ON THE EFFICIENCY OF A VERTICAL-AXIS PROPELLER

J. A. Sparenberg
Netherlands Ship Model Basin

In order to minimize the kinetic energy left behind in the wake of a vertical-axis propeller the oscillating motion of the propeller blades is discussed. The blades are assumed to be infinitely thin and infinitely long. The chord of the profiles is small with respect to the curvature of the cycloidal orbit of the blades. Also the case of a propeller with many blades or with a high rotational velocity is considered.

1. INTRODUCTION

The oscillatory motion of the blades of a vertical-axis propeller is governed by a system of rods and hinges [1]. This system prescribes the angle of incidence of the blades when they perform their rotational motion. The forces which act on the blades have been calculated by Isay [2], making approximations which give rise to some doubt and which we discuss at the end of section 6 of this paper.

One of the difficulties in a theoretical investigation is the complicated wake which is crowded with vortex layers. Another difficulty, which is discussed in section 2 is that for rigid blades with a finite chord no rigorous linearized theory for the perturbation velocities can be given. This is caused by the varying radius of curvature of the cycloidal orbits described by the blades.

In this paper we do not assume a prescribed oscillatory motion of the blades. We describe a method to construct the angle of incidence of the profile, as a function of position, in such a way that the kinetic energy which remains in the wake becomes a minimum. The following simplifying assumptions are made. The blades are infinitely thin, tip effects are ignored, and the chord of the blades is assumed to be small with respect to the smallest radius of curvature of the orbits of the blades.

The kinetic energy of the wake is expressed in terms of the bound vortices. Then by the Ritz-method this energy is minimized. The following property (section 4) of the bound vorticity of the blades, under the conditions just stated, will be proved: In order to obtain the highest efficiency, it is necessary that the sum of the bound vorticities of a blade in its two positions on a straight line parallel to the direction of translation of the propeller, be

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constant. For instance, this constant may be taken equal to zero; then the blades pull as much as they push. It is also possible to give the constant such a value that the profiles are more active when they are in the front position.

Also the case of a vertical-axis propeller with many blades or a high-rotational velocity is considered. Here the bound vorticity and the vortex layers in the wake are approximated by continuous vortex densities. Then we find the following condition on the bound vorticity: In order to obtain the highest efficiency it is necessary and sufficient that the difference of the bound vorticity density for two points lying on a straight line parallel to the direction of translation of the propeller is a constant (section 6). In this case it is possible to write down explicitly the angle of the incidence of a blade as a function of its position.

It is intended to compare in a future paper numerical results of this theory with experimental results.

2. STATEMENT OF THE PROBLEM

We consider an unbounded fluid which is at rest relative to a Cartesian coordinate system x, y, z. In the direction of the positive x axis moves a circle with radius R (see Fig. 1). Its centre is on the x axis and has a velocity U. On the circle are M equally spaced identical blades perpendicular to the x, y plane. The blades rotate with a constant angular velocity ω around C. Besides this they execute an oscillatory motion around the point T in order to preserve the desired angle of incidence. We simplify our considerations, as stated in the introduction, by assuming each blade to be two-sided and infinitely long.

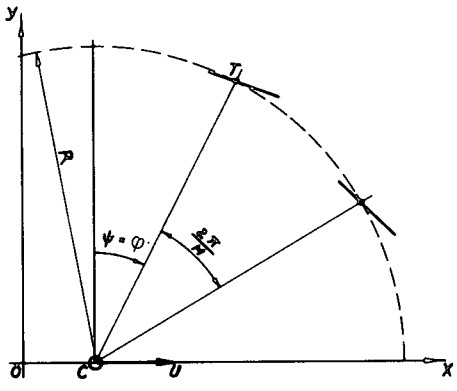


Fig. 1. Scheme of a vertical-axis propeller with M blades

The orbits of the pivotal points T of the M blades are the cycloids C_m with

$$x_m = R\left(\mu\psi + \sin\psi - \frac{2\pi m\mu}{M}\right), \qquad y_m = R\cos\psi, \qquad \mu = \frac{U}{\omega R}. \tag{2.1}$$

Several types of cycloids, as a function of μ, are drawn in Fig. 2.

As in lifting surface theory for airplane wings or ship screws we try to refer the profiles to a surface which moves through the fluid without disturbing it. For an airplane wing this surface is the projection of the wing on a certain adjacent flat plane with zero angle of attack. For a ship screw it is the projection of the blades on an adjacent helicoidal surface which does not disturb the incoming fluid when it rotates with the rotational velocity of the screw. In the case of a vertical-axis propeller a rigid reference surface does not exist because the curvature of the cycloids is not constant. For this reason we start with a deformable line segment $A - B$ covered with bound vorticity (Fig. 3), which moves exactly along the cycloidal orbit of the turning point T.

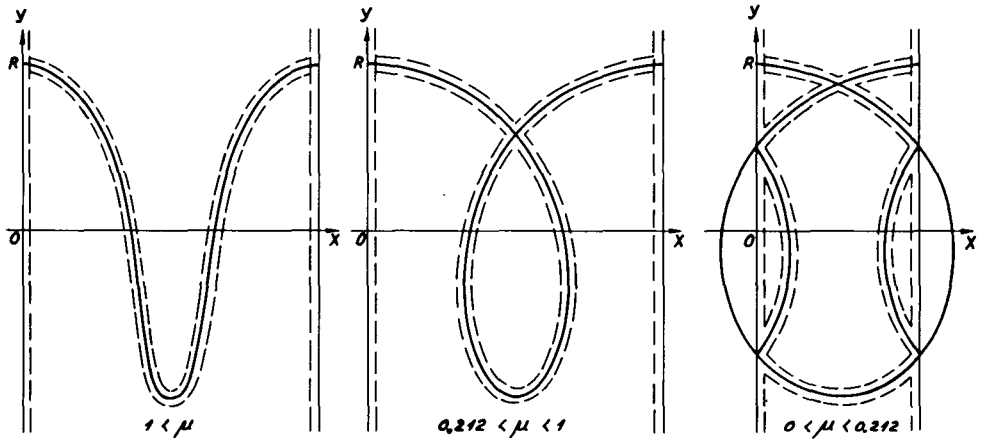


Fig. 2. Several types of cycloids

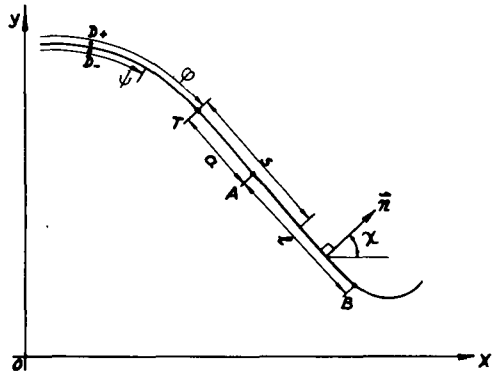


Fig. 3. The deformable profile $A - B$ along the cycloid

On this cycloid the turning point T corresponds to the parameter value $\psi = \varphi = \omega t$. Although the segment $A - B$ changes its radius of curvature we assume the distance s of some point of the profile to the point T, measured along the cycloid, to be constant. Trailing and leading edge are given by $s = a$ and $s = a + \ell$.

The magnitude of the velocity $V(\varphi)$ of the points of the profile is assumed to be equal to the velocity of the pivotal point T. Using the relation $\varphi = \omega t$ for the parameter of position of this point we find

$$V(\varphi) = \omega R \sqrt{1 + \mu^2 + 2\mu \cos \varphi} \,. \tag{2.2}$$

The density of the bound vortices $\Gamma(\varphi, s)$ is a function of the two variables φ and s with

$$\Gamma(\varphi, s) \neq 0, \qquad a \leq s \leq a + \ell. \tag{2.3}$$

In the following we will use the notation

$$\frac{\partial \Gamma(\varphi, s)}{\partial \varphi} = \dot{\Gamma}(\varphi, s), \qquad \frac{\partial \Gamma(\varphi, s)}{\partial s} = \Gamma'(\varphi, s). \tag{2.4}$$

When the elementary bound vortex $\Gamma(\varphi,s)\,ds$ changes its strength when φ increases, it leaves behind at the place φ,s free vorticity of the density

$$-\frac{\dot{\Gamma}(\varphi, s)\ ds}{R\sqrt{1 + \mu^2 + 2\mu \cos \varphi}}. \tag{2.5}$$

The total density $\gamma(\varphi,s)$ of the free vorticity which is left behind at the point φ,s by the bound vorticity of the profile which has passed this point is then given by

$$\gamma(\varphi, s) = -\int_a^{a+\ell} \frac{\dot{\Gamma}(\vartheta, \sigma)\ d\sigma}{R\sqrt{1 + \mu^2 + 2\mu \cos \vartheta}}. \tag{2.6}$$

After the partial differentiation in the integrand of (2.6) we have to consider ϑ as a function of φ, s, and σ by

$$\sigma = s + R \int_{\vartheta}^{\varphi} \sqrt{1 + \mu^2 + 2\mu \cos x}\ dx. \tag{2.7}$$

A necessary condition on the bound vorticity follows from the mean value K of the x component of the forces which act on the bound vortices. This mean value has to possess some prescribed value which depends on the velocity of the ship. Calculating the angle χ (Fig. 3) and using the fact that the profile has the velocity $V(\varphi)$ we find for the M blades by the law of Kutta-Joukowski

$$K = \frac{\rho\omega RM}{2\pi} \int_0^{2\pi} \int_a^{a+\ell} \Gamma(\varphi, s) \sin \vartheta \frac{\sqrt{1 + \mu^2 + 2\mu \cos \varphi}}{\sqrt{1 + \mu^2 + 2\mu \cos \vartheta}}\, ds\ d\varphi. \tag{2.8}$$

where $\vartheta = \vartheta(\varphi,s)$ by (2.7) with $\sigma = 0$.

Another formula which will be used later on gives the difference of the value of the velocity potential $\phi(x,y)$ across the cycloid (points D_+, D_-, Fig. 3) when the profile has passed. This difference at the place ψ is

$$\phi_+(\psi) - \phi_-(\psi) = \int_a^{a+\ell} \Gamma(\vartheta, \sigma)\ d\sigma. \tag{2.9}$$

where $\vartheta = \vartheta(\psi,\sigma)$, defined by (2.7) when we put $s = 0$ and $\varphi = \psi$ in that equation. In order to use a potential for the whole fluid it is necessary that the lines covered with free vorticity

$\gamma(\varphi,a)$ do not intersect each other. This can be obtained by a simple artifice; viz., in the neighbourhood of points of intersection of the cycloids we assume that the bound vorticity of the blades becomes zero. Then the bound vorticity of the profile is left behind as free vorticity; however, a little farther, free vorticity of opposite strength is created which compensates the effect of the first one. Then (2.9) can be used with arbitrary accuracy when the distance between the two free vorticities just mentioned becomes small enough.

We now introduce two independent order quantities, viz., the length ℓ of the chord of the profile and Γ the total strength of the bound circulation of the profile at a reasonable point. We consider the case

$$\mu \neq 1\,, \qquad \ell << R\,\frac{|1-\mu|^2}{1+\mu}, \qquad a \approx \mathbb{O}(\ell) \tag{2.10}$$

where $\mu \neq 1$ means that the cycloid has no cusp, and the second condition means that the chord of the profile is very small with respect to the smallest radius of curvature of the cycloid.

The function $\Gamma(\varphi,s)$ exhibits, as a function of s, large variations over the chord. In fact, it becomes infinite at the leading edge $s = a + \ell$ and zero at the trailing edge $s = a$. As a function of φ it is clear that the variations of $\Gamma(\varphi,s)$ are almost everywhere small when $\ell \to 0$ and φ changes with an amount of the order ℓ. In formulas:

$$\begin{aligned} &\Gamma(\varphi,\ s+\varepsilon) \neq \Gamma(\varphi,s) + \varepsilon\Gamma'(\varphi,s), \\ &\Gamma(\varphi+\varepsilon,\ s) \approx \Gamma(\varphi,\ s) + \varepsilon\dot{\Gamma}(\varphi,s), \qquad \varepsilon \approx \mathbb{O}(\ell)\,. \end{aligned} \tag{2.11}$$

From this we can develop, with respect to ℓ and Γ, Eqs. (2.6), (2.8), and (2.9), which are important for our theory. Only lowest order terms are taken into consideration.

We find from (2.6) for the density $\gamma(\psi)$ of the free vorticity at the end of the profile, $s = a$ in (2.6), within higher order terms

$$\begin{aligned} \gamma(\psi) \overset{\text{def}}{=} \gamma(\psi,a) &= -\int_a^{a+\ell} \frac{\dot{\Gamma}(\vartheta,\sigma)\ d\sigma}{R\sqrt{1+\mu^2+2\mu\cos\vartheta}} \\ &\approx -\frac{1}{R\sqrt{1+\mu^2+2\mu\cos\psi}}\int_a^{a+\ell}\dot{\Gamma}(\psi,\sigma)\ d\sigma \\ &\quad + \mathbb{O}(\ell\Gamma) \overset{\text{def}}{=} -\frac{\dot{\Gamma}(\psi)}{R\sqrt{1+\mu^2+2\mu\cos\psi}} + \mathbb{O}(\ell\Gamma). \end{aligned} \tag{2.12}$$

The mean value of the component K in (2.8) becomes

$$K \approx \frac{\rho\omega RM}{2\pi}\int_0^{2\pi}\Gamma(\varphi)\sin\varphi\ d\varphi + \mathbb{O}(\ell\Gamma), \tag{2.13}$$

while (2.9) changes into

$$\phi_+(\psi) - \phi_-(\psi) = \Gamma(\psi) + \mathbb{O}(\ell\Gamma) . \tag{2.14}$$

Hence with respect to the wake and to the mean value of K we can treat the profile with a sufficiently small chord as a concentrated vortex $\Gamma(\varphi)$. This is no longer true when we calculate the angle of incidence of the profile; then we have to take into account the variations of the induced velocities over the chord (section 5).

3. THE KINETIC ENERGY LEFT BEHIND IN THE WAKE

We want to derive the potential of the cycloidal free vortex layers with density $\gamma(\psi)$ (2.12). To this effect we start from the potential $\phi^*(x,y)$ of a row of equally spaced vortices of equal strength. The potential of such a configuration is known [3, p. 186]. Because in the remaining part of this paper we do not need the third space coordinate z, the symbol z will be used to denote the complex variable $x + iy$. We find

$$\phi^*(x,y) = Re \frac{\gamma}{2\pi i} \ln \sin \frac{\pi}{h}(z - \zeta_m), \tag{3.1}$$

where γ is the strength of the vortices, h their mutual separation distance, z the point in which the potential is considered and ζ_m the location of some vortex (see Fig. 4). It can be easily seen that this potential is not periodic, we find

$$\phi^*(x+h,y) - \phi^*(x,y) = \begin{cases} -\frac{1}{2}\gamma, & y > Im\ \zeta_m \\ +\frac{1}{2}\gamma, & y < Im\ \zeta_m \end{cases} . \tag{3.2}$$

When we take $h = 2\pi\mu R$ and let the points $\zeta_m = (\xi_m + i\eta_m)$ with $m = 1, \ldots, M$, describe the parts of the M cycloids lying in the interval $0 \leq x < 2\pi\mu R$, the other points of the rows describe the parts of the cycloids outside this interval. In this way we cover the whole wake of the propeller with free vortices. The potential $\phi(x,y)$ of the wake flow can be written as

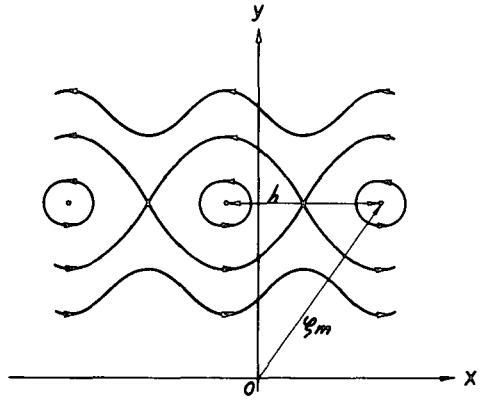


Fig. 4. Row of vortices with streamlines

$$\phi(x,y) = Re\ \frac{R}{2\pi i}\sum_{m=0}^{M-1}\int_0^{2\pi}\gamma(\vartheta)\ \ln\left(\sin\frac{(2-\zeta_m)}{2\mu R}\right)\sqrt{1+\mu^2+2\mu\cos\vartheta}\ d\vartheta \quad (3.3)$$

where

$$\zeta_m = (\xi_m + i\eta_m) = R\left(\mu\vartheta + \sin\vartheta - \frac{2\pi m\mu}{M} + i\cos\vartheta\right). \quad (3.4)$$

Because the intensity $\Gamma(\psi)$ of the bound vortex is a periodic function of ψ, the same holds for the free vortex density $\gamma(\psi)$, and besides this the mean of $\gamma(\psi)$ over a period $0 \le \psi < 2\pi$, or over an interval of length $2\pi\mu R$ in the x direction, is zero.

Then it follows from (3.2)

$$\phi(x+2\pi\mu R, y) - \phi(x,y) = 0\ , \qquad |y| > R\ . \quad (3.5)$$

When, however, $|y| < R$ we have for the mean vorticity between $y = \pm R$ and y over the period $2\pi\mu R$ in the x direction

$$\pm M\left\{\Gamma(2\pi-\psi) - \Gamma(\psi)\right\}, \qquad y = R\cos\psi\ . \quad (3.6)$$

Here by (3.2) we have

$$\phi(x+2\pi\mu R, y) - \phi(x,y) = M\left\{\Gamma(2\pi-\psi) - \Gamma(\psi)\right\}, \qquad |y| < R\ . \quad (3.7)$$

Our aim is to determine the function $\Gamma(\psi)$ in such a way that the energy left behind in the wake becomes as small as possible. To this end we consider the kinetic energy E of the fluid in the strip $0 \le x \le 2\pi\mu R$, $-\infty < y < +\infty$, which by Green's theorem can be written as

$$E = \frac{1}{2}\rho\iint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}dx\ dy = \frac{1}{2}\rho\int\phi\frac{\partial\phi}{\partial n}\,d\ell\ . \quad (3.8)$$

The surface integral extends over a vertical strip of width $2\pi\mu R$, the line integral along the vertical boundaries of the strip and along both sides of the cycloids (Fig. 2, dashed lines). The normal derivative is directed into the enclosed regions.

We consider first the integrals along both sides of the cycloids. These integrals for M different cycloids have the same magnitude; hence we find by (2.14) for their combined contribution E_c to the kinetic energy E,

$$E_c = -\frac{\rho RM}{2}\int_0^{2\pi}\Gamma(\psi)\ \frac{\partial\phi}{\partial n}(\psi)\sqrt{1+\mu^2+2\mu\cos\psi}\ d\psi \quad (3.9)$$

where the line of integration and the direction of the normal to this line are denoted in Fig. 5. As in (3.9) we will not mention in this and the next section the order of accuracy of the formulas. This will be considered again in section 5.

From (3.9) it follows that we have to determine the normal derivative $\partial\phi/\partial n$. The unit normal of the cycloid is

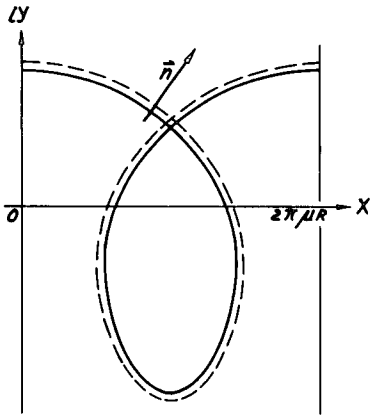


Fig. 5. The path of integration along the cycloid for $m = 0$

$$\vec{n} = (\sin\psi,\ \mu + \cos\psi)\frac{1}{\sqrt{1 + \mu^2 + 2\mu\cos\psi}}. \tag{3.10}$$

The velocity $\vec{v}(x,y)$ which belongs to the potential $\phi(x,y)$ (3.3) is

$$\vec{v} = -\frac{1}{4\pi\mu}\sum_{m=0}^{M-1}\int_0^{2\pi}\gamma(\vartheta)\cdot\left\{\frac{\sinh\frac{(y-\eta_m)}{\mu R} - \sin\frac{(x-\xi_m)}{\mu R}}{\cosh\frac{(y-\eta_m)}{\mu R} - \cos\frac{(x-\xi_m)}{\mu R}}\right\}\cdot\sqrt{1 + \mu^2 + 2\mu\cos\vartheta}\ d\vartheta \tag{3.11}$$

where ξ_m and η_m are defined in (3.4). By (3.10) and (3.11) we find for the normal velocity in a point x_o, y_o of the cycloid with $m = 0$ (2.1)

$$\frac{\partial\phi}{\partial n}(\psi) = -\frac{1}{4\pi\mu}\sum_{m=0}^{M-1}\int_0^{2\pi}\gamma(\vartheta)\cdot\left[\frac{\sin\psi\,\sinh\left(\frac{y_o-\eta_m}{\mu R}\right) - (\mu+\cos\psi)\sin\left(\frac{x_o-\xi_m}{\mu R}\right)}{\cosh\frac{(y_o-\eta_m)}{\mu R} - \cos\frac{(x_o-\xi_m)}{\mu R}}\right]\cdot\frac{\sqrt{1+\mu^2+2\mu\cos\vartheta}}{\sqrt{1+\mu^2+2\mu\cos\psi}}\,d\vartheta. \tag{3.12}$$

By (2.12), (3.9), and (3.12) we obtain

$$E_c = -\frac{\rho M}{8\pi\mu}\sum_{m=0}^{M-1}\int_0^{2\pi}\int_0^{2\pi}\Gamma(\psi)\,\dot{\Gamma}(\vartheta)\cdot\left[\frac{\sin\psi\,\sinh\frac{(y_o-\eta_m)}{\mu R} - (\mu+\cos\psi)\sin\frac{(x_o-\xi_m)}{\mu R}}{\cosh\frac{(y_o-\eta_m)}{\mu R} - \cos\frac{(x_o-\xi_m)}{\mu R}}\right]d\vartheta\,d\psi \tag{3.13}$$

where x_o, y_o and ξ_m, η_m are defined by (2.1) and (3.4).

Next we consider the contribution of the vertical boundaries of the strip (Fig. 2) to the line integrals (3.8) of the kinetic energy E. Because $\partial\phi(x,y)/\partial x$ is periodic for all values

of y and $\phi(x,y)$ is periodic for $|y| > R$ the contributions of the integrals cancel each other for $|y| > R$. This is no longer true for $|y| < R$. We obtain for the part E_v of E which belongs to the vertical boundaries

$$E_v = -\frac{1}{2}\rho \int_{-R}^{+R} \left\{\phi(0,y) - \phi(2\pi\mu R, y)\right\} \frac{\partial\phi(0,y)}{\partial x}\, dy\,. \tag{3.14}$$

Using (3.7) and (3.11) we find after some reduction

$$E_v = -\frac{\rho M}{8\pi\mu} \sum_{m=0}^{M-1} \int_0^{2\pi}\int_0^{2\pi} \frac{\Gamma(\psi)\,\dot{\Gamma}(\vartheta)\,\sinh\dfrac{(y-\eta_m)}{\mu R}\,\sin\psi}{\left\{\cosh\dfrac{(y-\eta_m)}{\mu R} - \cos\dfrac{\xi_m}{\mu R}\right\}}\, d\vartheta\, d\psi\,, \tag{3.15}$$

where $y = R\cos\psi$ and ξ_m and η_m are from (3.4).

Hence we find for the total kinetic energy

$$E = E_c + E_v = \int_0^{2\pi}\int_0^{2\pi} \Gamma(\psi)\,\dot{\Gamma}(\vartheta)\,L(\psi,\vartheta)\, d\psi\, d\vartheta \tag{3.16}$$

where

$$L(\psi,\vartheta) = -\frac{\rho M}{8\pi\mu} \sum_{m=0}^{M-1} \left[\frac{\sin\psi\,\sinh\dfrac{(y_o-\eta_m)}{\mu R} - (\mu+\cos\psi)\,\sin\dfrac{(x_o-\xi_m)}{\mu R}}{\cosh\dfrac{(y_o-\eta_m)}{\mu R} - \cos\dfrac{(x_o-\xi_m)}{\mu R}} + \frac{\sinh\dfrac{(y-\eta_m)}{\mu R}\,\sin\psi}{\cosh\dfrac{(y-\eta_m)}{\mu R} - \cos\dfrac{\xi_m}{\mu R}} \right] \tag{3.17}$$

in which $y = R\cos\psi$ and x_o, y_o and ξ_m, η_m are from (2.1) and (3.4).

It can be seen that for $\varphi = \vartheta$ this function possesses a singularity of the form

$$\lim_{\psi\to\vartheta} L(\psi,\vartheta) \approx \frac{\rho M}{4\pi}\,\frac{1}{(\psi-\vartheta)}\,. \tag{3.18}$$

Besides this singularity there are other ones which are the points of intersection of the cycloids with themselves ($\mu < 1$), with each other ($M > 1$), or with the vertical lines $x = 2k\pi\mu R$. The first term yields besides the singularity mentioned in (3.18) also singularities for

$$\begin{gathered} y_o = \eta_m, \qquad x_o - \xi_m = 2k\pi\mu R, \qquad 0 \le \psi, \\ \vartheta \le 2\pi, \qquad \psi \neq \vartheta, \qquad k = 0,\ 1,\ \ldots \end{gathered} \tag{3.19}$$

and the second term yields singularities for

$$y = \eta_m, \qquad \xi_m = 2k\pi\mu R, \qquad 0 \leq \psi, \qquad \vartheta \leq 2\pi, \qquad k = 0, 1, \ldots \tag{3.20}$$

where as many integers $k = 0, 1, \ldots$ have to be used as are compatible with the condition $0 \leq \psi, \vartheta \leq 2\pi$. The order of these latter singularities is

$$\mathcal{O}\left\{\left[(\psi_s - \psi)^2 + (\vartheta_s - \vartheta)^2\right]^{-1/2}\right\} \tag{3.21}$$

where ψ_s s, ϑ_s denotes the singular point in the ψ,ϑ plane.

4. THE VARIATIONAL PROBLEM

We want to minimize the energy E left behind in the wake over a length of a period $2\pi\mu R$. It seems most simple to start from (3.16) by using Ritz's method. First, however, a general property of the function $\Gamma(\psi)$ will be derived.

We can split $\Gamma(\psi)$ into an even and an odd function with respect to $\psi = \pi$,

$$\Gamma(\psi) = \Gamma_e(\psi) + \Gamma_o(\psi), \qquad \underset{o}{\Gamma_e}(\pi + \psi) = \pm \underset{o}{\Gamma_e}(\pi - \psi) \ . \tag{4.1}$$

The condition (2.13) then becomes

$$\frac{\rho\omega RM}{2\pi} \int_0^{2\pi} \Gamma_o(\psi) \sin\psi \, d\psi = K \tag{4.2}$$

which does not yield any restriction on $\Gamma_e(\psi)$. Using (4.1) the kinetic energy (3.16) becomes

$$E = \int_0^{2\pi} \int_0^{2\pi} \left\{\left[\Gamma_e(\psi)\,\dot{\Gamma}_e(\vartheta) + \Gamma_o(\psi)\,\dot{\Gamma}_o(\vartheta)\right] + \left[\Gamma_e(\psi)\,\dot{\Gamma}_o(\vartheta) + \Gamma_o(\psi)\,\dot{\Gamma}_e(\vartheta)\right]\right\} L(\psi,\vartheta)\, d\psi\, d\vartheta \ . \tag{4.3}$$

From (3.17) it can be derived that

$$L(\psi,\vartheta) = -L(2\pi - \psi,\ 2\pi - \vartheta) \ . \tag{4.4}$$

By this (4.3) reduces to

$$E = \int_0^{2\pi} \int_0^{2\pi} \left[\Gamma_e(\psi)\,\dot{\Gamma}_e(\vartheta) + \Gamma_o(\psi)\,\dot{\Gamma}_o(\vartheta)\right] L(\psi,\vartheta)\, d\psi\, d\vartheta \ . \tag{4.5}$$

Both terms between brackets in (4.5) yield a positive contribution to E, because each of them represents the kinetic energy which belongs to some bound vortex function $\Gamma_e(\psi)$ or $\Gamma_o(\psi)$.

From this result and from (4.2) we find that we have to take

$$\Gamma_e(\psi) \equiv 0 . \tag{4.6}$$

Hence $\Gamma(\psi)$ has to be an odd function with respect to ψ. We suppose

$$\Gamma(\psi) = \Gamma_o(\psi) = \sum_{n=1}^{N} a_n \sin n\psi . \tag{4.7}$$

Condition (2.13) yields

$$a_1 = \frac{2K}{\rho\omega RM} . \tag{4.8}$$

Using (4.7) the energy E becomes

$$E = \sum_{p,\,q=1}^{N} q a_p a_q \int_0^{2\pi}\int_0^{2\pi} \sin p\psi \cos q\vartheta \, L(\psi,\vartheta) \, d\psi \, d\vartheta . \tag{4.9}$$

Differentiation of this expression with respect to a_n ($n = 2, \ldots, N$) yields

$$\frac{\partial E}{\partial a_n} = \sum_{q=1}^{N} a_q \int_0^{2\pi}\int_0^{2\pi} (q \sin n\psi \cos q\vartheta + n \sin q\psi \cos n\vartheta) \, L(\psi,\vartheta) \, d\psi \, d\vartheta = 0, \qquad (n = 2, \ldots, N). \tag{4.10}$$

These are $N-1$ linear equations for the $N-1$ unknowns $a_2, \ldots, a_N$.

Without altering the kinetic energy in the wake we may add a constant value to the circulation $\Gamma(\psi)$.

From the fact that the optimum bound vorticity $\Gamma(\psi)$ has to be an odd function of ψ plus an arbitrary constant we find the following property, mentioned already in the introduction,

$$\Gamma(\psi) + \Gamma(-\psi) = \text{const.} \tag{4.11}$$

5. THE DETERMINATION OF THE ANGLE OF INCIDENCE

Our aim is to derive formulas which determine the angles of incidence of the profiles as functions of their position.

Because all blades are equivalent we consider the profile whose turning point T moves along the cycloid C_o. On this profile we introduce a parameter s^* (Fig. 6) which measures the distance of a point of the profile to the turning point T; s^* is positive in the direction of $V(\varphi)$. We can calculate the normal velocities on the profile caused by the translation $V(\varphi)$ and a rotation around T, which have to be compensated by a suitable vortex distribution on the profile. Next we assume a kind of linearisation procedure which consists of compensating the normal velocities not on the real profile, but on the cycloid. The points

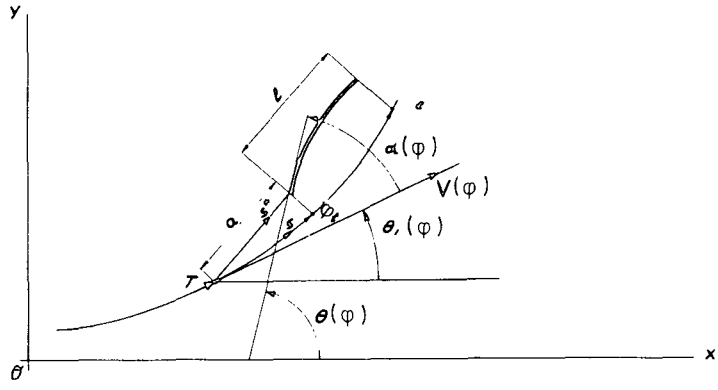


Fig. 6. The profile in the neighbourhood of the cycloid

of the real profile are represented by points on the cycloid in such a way that $s^* = s$. The leading and trailing edge of the profile are denoted on the cycloid by φ_ℓ and φ_t.

First we consider the velocities induced by the bound vorticity of the other blades and by the free vorticity along the cycloids C_m, $m = 1, \ldots, M-1$. It is clear that for these velocities and hence for their resultant component $v_{n,1}(\varphi_o, s)$ normal to the profile, we have the relation

$$v_{n,1}(\varphi, s) = \mathbb{Q}(\Gamma), \qquad a \leq s \leq a + \ell. \tag{5.1}$$

At the end of this section it will turn out that this information about these velocities is sufficient with respect to the accuracy with which we have approximated the basic formulas (2.6), (2.8), and (2.9).

Next we consider the velocity $\vec{v}_z(\varphi, z)$ with components $v_{x,z}$ and $v_{y,z}$ induced by the free vorticity along C_o,

$$v_{x,z} - iv_{y,z} = -\frac{1}{2\pi i} \int_{-\infty}^{\varphi_t} \frac{\Gamma(\vartheta)\, d\vartheta}{[x - \zeta(\vartheta)]}$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{\zeta(\varphi_t)} \frac{\dot{\Gamma}(\vartheta)}{[z - \zeta(\vartheta)]} \frac{d\zeta}{R(\mu + \cos\vartheta - i \sin\vartheta)}. \tag{5.2}$$

The second integral formulation has the form of a Cauchy integral. We have to consider the behaviour of this integral when z is in the neighbourhood of $\zeta(\varphi_t)$. This is a well-known problem [4, ch. 4]; we find

$$v_{x,z} - iv_{y,z} = \frac{1}{2\pi i} \frac{\dot{\Gamma}(\varphi_t) \ln [z - \zeta(\varphi_t)]}{R(\mu + \cos\varphi_t - i \sin\varphi_t)} + \psi(z) \tag{5.3}$$

where $\psi(z)$ is a bounded function for $z \to \zeta(\varphi_t)$, which is of $\mathbb{O}(\Gamma)$. Making use of (3.10) we find for the normal component $v_{n,z}(\varphi,s)$ of this velocity,

$$v_{n,z}(\varphi, s) = \frac{1}{2\pi} \frac{\dot{\Gamma}(\varphi_t) \ln (s - a)}{R\sqrt{1 + \mu^2 + 2\mu \cos \varphi_t}}$$

$$+ \mathbb{O}(\Gamma) \overset{\text{def}}{=} w(\varphi) \ln \left(\frac{s - a}{R} \right) + \mathbb{O}(\Gamma) . \tag{5.4}$$

It will be assumed that the profile is infinitely thin and cambered. On the profile we assume a vortex layer with density $\Gamma_{tot}(\varphi,s)$ which represents the sum of bound and free vorticity. Hence we have the following equation:

$$Re\left[\frac{1}{2\pi i} \int_a^{a+\ell} \frac{\Gamma_{tot}(\varphi,\sigma)\, d\sigma}{[z(s) - \zeta(\sigma)]} \cdot \vec{n}(s) \right] = - V(\varphi) \, [\alpha(\varphi) + f'(s)] + \omega\theta'(\varphi)s$$

$$- w(\varphi) \ln \left(\frac{s - a}{R} \right) + \mathbb{O}(\Gamma) \tag{5.5}$$

where $\alpha(\varphi)$ is the angle of incidence of the profile, $f(s)$ is the camber, and $\theta(\varphi)$ represents the angle of the profile with respect to a line of fixed direction, for instance the x axis (Fig. 6). The angle of incidence $\alpha(\varphi)$ can be defined as the angle between the tangent to the profile at the trailing edge and the velocity $V(\varphi)$. The angles α, θ, and θ_1 to be defined in (5.6) are positive in the counterclockwise direction. The angle $\theta(\varphi)$ can be split into two parts:

$$\theta(\varphi) = \theta_1(\varphi) + \alpha(\varphi) = - \tan^{-1} \frac{\sin \varphi}{\mu + \cos \varphi} + \alpha(\varphi) . \tag{5.6}$$

For small values of ℓ Eq. (5.5) can be written in the form

$$\frac{1}{2\pi} \int_a^{a+\ell} \frac{\Gamma_{tot}(\varphi,\sigma)\, d\sigma}{(s - \sigma)} = - V(\varphi) \, [\alpha(\varphi) + f'(s)] + [\omega\theta_1'(\varphi) + \omega\alpha'(\varphi)]s$$

$$- w(\varphi) \ln \left(\frac{s - a}{R} \right) + \mathbb{O}(\Gamma) . \tag{5.7}$$

The inversion of this integral equation is well known [4, section 88]:

$$\Gamma_{tot}(\varphi, s) = \frac{\beta(\varphi)}{\pi\sqrt{s - a}\,\sqrt{a + \ell - s}} + \frac{2}{\pi\sqrt{s - a}\,\sqrt{a + \ell - s}} \int_a^{a+\ell} \Big\{ V(\varphi) \, [\alpha(\varphi) + f'(\sigma)]$$

$$- [\omega\theta_1'(\varphi) + \omega\alpha'(\varphi)]\sigma + w(\varphi) \ln \left(\frac{\sigma - a}{R} \right)$$

$$+ \mathbb{O}(\Gamma) \Big\} \frac{\sqrt{\sigma - a}\,\sqrt{a + \ell - \sigma}}{(s - \sigma)} \, d\sigma \tag{5.8}$$

where

$$\beta(\varphi) = \int_a^{a+\ell} \Gamma_{tot}(\varphi, s)\, ds. \tag{5.9}$$

We consider the asymptotical representation of $\Gamma_{tot}(\varphi,s)$ in the neighbourhood of $s = a$. After some analysis we find from (5.8)

$$\lim_{s \to a} \Gamma_{tot}(\varphi, s) \approx \frac{\beta(\varphi)}{\pi\sqrt{\ell}\,\sqrt{s-a}} - \frac{2}{\pi\sqrt{\ell}\,\sqrt{s-a}} \left(\int_a^{a+\ell} \left\{ V(\varphi)\,[\alpha(\varphi) + f'(\sigma)] - [\omega\theta_1'(\varphi) + \omega\alpha'(\varphi)]\sigma + w(\varphi)\,\ln\left(\frac{\sigma-a}{R}\right) + \mathbb{O}(\Gamma) \right\} \frac{\sqrt{a+\ell-\sigma}}{\sqrt{\sigma-a}} d\sigma + \pi^2\, w(\varphi)\,\sqrt{\ell}\,\sqrt{s-a} \right). \tag{5.10}$$

We have to satisfy the condition that the velocity of the fluid remains finite at the trailing edge. This is equivalent to the demand that the total vorticity $\Gamma_{tot}(\varphi,s)$ for $s = a$ equals the free vorticity in the wake. Hence we find from (5.10)

$$\beta(\varphi) = 2 \int_a^{a+\ell} \left\{ V(\varphi)\,[\alpha(\varphi) + f'(\sigma)] - [\omega\theta_1'(\varphi) + \omega\alpha'(\varphi)] + w(\varphi)\,\ln\frac{(\sigma-a)}{R} + \mathbb{O}(\Gamma) \right\} \frac{\sqrt{a+\ell-\sigma}}{\sqrt{\sigma-a}}\, d\sigma\,. \tag{5.11}$$

and

$$-2\pi\, w(\varphi) = \gamma(\varphi_t)\,. \tag{5.12}$$

Equation (5.11) determines $\beta(\varphi)$ in terms of the still unknown angle of incidence $\alpha(\varphi)$, while (5.12) is satisfied automatically (Eqs. (2.12) and (5.4)).

The last condition we have to satisfy states that the integral over the bound vorticity of the profile has to have the known value $\Gamma(\varphi)$. Hence first we have to calculate the bound vorticity $\Gamma(\varphi,s)$. We start from the definition of $\Gamma_{tot}(\varphi,s)$

$$\Gamma_{tot}(\varphi, s) = \gamma(\varphi, s) + \Gamma(\varphi, s) \tag{5.13}$$

where $\gamma(\varphi,s)$ is the free vorticity passing along the profile. Using (2.6) we obtain

$$\Gamma_{tot}(\varphi, s) = -\int_a^{a+\ell} \frac{\dot{\Gamma}(\varphi,\sigma)\, d\sigma}{R\sqrt{1 + \mu^2 + 2\mu\cos\vartheta}} + \Gamma(\varphi, s) + \mathbb{O}(\Gamma) \tag{5.14}$$

where ϑ follows from (2.7). However, the first term on the right-hand side of (5.14) is $\mathbb{O}(\Gamma)$. Hence we find, making use of (5.9),

$$\Gamma(\varphi) = \int_a^{a+\ell} \Gamma(\varphi, s)\, ds \approx \int_a^{a+\ell} \Gamma_{tot}(\varphi, s)\, ds$$

$$+ \mathbb{O}(\ell\Gamma) = \beta(\varphi) + \mathbb{O}(\ell\Gamma). \tag{5.15}$$

From (5.11) we obtain

$$\alpha(\varphi) - \frac{\ell\omega}{V(\varphi)}\left(\frac{a}{\ell} + \frac{1}{4}\right)\alpha'(\varphi) = \frac{\Gamma(\varphi)}{\pi\ell\, V(\varphi)} - \frac{2}{\pi}\int_0^1 f'(\sigma\ell + a)\,\frac{\sqrt{1-\sigma}}{\sqrt{\sigma}}\,d\sigma$$

$$+ \frac{\ell\omega}{V(\varphi)}\left(\frac{a}{\ell} + \frac{1}{4}\right)\theta_1'(\varphi) - \frac{\omega}{2\pi}\,\frac{\dot{\Gamma}(\varphi)}{V^2(\varphi)}\,\ln\frac{\ell}{R} + \mathbb{O}(\Gamma). \tag{5.16}$$

Because $\alpha = \mathbb{O}(\Gamma\ell^{-1})$ the second term on the left-hand side can be disregarded. Besides this by the choice $a = -(1/4)\ell$ the coefficient of this term as well as the coefficient of $\theta_1'(\varphi)$ vanishes. Then equation (5.16) changes into

$$\alpha(\varphi) = \frac{\Gamma(\varphi)}{\pi\ell\, V(\varphi)} - \frac{\omega}{2\pi}\,\frac{\dot{\Gamma}(\varphi)}{V^2(\varphi)}\,\ln\frac{\ell}{R}$$

$$- \frac{2}{\pi}\int_0^1 f'\left[\ell\left(\sigma - \frac{1}{4}\right)\right]\frac{\sqrt{1-\sigma}}{\sqrt{\sigma}}\,d\sigma + \mathbb{O}(\Gamma). \tag{5.17}$$

Hence, for a more easy analysis of the propeller, it can be recommended to place the turning point T at one quarter of the chord length from the trailing edge.

From this we see that, within the accuracy of the theory, we do not have to take into account the induced velocities (5.1), which give rise to a change of the angle of attack by an amount of $\mathbb{O}(\Gamma)$.

6. VERTICAL-AXIS PROPELLERS WITH AN INFINITE NUMBER OF BLADES OR QUICKLY ROTATING PROPELLERS

When the number of blades or the rotational velocity of the propeller increases, the wake of the propeller becomes crowded with free vortex layers. Hence it seems natural to replace this complicated system by a continuous distribution of free vorticity. The bound vorticity can be replaced by a continuous distribution over the circumference of the circle. We assume for its strength per unit length the function $\Gamma_c(\varphi)$, where the index c indicates a continuous distribution.

For the free vorticity per unit length in the x direction, which is left in the strips between y and $\pm R$ (see Fig. 7) we find

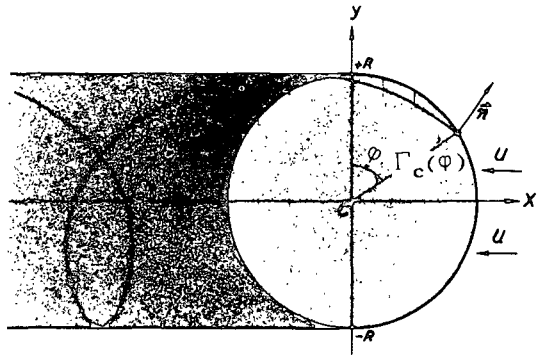


Fig. 7. The propeller with a continuous vortex distribution

$$\mp \frac{1}{\mu} \{\Gamma_c(\varphi) - \Gamma_c(2\pi - \varphi)\} , \qquad y = R \cos \varphi . \tag{6.1}$$

The velocity induced by two infinite strips with vorticity of strength given in (6.1) at $y = R \cos \varphi$ is

$$- \frac{1}{\mu} \left[\Gamma_c(\varphi) - \Gamma_c(2\pi - \varphi)\right] . \tag{6.2}$$

By this the energy in an arbitrary rectangle of unit length in the x direction between $y = \pm R$ and far behind the propeller becomes

$$E = \frac{\rho R}{2\mu^2} \int_0^{\pi} \left[\Gamma_c(\varphi) - \Gamma_c(2\pi - \varphi)\right]^2 \sin \varphi \, d\varphi . \tag{6.3}$$

The condition for the total component in the x direction of the force acting on the bound vorticity is

$$\rho \omega R^2 \int_0^{2\pi} \Gamma_c(\varphi) \sin \varphi \, d\varphi = K . \tag{6.4}$$

We have now to minimize (6.3) under the condition (6.4). Introducing the perturbed bound vorticity

$$\Gamma_c(\varphi) + \varepsilon_1 g_1(\varphi) + \varepsilon_2 g_2(\varphi) , \tag{6.5}$$

where $g_1(\varphi)$ and $g_2(\varphi)$ are arbitrary functions of φ, into (6.3) we have to differentiate E with respect to ε_1, while condition (6.4) must be taken into account. Applying the multiplication method of Lagrange we obtain

$$\frac{\rho R}{\mu^2}\int_0^{\pi}\left[\Gamma_c(\varphi)-\Gamma_c(2\pi-\varphi)\right]\left[g_1(\varphi)-g_1(2\pi-\varphi)\right]\sin\varphi\,d\varphi$$

$$+\lambda\rho\omega R^2\int_0^{\pi} g_1(\varphi)\sin\varphi\,d\varphi=0 \qquad (6.6)$$

where λ is a still unknown constant.

This equation can be written in the form

$$\frac{1}{\mu^2}\left(\int_0^{\pi}-\int_{\pi}^{2\pi}\right)\left[\Gamma_c(\varphi)-\Gamma_c(2\pi-\varphi)\right]g_1(\varphi)\sin\varphi\,d\varphi$$

$$+\lambda\omega R\int_0^{2\pi} g_1(\varphi)\sin\varphi\,d\varphi=0\,. \qquad (6.7)$$

Because $g_1(\varphi)$ is arbitrary we have

$$\frac{1}{\mu^2}\left[\Gamma_c(\varphi)-\Gamma_c(2\pi-\varphi)\right]=\mp\lambda\omega R,\qquad \begin{matrix}0<\varphi<\pi\\ \pi<\varphi<2\pi\end{matrix}\,. \qquad (6.8)$$

From this it follows that the only condition which we have to satisfy is: the difference of the bound vorticity at the front and the back of the circle for the same values of y is a constant. Hence we may for instance assume $\Gamma_c(\varphi)=-\Gamma_c(2\pi-\varphi)=\text{const}$. From (6.4) it follows

$$\Gamma_c(\varphi)=\pm\frac{K\mu}{4\rho RU},\qquad \begin{matrix}0<\varphi<\pi\\ \pi<\varphi<2\pi\end{matrix}\,. \qquad (6.9)$$

This agrees with the result of section 4 where it was found that the optimum bound vorticity was an odd function of φ plus some arbitrary constant. In this case of continuous vorticities it is allowed to add an arbitrary even function of φ to the distribution in (6.9).

From the above it follows that the vorticity in the wake is concentrated wholly on the lines $y=\pm R$. The strength γ per unit length in the x direction becomes

$$\gamma=\mp\frac{K}{2\rho RU},\qquad y=\pm R\,. \qquad (6.10)$$

This means that the induced velocity u in the wake is, parallel to the x axis, independent of y and amounts to

$$u=-\frac{K}{2\rho RU}\,. \qquad (6.11)$$

This result can be checked easily by calculating the external force necessary for the increase of momentum which follows from (6.11).

Isay [2] uses a formula, Eq. (1) in his work, which is not in agreement with our formula (6.1). He gives also another formula, Eq. (12), which he thinks to be faulty, but which, in the opinion of the author, is the correct one. That something is wrong with Eq. (1) of Isay follows from the fact that in it the velocity of the incoming fluid does not occur. It is clear that when this velocity increases, the density of the free vorticity in the wake decreases, when the bound vorticity distribution remains the same.

7. THE ANGLE OF INCIDENCE IN THE CASE OF CONTINUOUS DISTRIBUTIONS

We first consider the induced velocity by the two half infinite equidistant rows of vortices which form the wake. By elementary calculations we find that the component of the velocity $v_{n,1}$, in the direction $\vec{n}$, normal to the cycloid, at some point φ of the circle (Fig. 7) amounts to

$$v_{n,1} = \frac{-K}{8\rho RU\sqrt{1+\mu^2+2\mu\cos\varphi}}\left[\beta^*\sin\varphi + \frac{1}{\pi}(\mu+\cos\varphi)\ln\left(\frac{1+\cos\varphi}{1-\cos\varphi}\right)\right],$$

$$\beta^* = \begin{matrix}1, & 0<\varphi<\pi\\ 3, & \pi<\varphi<2\pi\end{matrix}. \qquad (7.1)$$

The component of the velocity normal to the cycloid, induced by the bound vorticity (6.9) on the circle, becomes

$$v_{n,2} = -\frac{K\mu}{8\pi\rho RU}\,\frac{(1+\mu\cos\varphi)}{\sqrt{1+\mu^2+2\mu\cos\varphi}}\ln\left(\frac{1+\cos\varphi}{1-\cos\varphi}\right). \qquad (7.2)$$

Finally we write down the normal component of induced velocities by a constant density Γ_c^* of bound vorticity on the circle,

$$v_{n,3} = \frac{\Gamma_c^*\mu\sin\varphi}{2\sqrt{1+\mu^2+2\mu\cos\varphi}}. \qquad (7.3)$$

The velocities (7.2) and (7.3) are the mean values of the velocity components normal to the cycloid in points just outside and just inside the circle. Hence we find for the total induced normal velocity

$$v_n = v_{n,1} + v_{n,2} + v_{n,3}. \qquad (7.4)$$

When the propeller consists of M blades, each blade has to possess a bound vorticity

$$\Gamma(\varphi) = \frac{2\pi R}{M}\left(\pm\frac{K\mu}{4\rho RU} + \Gamma_c^*\right), \qquad \begin{matrix} 0 < \varphi < \pi \\ \pi < \varphi < 2\pi \end{matrix} . \tag{7.5}$$

The constant bound vorticity Γ_c^* is still arbitrary and can be used also in this case to determine the activity of the blades in the front or in the back position.

With respect to the tangent at the cycloid we obtain for the angle of incidence $\alpha(\varphi)$

$$\alpha(\varphi) = \alpha_o - \frac{v_n(\varphi)}{V(\varphi)} + \frac{\Gamma(\varphi)}{\pi V(\varphi)\,\ell} \tag{7.6}$$

where α_o is the angle of zero lift, $V(\varphi)$ follows from (2.2), and ℓ is the length of the chord of the profile. The angles $\alpha(\varphi)$ and α_o are positive when they open to the left with respect to the direction of $V(\varphi)$.

Because we have not taken into account the velocities induced by the oscillating motion of the blades we have to place the turning point T of the profiles at one quarter of the chord length from the trailing edge, as follows from the end of section 5.

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DISCUSSION

G. Weinblum (Institut für Schiffbau, University of Hamburg)

I wish to raise the question, who should decide when mathematicians disagree? I have not understood what you have done. Have you calculated some numerical values so that we can compare what comes out?

J. A. Sparenburg

This is a case in which mathematicians do not agree upon the interpretation of the basic formulae needed for the theory. In my opinion the taxation of the various ways of approach has to be made by executing experiments, not only of the over-all results but also of the basic formulae. We have made some calculations based on the simple theory of sections 6 and 7. Before publishing results, however, we shall compare them with results based on the more refined theory of sections 2-5 and on experiments.

W. H. Isay (Institute of Applied Mathematics of the German Academy of Science)

In reply to Dr. Sparenberg's discourse on my papers* dealing with the theory of the Voith-Schneider propeller I should like to remark the following:

Equation (12) of my paper II mentioned by Dr. Sparenberg is obtained by the usual methods of potential flow analysis if it is assumed that the free vortices move to infinity downstream of the propeller without losing their intensity. In reality, however, these vortices will decay in the real turbulent flow downstream of the propeller and both their influence and their induced velocity will decrease accordingly. The use of Eq. (12) will therefore yield grossly erroneous results, particularly for the blades on the downstream half of the propeller which are hit by the free vortices produced by the blades on the upstream half of the propeller. This is because the influence of the free vortices is greatly exaggerated by Eq. (12) of potential theory.

On the other hand, any attempt to arrive at an actually comprehensive theoretical description of the turbulent mixing and decay of the free vortices would have little chance to succeed as the process is not only different at different points but also subject to random effects. It was therefore deemed important to evolve a physically reasonable equivalent representation of the velocity field of free vortices which may be used also for numerical calculations without excessive difficulty. Equation (1) of my paper II, or Eq. (3) of my paper I, is satisfactory under this aspect though naturally different from the conventional formulae of potential theory. This formula yields useful results which are in satisfactory agreement with force measurements. A full discussion of the difference between Eqs. (1) and (12) along with numerical examples can be found in sections 2, 5, and 6 of my paper II.

The problem thus encountered will be of interest with all types of propellers and turbo equipment in which part of the blades operate within the wake vortices produced by other blades. This, however, shall not imply that the method I have shown for Voith-Schneider propellers can be applied without change to other types of hydrodynamic machinery.

Contrary to the opinion held by Dr. Sparenberg the induced velocity u_f of the free vortices decreases with increasing incoming velocity u_o also in the case of Eq. (1) of my paper II. This is caused by the resulting change in the distribution of circulation as can be readily seen from the numerical examples given in my papers. Furthermore, this reduction in u_f, for increases in u_o, is not more pronounced with Eq. (12) than it is with Eq. (1) as might erroneously be concluded when considering the initial factor $\omega R/u_o$ without at the same time observing the behaviour of the circulation. The latter becomes apparent only from the boundary condition of flow past the profile, i.e., from the integral equation.

Naturally it is possible also with my theory to determine – similarly as has been shown for the problem treated by Dr. Sparenberg in his paper – the distribution of the angle of incidence for a prescribed relationship for the change of blade circulation as the blade proceeds on the propeller circle. Also in this instance the difference between Eqs. (1) and (12) of my paper II becomes apparent. If, for example, a simple cosine law is prescribed for blade circulation (in which case the propulsive force produced by the upstream and downstream halves of the propeller would be about equal), then the incidence angle distribution obtained from

*(I) "Zur Behandlung der Strömung durch einen Voith-Schneider-Propeller mit kleinem Fortschrittsgrad," Ing. Arch. 23:379 (1955); (II) "Zur Berechnung der Strömung durch Voith-Schneider-Propeller," Ing. Arch. 24:148 (1956); (III) "Der Voith-Schneider-Propeller im Nachstrom eines Schiffsrumpfes," Ing. Arch. 25:303 (1957); (IV) "Ergänzungen zur Theorie des Voith-Schneider-Propellers," Ing. Arch. 26:220 (1958).

Eq. (1) is quite reasonable. In fact it is found that, apart from the obvious change in sign, the angle of incidence for the blades on the downstream half of the propeller must be about 1.3 to 1.8 the value for the upstream half, depending on the advance ratio and the total lift. On the other hand, if Eq. (12) is employed, the resulting incidence angle pattern for the downstream side is completely senseless as it includes abrupt changes and even infinities.

It must further be noted that one might consider describing the mixing and decay of the free vortices not by Eq. (1), but by introducing the known Lamb's law* describing the decay of vortices in a real liquid into Eq. (12). I have abstained from this for two reasons: First, Lamb's law holds only for isolated vortices in purely laminar flow whereas propeller flow is turbulent, and second, the mathematical realization of this method would have been rather complicated without enabling a definitely better physical description of the problem.

While this problem is discussed here to such an extent, it is scarcely considered in Dr. Sparenberg's paper. Equations (5.1), (5.2), and (5.3) used therein for angle of incidence determination will render approximate solutions for the upstream half of the propeller blade circle only. One must keep in mind that the blades on the downstream side are exposed to the free wake vortices induced by the upstream blades, and that on the downstream side the velocity field of these free vortices cannot justly be omitted from the boundary condition for the flow past the profile since it exerts a decisive influence.

In closing I should like to remark that in my opinion the theory of quickly rotating propellers evolved by Dr. Sparenberg in sections 6 and 7 with circulation (6.9) and free wake vortices (6.10) has the disadvantage of introducing excessive simplification. It is therefore probable that this theory will not render physically reasonable results. This can be seen already from the fact that Eq. (7.6) for incidence angle α yields $\alpha = \infty$ exactly for $\varphi = 0$ and $\varphi = \pi$, whereas in reality the value of α will be relatively small or even zero for these values of φ.

J. A. Sparenburg

Equation (1) in paper II by Prof. Isay is the basis of his theoretical work. This formula describes the velocities induced by free vortices which are shed by a system of rotating and periodically varying bound vortices, placed in a homogeneous stream. The density of the free vorticity decreases when the velocity of the incoming stream increases. Hence it may be expected that the induced velocities of the free vorticity decrease in this case. However, in the above mentioned Eq. (1) the incoming velocity does not occur, which is surprising. The remark of Prof. Isay, that the increase of the incoming velocity causes a decrease in the free vorticity by changing the bound vorticity, describes a secondary effect. Here the basis formula itself and not its relation to a special boundary value problem is discussed.

Prof. Isay further questions the validity of author's formulae (5.1), (5.2), and (5.3) for the downstream side of the circle. When we look at (5.2), for instance, we see that a pole of the integrand crosses the line of integration when the point z, where we consider the induced velocities, crosses the vortex layer. This means a discontinuity of the value of the integral, which corresponds to the discontinuity of the velocities at the vortex layer. Hence with respect to potential theory, this formula is valid also in the downstream region.

*H. Lamb, "Lehrbuch der Hydrodynamik," 2nd ed., Leipzig, 1931, p. 669

The remark of Prof. Isay that the theory for the quickly rotating propeller (sections 6 and 7 of my paper) is too crude may be true. It does not follow, however, from the fact that the angle of incidence α becomes infinite at the places $\varphi = 0$ and $\varphi = \pi$. The regions where α is large are so small that they can be neglected by the realisation of a model. As is well known in many linearised theories, there occur singularities at points where the theory does not hold, for instance, the endpoints of the range of contact by indentation problems and the leading edge by lifting surface theory. This does not affect the applicability of these theories, when the influence of the singularities is restricted within narrow limits. Moreover the results of the simple theory are not devoid of sense and show the behaviour mentioned by Prof. Isay in his contribution.

It is, as Prof. Isay stresses, highly recommendable to investigate the interaction of turbulence and free vorticity. However, in my opinion this must be done on the basis of experiments or theory and not on the basis of a law stated a priori.

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