possible to obtain very uniform fingers (Fig. 4 shows an example with water penetrating a narrow channel filled with an oil), and that the value of λ was always close to 1/2, provided that the speed of the finger was not too slow.

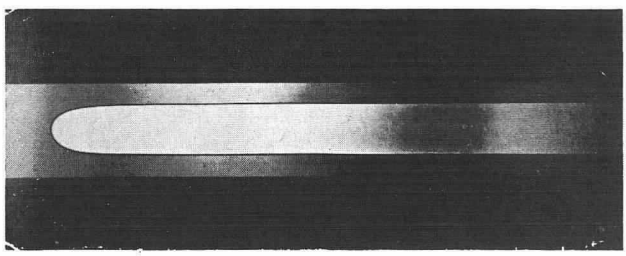

Fig. 4 - A photograph showing a uniform finger of water penetrating a channel filled with oil

At low speeds, the value of λ increased from 0.5 to the limit 1.0 as the velocity tended to zero, but it was found that the wider forms did not conform to the corresponding profiles calculated from Eq. (7) with the same value of λ. This indicated that surface tension, or some equivalent surface stress, was affecting the shape of the interface, and that the boundary condition (Eq. 4) is not applicable when u is too small (as is indeed to be expected according to the previous discussion). Restricting attention to the case of cavity flows in which the viscosity of the fluid in the finger is negligible, it would then be expected on dimensional grounds that the value of λ in a channel of given shape would be a function of $\mu U/T$ only, when the fluid wets the flat sides of the channel. The results of some experiments are shown in Fig. 5 and justify this expectation.

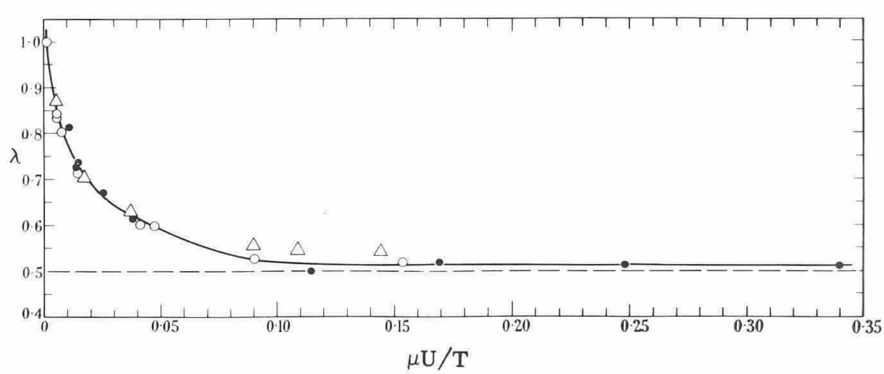

Fig. 5 - Dependence of λ on $\mu U/T$

The value of λ rapidly decreases to 1/2 as $\mu U/T$ increases, i.e., as the effect of interfacial stress in determining the shape of the interface becomes less important in comparison with that of the viscous stress. This indicates that when the physical conditions are such that the boundary condition (Eq. 4) is a reasonable representation of the actual physical conditions at the interface, then the only one of the infinity of possible shapes given by Eq. (7) that can actually occur is that for which $\lambda = 1/2$, a conclusion which is supported by the evidence of Fig. 3.

The authors (5) were unable to put forward any theoretical reason for this deduction from observation. Further work still in progress has not yet succeeded in accounting for this phenomenon, but a number of mathematical results, which however lack an obvious physical significance, have been discovered which single out $\lambda = 1/2$. These will be described in the following section.

SOME MATHEMATICAL RESULTS WHICH DISTINGUISH $\lambda = 1/2$

1. The first result is of a geometrical character. Consider the area between the surface of the cavity, its asymptotes $y = \pm\lambda$, and the tangent at the vertex of the finger. This area is, according to Eq. (7),

$$A = 2\int_0^\lambda x\,dy = -\frac{2(1-\lambda)}{\pi}\int_0^\lambda \log\frac{1}{2}\left(1+\cos\frac{\pi y}{\lambda}\right)dy$$

$$= \frac{4\log 2}{\pi}\lambda(1-\lambda)$$

Hence, the arbitrary criterion that A is a maximum gives $\lambda = 1/2$!

2. An exact unsteady solution, under the same simplifying assumptions about the physical conditions at the interface, has been discovered (8) which represents the change with time of the interface between a cavity and a viscous fluid in a channel in a Hele-Shaw cell. Initially, the shape of the interface is such that it is almost flat and extends right across the channel; as the time becomes large, the shape tends to that of a long finger propagating steadily through the channel. That is, the solution represents the growth of a finger from a flat interface which is disturbed initially in a particular way; by reflecting in the channel walls we have a solution for the growth of equal and equally spaced fingers from an unstable interface.

There is a singly infinite family of these unsteady solutions corresponding to different values of the ratio of the width of the final finger and that of the channel; this parameter is denoted by λ. It was shown that the shape of the interface for each value of λ is obtained by eliminating the real variable s from

$$x + iy = s + \frac{2}{\pi}(1-\lambda)\log\frac{1}{2}\left\{1 + a(\tau)e^{-i\pi s}\right\} + \text{constant},$$

where $a(\tau)$ is a function of the time τ which is small when $\tau = 0$ and which tends to one as $\tau \to \infty$ (giving Eq. (7) in the limit). It was also shown that

$$\log(1-a) \propto -\frac{\tau}{\lambda(1-\lambda)} \quad \text{as} \quad \tau \to +\infty,$$

so that the rate at which $a \to 1$, which corresponds intuitively with the rate at which the final asymptotic state is attained, is a minimum when $\lambda = 1/2$, thus distinguishing the finger which is half the width of the channel.

Actually, this result is puzzling, since it might intuitively be expected that the actual shape would correspond to a maximum rate of formation.

3. Consider the mean (two-dimensional) velocity field in a channel due to the supposition of a uniform mean velocity $V/2$ and a "source" of strength $2V$, i.e., a source of viscous fluid which gives rise to a (two-dimensional) distribution of mean velocity whose magnitude is $V/\pi r$ near the source and radially outwards. Using the notation from the preceding section, it is easily seen that the complex potential $\omega = \varphi + i\psi$ is related to $z = x + iy$ by

$$\omega = Vz + \frac{V}{\pi} \log \left(2 \sinh \frac{1}{2} \pi z\right) = \frac{V}{\pi} \log (e^{\pi z} - 1) . \tag{8}$$

At $z = -\infty$, the mean velocity is zero, at $z = +\infty$ the mean velocity is uniform of magnitude V.

The equation of the equipotential $\varphi = 0$ is

$$x = \frac{1}{\pi} \log 2 + \frac{1 - \frac{1}{2}}{\pi} \log \frac{1}{2} \left(1 + \cos \pi y \Big/ \frac{1}{2}\right) ,$$

and, on it, $\psi = 2Vy$. Hence, the mean velocity field in a channel due to a "source" of strength V moving with velocity $2V$ and a uniform mean velocity $1/2\ V$ is identical with the mean velocity field in a viscous fluid being driven through the channel by a finger whose width is half that of the channel (provided the conditions at the interface may be represented by the simplifying assumptions), which is a mathematical feature to distinguish $\lambda = 1/2$.

All the flows with $\lambda = 1/2$, that is, for all combinations of viscosities, densities, and various values of t, can be so obtained by a superposition of a uniform stream and a moving source of appropriate magnitudes. Zhuralev (9) has pointed out that two-dimensional cavity flows in a porous medium (which is mathematically analogous to flow in a Hele-Shaw cell) maybe constructed by a suitable superposition of sources and sinks and he refers to the solution (Eq. 8). It is, in fact, possible to demonstrate that the flows with $\lambda \neq 1/2$ can be constructed in a similar way, but it turns out that the system of sources required is much more complicated involving line sources and line dipoles instead of the simple source which suffices for $\lambda = 1/2$.

4. Before we give the next feature, it is first necessary to describe briefly a stability investigation. The stability to small disturbances of the interface of a cavity penetrating a channel was investigated theoretically since it was thought that this might throw some light on the occurrence of $\lambda = 1/2$ in practice. Taking the origin at a point fixed in space (and neglecting hydrostatic pressures), the steady mean motion is given by (τ denotes time)

$$z_0 = \frac{\omega}{V} + U\tau + \frac{2}{\pi}(1 - \lambda) \log \frac{1}{2}\left\{1 + e^{-\pi\omega/V}\right\}, \qquad V = U\lambda, \tag{5'}$$

where the $U\tau$ term arises because the origin is fixed in space and not fixed relative to the bubble as in the derivation of Eq. (5), and z is taken as a function of ω, so that in fact we are working in the potential plane. Suppose the motion is slightly disturbed, so that it is given by

$$z = z_0(\omega) + \epsilon z_1(\omega, \tau), \qquad z_1 = x_1(\varphi,\psi) + iy_1(\varphi,\psi),$$

where ϵ is small. The conditions that the disturbance vanishes at infinity and that the walls of the channel are streamlines are $x_1 \to 0$ as $\varphi \to \pm\infty$ and $y_1 = 0$ on $\psi = \pm V$, respectively. If we assume that the boundary conditions at the interface are unaltered, i.e., the interface is taken as $\varphi = 0$ and the value of t is supposed to remain unaltered, then it can be shown that the condition satisfied by x_1 and y_1 on the interface is

$$\frac{\lambda}{V}\frac{\partial x_1}{\partial \tau} + \frac{1}{\lambda}\frac{\partial x_1}{\partial \varphi} + \frac{1-\lambda}{V}\frac{\sin(\pi\psi/V)}{1+\cos(\pi\psi/V)}\frac{\partial y_1}{\partial \tau} = 0 \qquad \text{or} \quad \varphi = 0,$$

to the first order in ϵ. It is then easy to show that

$$z_1^{(n)}(\omega,\tau) = e^{n\pi V\tau/\lambda}\sum_{r=0}^{n} a_{rn}\, e^{-r\pi\omega/V}$$

satisfies all the necessary conditions, and that any disturbance which leaves the straight sides unaltered at ∞ (i.e., which vanishes as $x \to -\infty$) can be expressed as a sum of disturbances of this type. The a_{rn} are real constants determined by a set of simultaneous equations. Since the amplification factor is positive, the conclusion of this analysis is that all possible shapes calculated using Eq. (7) are unstable; a conclusion which is manifestly wrong (see, for example, Fig. 4).

This indicates that surface tension, or some other effect of interfacial stress, must stabilize the motion in some way, even when $\mu U/T$ is large, and that it is not legitimate to discuss the stability under the same simplifying assumptions about the physical conditions at the interface which were used in calculating the steady motion. To explain why surface tension can be neglected in calculating the steady shape, but yet is apparently important in its effect on the stability, is a problem which has not yet been resolved. Partly, the difficulty is that our knowledge of the actual physical conditions at the interface is limited, though it is hoped that further work will clarify this, and also throw light on the mechanism which selects the value of λ. It should be noted that, along the straight sides of the finger, the velocity of the fluid is exponentially small so that $\mu U/T$, where U is a local velocity, must eventually be small; and the surface tension effect may be expected to be large relative to the viscous stress along the straight sides, although it will still be small relative to the viscous stresses at the nose.

In this connection, a mathematical feature which distinguishes $\lambda = 1/2$ of the stability analysis given above may possibly be relevant. If we express the velocity potential and stream function in terms of the physical coordinates in the region of the straight sides of the finger (i.e., as $x \to -\infty$), it can be shown that the perturbation of the pressure behaves as

$$e^{\frac{\pi x}{2(1-\lambda)}}\cos\frac{\pi(1-y)}{2(1-\lambda)}\, f(\tau) + e^{\frac{\pi x}{(1-\lambda)}}\cos\frac{\pi(1-y)}{(1-\lambda)}\, g(\tau) + O\left(e^{\frac{3\pi x}{2(1-\lambda)}}\right)$$

where $f(\tau)$ and $g(\tau)$ are exponentially growing functions of the time. It so happens that $f(\tau) \equiv 0$ for any disturbance when $\lambda = 1/2$, but the physical significance of this is not yet clear.

MOTION OF A FINITE BUBBLE IN A CHANNEL

In a further attempt to find an explanation for $\lambda = 1/2$, we were led to the theoretical consideration of the steady motion through a channel in a Hele-Shaw cell of a bubble of finite area bounded by a closed curve symmetrical about the centre line of the channel, under the same assumptions about the physical conditions at the interface as were used for calculating the shape of the long fingers. Restricting attention to the case in which the fluid in the bubble is of negligible viscosity and hydrostatic pressures can be neglected, a solution is required of the free boundary problem formulated earlier in this paper, except that the interface has now to be closed instead of infinite. For brevity we shall omit the analytical details and confine ourselves to quoting the final result.

It may be verified that

$$z = \omega - \frac{4}{\pi}\frac{(U-1)}{U}\tanh^{-1}\left[\frac{\tan\frac{1}{4}\pi U\lambda}{\tan\frac{1}{2}\pi U\lambda}\left\{\tanh\frac{1}{2}\pi\omega - \left(\tanh^2\frac{1}{2}\pi\omega + \tan^2\frac{1}{2}\pi U\lambda\right)^{1/2}\right\}\right]$$

gives the velocity potential and stream function of the mean velocity when a finite bubble bounded by the curve

$$x = \frac{2}{\pi}\frac{U-1}{U}\tanh^{-1}\left[\sin^2\left(\frac{1}{2}\pi U\lambda\right) - \cos^2\left(\frac{1}{2}\pi U\lambda\right)\tan^2\left(\frac{1}{2}\pi U y\right)\right]^{1/2}$$

moves steadily with velocity U, the velocity at infinity ahead of and behind the bubble being uniform and equal to unity. The bubble is symmetrical about its centre, its length is

$$\frac{4}{\pi}\frac{U-1}{U}\tanh^{-1}\sin\left(\frac{1}{2}\pi U\lambda\right) = L, \qquad \text{say,}$$

its maximum width is 2λ, and its area is

$$\frac{16}{\pi}\frac{U-1}{U^2}\tanh^{-1}\tan^2\left(\frac{1}{4}\pi U\lambda\right) = S, \qquad \text{say.}$$

If $U\lambda \to 1$, keeping U fixed, the area of the bubble tends to ∞, and it is easily shown that we can recover Eqs. (5) and (7). This is, of course, only to be expected since a long finger may clearly be regarded as the front of a large bubble. It is clear that the motion is mathematically not unique. For a bubble of given area and for given conditions at infinity, there is again a singly infinite family of mathematically possible shapes.

When λ is small compared with one and U remains finite, the dimensions of the bubble are small compared with the width of the channel, so that this is the case of the motion of a small bubble. The equation of the interface reduces to

$$\frac{x^2}{(U-1)^2} + y^2 = \lambda^2$$

so that a small bubble is (according to the analysis) an ellipse of axis ratio $U-1$, where U is really the ratio of the velocity of the bubble to the velocity at infinity. When $U = 2$, the interface is circular.

Now the analysis has so far been based on the assumption that the pressure drop across the interface is given by Eq. (4). For small bubbles, surface tension is likely to be of some effect, in which case the pressure drop is more likely to be given by Eq. (3). It is worth noting that the above solution for small bubbles is valid exactly using the boundary condition (3) when the bubble is circular. Furthermore, the effect of surface tension may be expected to tend to reduce the length of the perimeter and make a small bubble circular; and we should therefore expect the motion of a small bubble to be physically unique with the bubble circular and moving with twice the velocity at infinity (if $t = 1$).

Visual observations of small bubbles indicate that they are indeed circular when sufficiently small. When larger, they are often pear shaped with a rounded front and pointed back; some, on the other hand, seem to be ovoid with the sharper end pointed in the direction of motion. These latter shapes are not predicted by the analysis and are probably due to the physical conditions at the retreating interface over the back of the bubble being different from those at the front; there is, however, no clear evidence on this point. When the dimensions of the bubble become comparable with the width of the channel, the front of the bubble resembles closely that of a long finger, but the shape at the back is different, for probably the same reason.

Returning to the exact solution for a bubble of arbitrary size, a particular shape is singled out by making the arbitrary hypothesis that $U\lambda$ should be a minimum for a bubble of given size. It is easily seen that this gives $U = 2$. When the bubble is small, the circular bubble is singled out; when the bubble is large, i.e., $U\lambda \approx 1$, this gives $\lambda \approx 1/2(!)$ and in the limit the finger which is half the width of the channel.

The product $U\lambda$ of the velocity of the bubble and its maximum width does not have a clear physical significance, although it may be identified intuitively with the rate at which fluid is pushed aside by the bubble. Attempts to give this result a sound physical basis have so far failed.

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DISCUSSION

G. Birkhoff (Harvard University)

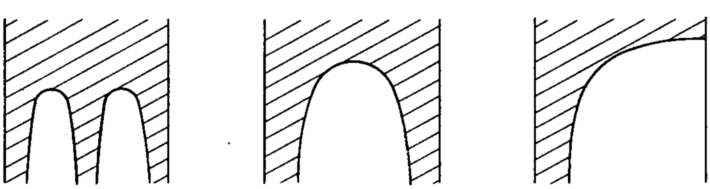
I agree that surface tension is probably an important stabilizing factor in the experiments considered, especially for shorter wavelengths as in the experiments of Lewis (1950) on ordinary Taylor instability. In analyzing the experimental data, it would be interesting to tabulate the dimensionless parameter

$$\mu V/(\rho - \rho')\, g d^2 = F/[(1 - \rho'/\rho) R_e] .$$

This, roughly, expresses the ratio: (viscous pressure gradient)/(gravity pressure gradient), if d is the plate spacing (about 0.09 cm). A casual inspection of Fig. 13 of Saffman-Taylor (1958) suggests that, perhaps, gravity might explain the marked deviation from (17).

As regards λ, it would be interesting to see how stable the value $\lambda = 1/2$ was against its initial width, which could be controlled by forcing an air bubble into a channel from a narrow orifice. It would also be interesting to see how stable the centering of the bubble about the channel axis was, by forcing air from an orifice near one edge. Further, it would be interesting to know how stable the bubble shape is against small variations in d. For example, if the plates are not parallel how great is the tendency for the bubble to migrate towards the side where the spacing is greater?

Finally, I should like to mention that the first demonstration of non-uniqueness in such problems was made by Carter and myself* using simple considerations of symmetry (see sketch). I should also like to record my personal impression that Garabedian's discussion is not rigorous. As this point has been explained† I shall not discuss it further here.



P. G. Saffman

With regard to Professor Birkhoff's first comment, the plane of the Hele-Shaw apparatus was horizontal in most of the experiments. The dimensionless parameter $\mu V/(\rho - \rho')\, g d^2$ is not relevant to these experiments, and in particular the deviation shown in Fig. 13 from the calculated shapes is not capable of being explained by gravity effects. Indeed, the same family of shapes is given by the analysis irrespective of whether the plane of the apparatus is vertical or horizontal although the

*G. Birkhoff and D. Carter, Appendix D of Rep. LA-1927 of the Los Alamos Scientific Labs.

†G. Birkhoff and D. Carter, Arch. Ratl. Mech. Anal. 6:769-780 (1957).

velocity field is different. For example, in the case when the viscosity of the fluid inside the finger is negligible, the velocity of the fluid outside adjacent to the straight sides of the finger is at rest relative to the plates if the plane of the apparatus is horizontal, but not if it is vertical.

Systematic information on the stability of $\lambda = 1/2$ against changes in the initial conditions or small variations in d is not available. It should be noted that λ is expected to be a function of the spacing d, as well as of $\mu U/T$.

In connection with the sketches shown by Professor Birkhoff, we should like to mention that we have found exact solutions in closed form* giving asymmetric finger shapes, and there is a whole series of these shapes which go continuously from the symmetrical one shown in the middle sketch to the shape shown in the third sketch.

*G. I. Taylor and P. G. Saffman, (to appear, 1959) Quart. J. Mech. Appl. Math.

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