XI

FREE-STREAMLINE THEORY AND STEADY-STATE CAVITATION

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The following is a brief review of several aspects of free streamline theory, with particular reference to steady state cavitation [1].

Cavity models

The Helmholtz concept of free streamline long had its principal applications in the theory of wakes and jets. The well-known Kirchhoff model of the infinite wake, in which free streamlines detach from a body and enclose a constant pressure stagnation region, provided a theory of fluid resistance within the framework of classical hydrodynamics, but was recognized, practically from inception, to be unrealistic in its essential features [2]. It has recently become apparent that the natural application of free streamline theory is to the phenomena of cavitation rather than wakes.

In steady state cavitation, as observed in water tunnel studies, a vapor-filled cavity forms behind a body past which the fluid is moving at sufficiently high speed, the cavity being essentially at the vapor pressure of the liquid and its boundary a constant pressure free surface. The conditions here differ from those of the classical wake theory in that the cavity pressure p_c is below rather than equal to the static value p_∞. The pressure difference is usually measured in non-dimensional units by the *cavitation parameter*

$$\sigma = \frac{p_\infty - p_c}{\frac{1}{2}\rho u^2}, \tag{1}$$

where u is the free stream velocity and ρ is the fluid density. σ is the basic similarity parameter of the theory of cavitation, and in the absence of gravity, viscosity, etc., all non-dimensional quantities are functions of it alone.

Observed cavities have positive cavitation number and of course finite dimensions. It is the function of the theory to describe these flows and the associated physical quantities. The limiting case $\sigma = 0$—which yields an infinite cavity corresponding to the classical wake—often proves useful in studying the finite cavity. Mathematically, the essential difference between the problems of cavitation and the traditional problems of hydrodynamics is that in the former the shape of the flow region is unspecified and has to be determined from the condition that the pressure, and hence flow speed, is constant on the free surface.

In attempting to describe a finite cavity within classical hydrodynamics—that is, under the assumptions of steady, gravity free, irrotational flow—we run at once into the paradox that such a flow cannot be realized physically. To be specific, let us consider the behavior of the free streamlines that detach from a body in cavitational flow in an unbounded stream. Because the minimum pressure, and therefore the maximum speed, must occur on the free streamlines (the cavitation hypothesis), it follows that these curves are convex to the flow [3]. This allows only the following possibilities: *i.* The free streamlines may extend to infinity downstream without intersecting one another; this can be shown impossible except when $\sigma = 0$. *ii.* They may intersect, which obviously contradicts the continuity of the flow (and can also be shown mathematically untenable if they are considered to overlap). *iii.* The streamlines turn back, forming a re-entrant jet which, if continued, must pass through the rear of the body or through the cavity wall (Fig. 1). The last alternative is the only one mathematically consistent with the theory but is of course non-physical. Thus, under the present assumptions, it is impossible in principle to satisfy the physical requirements of steady state cavitation.

Water tunnel and water entry experiments reveal clearly the existence of the re-entrant jet. However, the jet is greatly weakened by turbulent mixing in the stagnation region, and in larger cavities it can be seen striking the cavity walls and then being swept away by the flow. To all intents and purposes these flows may be considered stationary. In smaller cavities, or in flows past obstacles with large afterbodies, the jet can be observed (in high speed photographs) distorting the cavity and giving rise to an unsteady pulsating motion.

Another approach to the problem of the finite cavity is through the artificial but conceptually simpler Riabouchinsky model. This avoids the non-physical doubly-covered flow plane by introducing the mirror image of the obstacle as a streamline of the flow (Fig. 2). As far as quantitative results are concerned it is a matter of indifference which cavity model is adopted, for the Riabouchinsky and re-entrant jet flows yield almost identical values of drag and cavity dimensions. From this it appears that conditions at the rear of the cavity have but little effect on gross flow quantities.

In the two-dimensional case the calculation of these flows is reduced by the hodograph method to a simple exercise in conformal mapping. Consider first the symmetric re-entrant jet flow past a flat plate, shown in Fig. 1 with its image planes. Let $f = \varphi + i\psi$ be the complex potential and let the cavity speed be unity. Then one finds:

$$f'(s) = M\frac{s(s^2 - b^2)}{s^2 - h^2}, \qquad M \quad \text{real}, \tag{2}$$

$$f'(z) = \left(\frac{s+1}{s-1}\right)^{\frac{1}{2}}\left(\frac{b-s}{b+s}\right), \tag{3}$$

where $b = -2h^2 - h + 2$, and the cavitation parameter is given by

$$\sigma = \frac{-4h^3 + h^2 + 4h - 2}{(h^2 + h + 1)^2}. \tag{4}$$

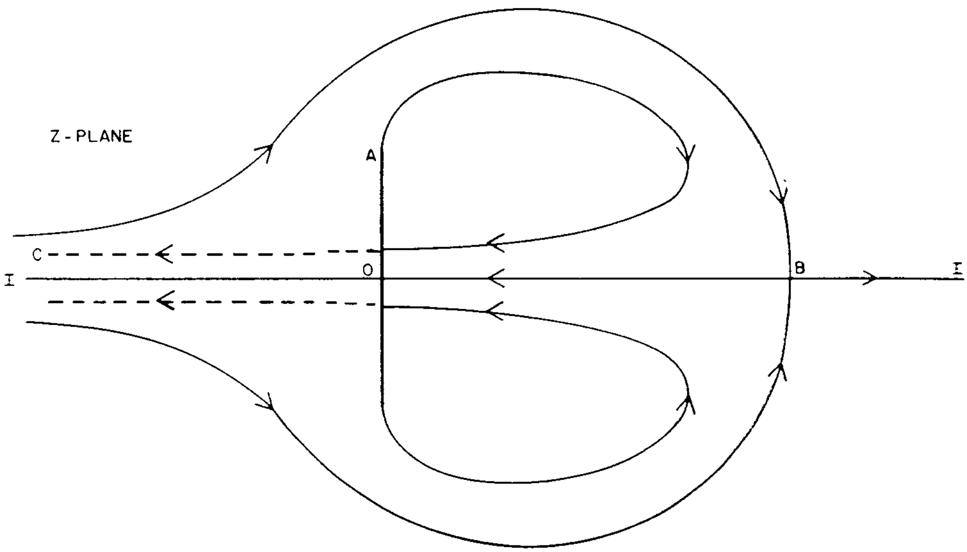
The expression for the drag coefficient obtained from (2) and (3) by integrating the pressure difference $p - p_c$ over the front face is

$$C_D(\sigma) = (1+\sigma)\left[1 - \frac{\int_{-1}^{0}\left(\frac{1+s}{1-s}\right)^{\frac{1}{2}}\left(\frac{s-b}{s^2-h^2}\right)^2 sds}{\int_{-1}^{0}\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}}\left(\frac{s+b}{s^2-h^2}\right)^2 sds}\right]. \tag{5}$$

This gives the asymptotic formula, valid for small σ,

$$C_D(\sigma) \sim \frac{2\pi}{4+\pi}(1+\sigma). \tag{6}$$

FIG 1A



RE-ENTRANT JET CAVITY

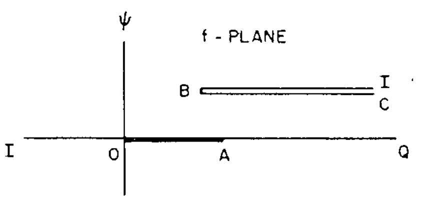


FIG 1B

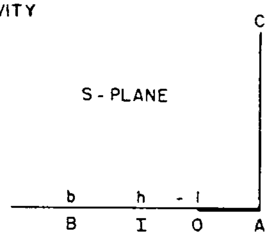


FIG 1C

A more precise estimate is

$$C_D(\sigma) \sim \frac{2\pi}{4+\pi}\left[\,1+\sigma+\frac{\sigma^2}{8(\pi+4)}\,\right]. \qquad (7)$$

The formula (6) underestimates within .8 percent at $\sigma = 1$ and improves in accuracy as $\sigma \to 0$. If the flat plate is replaced by a symmetric wedge of semiangle $\alpha\pi$ the exponent $\frac{1}{2}$ in (3) and (5) is replaced by α.

In asymmetric flow without circulation past an arbitrary obstacle of diameter l, the drag and lift coefficients are related to the jet width J and direction γ by the formulas

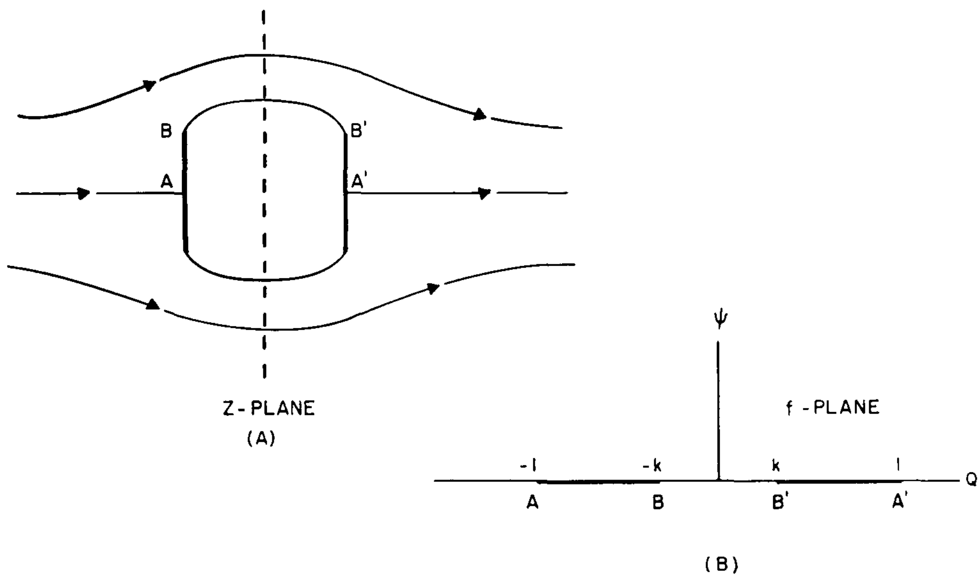
$$C_D(\sigma) = 2\frac{J}{l}\sqrt{1+\sigma}\,(1-\sqrt{1+\sigma}\cos\gamma), \qquad C_L(\sigma) = -\frac{2J}{l}\sin\gamma. \qquad (8)$$

Calculations of asymmetric re-entrant jet flows have not yet been carried out in specific instances.

Turning to the Riabouchinsky flow past the plate (Fig. 2), and assuming the flow so normalized that $f = -1, +1$ at the front and rear stagnation points, one finds

FIG. 2

RIABOUCHINSKY CAVITY



$$\frac{dz}{df} = \frac{\sqrt{f^2 - k^2} + f\sqrt{1 - k^2}}{k\sqrt{f^2 - 1}} \tag{9}$$

where

$$\sigma = \frac{2k'}{1 - k'}, \qquad \text{and} \quad k' = \sqrt{1 - k^2}. \tag{10}$$

The various flow quantities can be expressed in terms of the complete elliptic integrals of first and second kinds, denoted in the usual notation by $E = E(k)$, $E' = E(k')$, $K = K(k)$, $K' = K(k')$. Thus, we have: the ratio of cavity width d to plate length l,

$$\frac{d}{l} = \frac{k' + E' - k^2K'}{k'^2 + E' - k^2K'}; \tag{11}$$

the ratio of cavity length h to plate length,

$$\frac{h}{l} = \frac{E - k'^2K}{k'^2 + E' - k^2K'}; \tag{12}$$

the drag coefficient,

$$C_D = 2(1 + \sigma)\left(\frac{E' - k^2K'}{k'^2 + E' - k^2K'}\right). \tag{13}$$

From the asymptotic properties of elliptic integrals as $k' \to 0$ one obtains (6) and (7) again, and also

$$\frac{d}{l} \sim \frac{4}{4+\pi}\left(\frac{2+\sigma}{\sigma} + \frac{\pi}{4}\right) \tag{14}$$

$$\frac{h}{l} \sim \frac{4}{4+\pi}\left[\left(\frac{2+\sigma}{\sigma}\right)^2 - \tfrac{1}{2}\log 4\left(\frac{2+\sigma}{\sigma}\right) - \tfrac{1}{4}\right]. \tag{15}$$

These values are accurate within .7 percent at $\sigma = 1$ and improve rapidly for smaller σ.

(6) is a special case of the formula

$$C_D(\sigma) \sim C_D(0)(1+\sigma), \tag{16}$$

which seems to be generally valid for both plane and axially symmetric cavities having fixed separation points [4], and is in good agreement with observation. The error term in (16) grows more rapidly with σ in the axially symmetric case. Experimental results indicate that the formula $C_D(\sigma) \sim C_D(0)(1+\alpha\sigma)$ should replace (16) in flows past bodies having variable points of separation (such as the cylinder and sphere).

Formulas analogous to (14) and (15) for the axially symmetric Riabouchinsky flow past a body of diameter l are

$$\frac{d}{l} \sim \sqrt{\frac{C_D(\sigma)}{\sigma}} \quad \text{and} \quad \frac{h}{l} \sim \sqrt{\frac{C_D(\sigma)}{\sigma^2}}\log\frac{1}{\sigma}. \tag{17}$$

From these it is apparent that the plane cavity has the larger length-width ratio for fixed cavitation number, whereas the axially symmetric cavity is the flatter of the two (as measured by d/l). The derivation of (17) is based on a convincing but heuristic comparison between cavities and flows past ellipsoids [5].

In the limiting case $\sigma = 0$, when the cavity is infinite, its asymptotic shape is given by

$$y \sim Cx^{\frac{1}{2}} \quad \text{in plane flow;}$$

$$y \sim \frac{Cx^{\frac{1}{2}}}{(\log x)^{\frac{1}{4}}} \quad \text{in axially symmetric flow [6].} \tag{18}$$

The drag is proportional to C^2 in the former case and to C^4 in the latter.

The theory of cavitational flow past smooth obstacles—that is, obstacles without corners or sharp edges—still has important gaps. The problem here is that the separation points of the free streamlines are not known a priori and must be determined as part of the solution; at the same time, it is not entirely clear what conditions are needed to determine a unique and physically acceptable solution. It is well known that a free streamline at detachment has either infinite curvature or the same curvature as the body from which it separates. The latter type of detachment will be called *smooth.* A flow will be considered *physically acceptable* if the free streamlines do not intersect the body or themselves and if the maximum speed occurs on the free boundary (as required by cavitation). Since a free streamline in cavitating flow is convex, its detachment (from a smooth body) must evidently be smooth if the flow is to be physically acceptable, but this condition is certainly not sufficient, as one sees by counterexample.

It is known [7] that within the class of bodies for which physically acceptable infinite cavity flows always exist are those having non-decreasing curvature on both sides of the forward stagnation point (Fig. 3). However, it is a simple matter to exhibit other convex bodies for which physically acceptable solutions of this type do not exist. In this connection the flow past an asymmetric wedge is instructive. (The wedge can be looked upon as an approximation to a convex obstacle with large curvature at the vertex.) The various possibilities when $\sigma = 0$ are the following: *i.* The

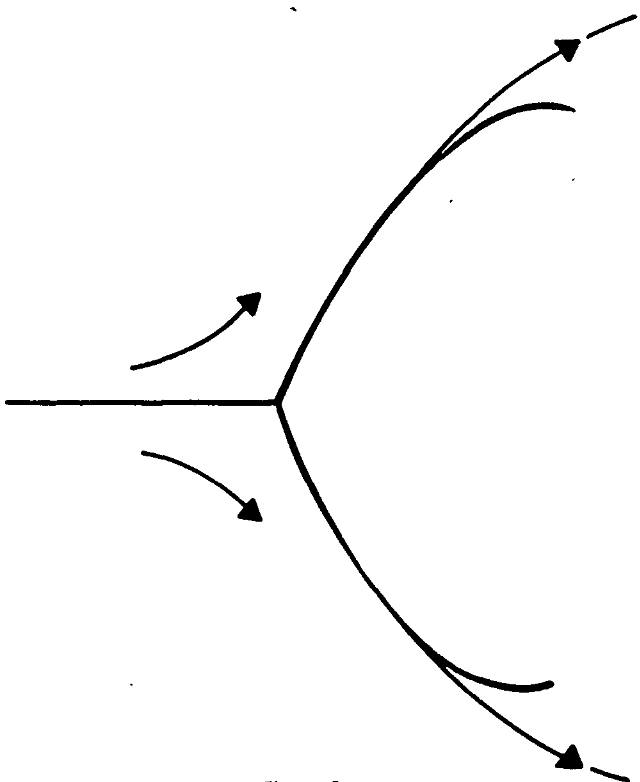


Figure 3.

dividing streamline meets the body at the vertex, and free streamlines detach from the ends forming an infinite cavity; in this case the ratio of the two lengths cannot be arbitrary and hence the flow is exceptional. *ii.* The free streamlines again detach from the ends, but the dividing streamline meets the wedge at a point other than the vertex; in which case the velocity is infinite at the corner and flow is physically unacceptable. *iii.* One of the free streamlines detaches from the vertex, determining a solution that is simply the flow past an inclined flat plate; this flow is not always acceptable since the free streamline may intersect the second side of the wedge. A way out of this difficulty is to allow the free streamline detaching from the vertex to reverse direction, forming a re-entrant jet (Fig. 4), while the portion of the flow not included in the jet bounds an infinite cavity whose free streamlines separate from the ends. Except for the feature of the two-sheeted flow plan, the latter model satisfies the physical requirements and, furthermore, is observed experimentally [8]. Its basic idea can most likely be extended to arbitrary curved obstacles to provide physically acceptable solutions which are otherwise lacking without the jet. In these cases the flow would contain multiple cavities. Such a theory is still unexplored.

Calculations of axially symmetric flows

Developments in the theory of axially symmetric cavitational flow reverse the customary order of things in that the general theory is already well advanced and boasts numerous qualitative results on existence, uniqueness, geometric behavior of the free streamlines, extremal properties, etc., while, on the other hand, useful explicit solutions are still unknown. (There do exist explicit solutions but without practical

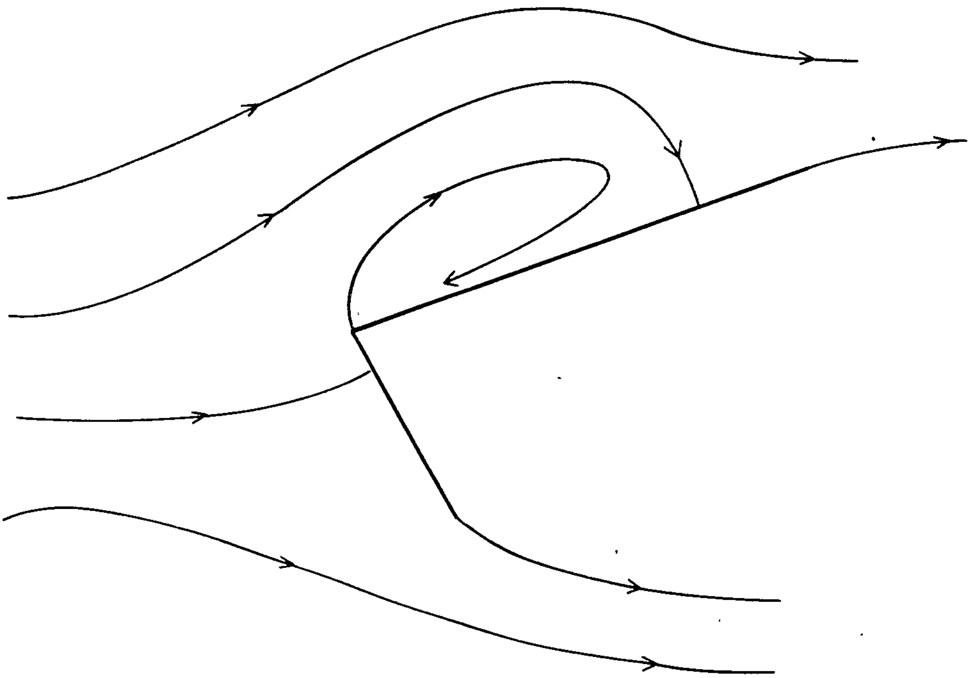


Figure 4.

significance.) This may be in the nature of things and it is quite possible that such solutions are simply unattainable.

It is therefore natural to fall back on numerical methods, and with the increasing emergence of high speed computing machines it is appropriate that methods be developed for the calculation of cavity flows. At the present time, the labor in these calculations is formidable, and most questions of convergence and accuracy remain unanswered.

An interesting and valuable illustration is provided by the problem of the *vena contracta,* which is perhaps the best studied of the axially symmetric free surface problems. The experimental value for the contraction coefficient of a jet issuing from a circular orifice in a plane wall is in the neighborhood of .61. This figure is of particular interest because it coincides with the known theoretical value $\pi/(\pi + 2) \approx .611$ of the contraction coefficient of a plane jet issuing from a slot. This agreement has stimulated the conjecture that the theoretical values of corresponding plane and axially symmetric contraction coefficients are indeed the same. Trefftz [9], in calculations we shall not discuss here, first gave strong support to the conjecture by arriving at the value .61 for the axially symmetric jet, and later several calculations by the relaxation method yielded the same result [10]. However, Garabedian [11], using an altogether different method, recently obtained the value .58. This difference is large enough to raise serious doubts concerning the conjecture and to spur inquiry as to the sources of the discrepancy.

Garabedian's point of departure is the concept of axially symmetric flow in a space of $\varepsilon + 2$ dimensions. The stream function of such a flow obeys the equation

$$\psi_{xx} + \psi_{yy} - \frac{\epsilon}{y}\psi_y = 0, \tag{19}$$

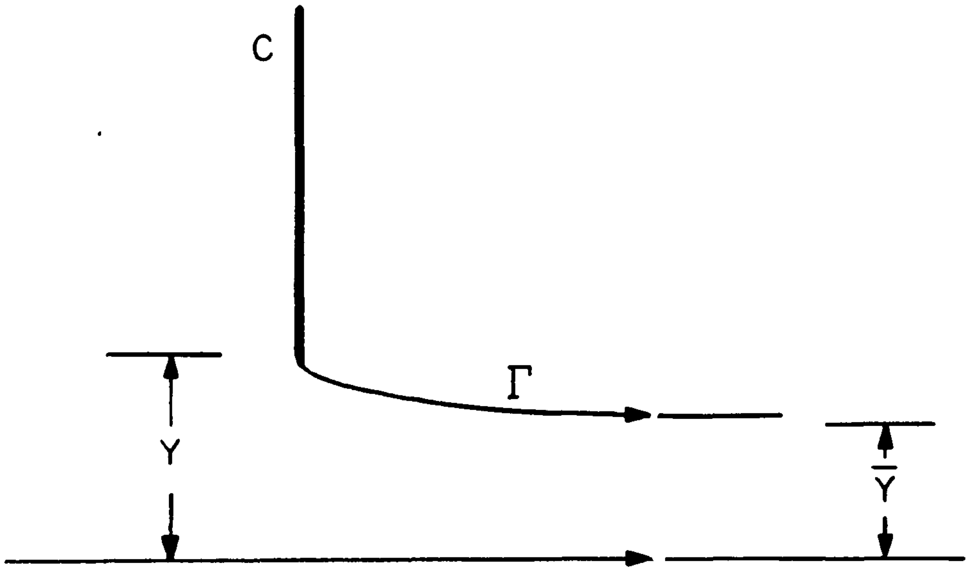


Figure 5.

where x is the axial and y the radial coordinate in the meridian plane. One can now formulate the problem of jet flow in terms of ε; namely, a curve $\Gamma(\varepsilon)$ (Fig. 5) and a stream function $\psi(x, y; \varepsilon)$ are sought such that $\psi = 0$ on the axis of symmetry, $\psi =$ const. on the bounding streamline $C + \Gamma$, and $\frac{1}{y^\varepsilon}\frac{\partial\psi}{\partial n} = 1$ on Γ. Of course, $\varepsilon = 1$ corresponds to axially symmetric flow and is the case of immediate physical interest. The contraction coefficient as a function of ε is $C_c(\varepsilon) = (\bar{Y}(\varepsilon)/Y)^{1+\varepsilon}$ where Y and $\bar{Y}$ are the orifice and jet radii (cf. Fig. 5). If it were indeed true that $C_c(1) = C_c(0) = \pi/(\pi + 2)$, one might conjecture that $C_c(\varepsilon)$ remains constant, and hence that $C'_c(\varepsilon)_{\varepsilon=0} = 0$. However, this quantity can be accurately computed and is equal to $-.057$. Garabedian arrives at his estimate $C_c(1) = .58$ by cubic interpolation, using the exact values of $r = \bar{Y}/Y$ at $\varepsilon = -1$, 0, and ∞ where $r = 0$, $\pi/(\pi + 2)$, and 1, respectively, and the computed quantity $\frac{d}{d\varepsilon}(\bar{Y}/Y)_{\varepsilon=0} = .243$. Even without the latter figure, interpolation gives $C_c(1) = .586$, an indication that .58 is probably accurate.

Turning to the relaxation method, we recall that the procedure here is to cover the flow region with a rectangular mesh and to replace the differential equation by an appropriate difference equation. In the problem at hand, the free streamline must be chosen so that the velocity computed by finite differences is constant on the curve. This requires a trial and error procedure which finally yields a curve on which the computed speed variation is considered sufficiently small. Unfortunately, it is not entirely clear what should be considered "sufficiently small" in a free boundary problem, especially since the computed velocity fluctuation can be insensitive to relatively large displacements of the free streamline. That this should be so can be anticipated for theoretical reasons, but is also confirmed by specific calculations which compare an exact solution for plane Riabouchinsky flow with one computed by the finite difference method [12]. The speed variations on the free streamline in the latter calculation were considerably smaller percentagewise than the error in the computed height above the separation point. The principal source of error in finite difference calculations is probably the infinite curvature of the free streamline at separation, for small errors in shape here are reflected in relatively large errors in the downstream position of the free streamline—without necessarily incurring large velocity fluctuations downstream. Good accuracy at detachment requires careful refinement of the mesh or special handling of the flow at separation. The importance of truncation error has still to be clarified.

Although several approximate methods have been applied to cavity flows with axial symmetry [13], calculations based on exact theory are quite rare. Recently, using the scheme of dimensional interpolation outlined above, Garabedian obtained the value $C_D = .827$ for the drag coefficient of the circular disk in infinite cavity flow. The extrapolated experimental data favors a value between .80 and .81, but its spread includes .827 at the upper end. In his treatment of the Riabouchinsky cavity, for which the preceding method is unsuitable, Garabedian [14] has developed an iteration procedure based on linearization of the original boundary value problem. Namely, he shows that on any curve Γ which is displaced normally by an amount δn from the free streamline, the correct stream function satisfies the condition

$$\frac{1}{y}\frac{\partial\psi}{\partial n} + \frac{\kappa}{y}\psi = 1 \tag{20}$$

within an error of the order $(\delta n)^2$, where κ is the curvature of Γ. An approximate solution of the flow problem is thus provided by a stream function ψ satisfying (20) on the starting approximation Γ; the next approximation to the free streamline—and the starting point of the next cycle—is the curve on which $\psi = 0$. In the calculation of flow past a disk, Garabedian finds $C_D(\sigma)/(1 + \sigma) = .865$ when $\sigma = .2235$. This is a few percent higher than the experimental value, but confirms the observed increase in slope of the $C_D(\sigma)$ curve. His figure 2.30 for the ratio of cavity to disk diameter at this value of σ is in close agreement with observation.

Comparison methods

A fruitful source of qualitative results has been the *comparison method* that was initiated by Lavrentieff and later extended by Gilbarg and Serrin [15]. This method features simple geometric arguments in the physical plane, which are often equally

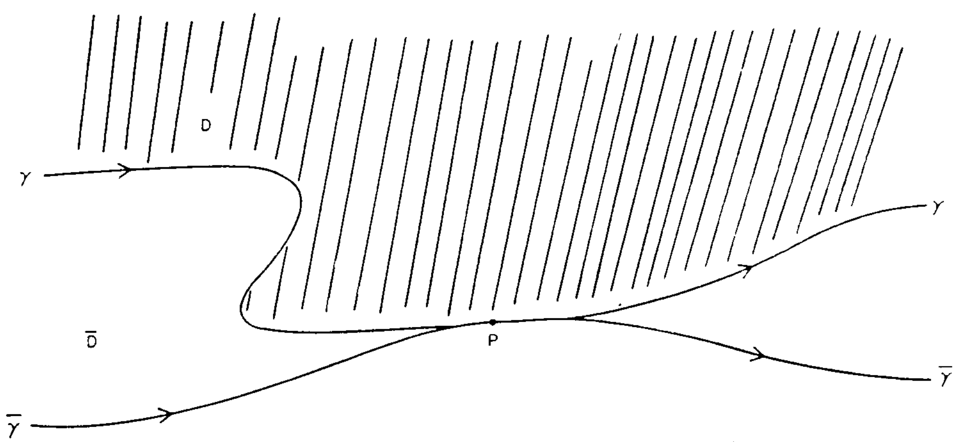


Figure 6.

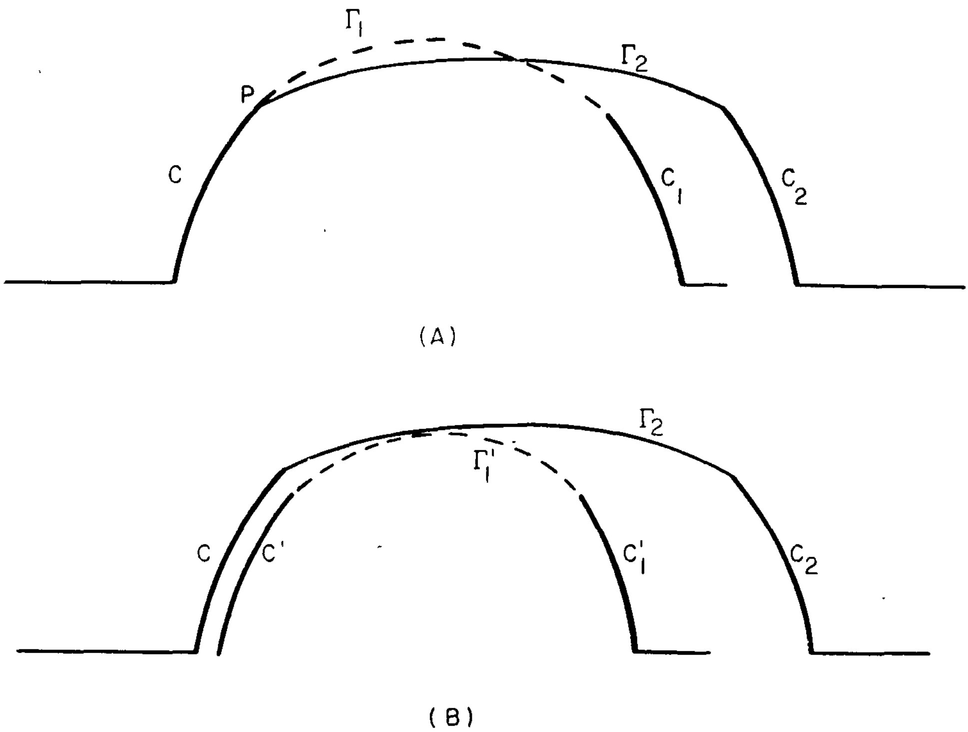


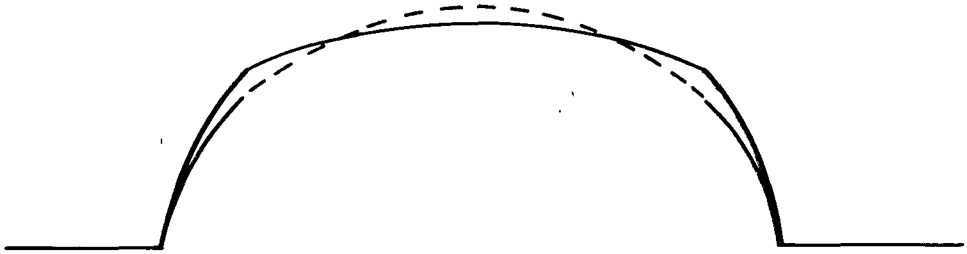
Figure 7.

effective in axially symmetric and compressible flow problems. Basic to the approach are certain comparison theorems, of which the following is an important example.

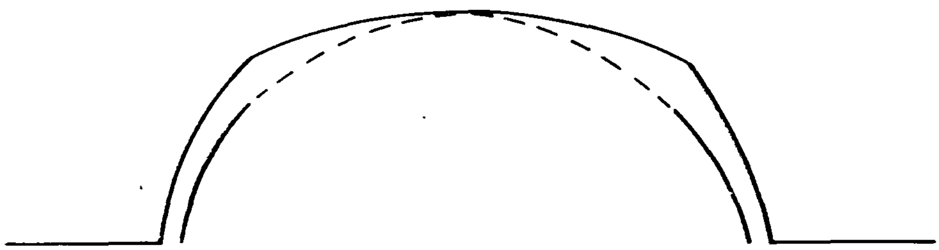
Let D and $\overline{D}$ be flow regions of two (plane or axially symmetric) flows having uniform velocities q_0, $\overline{q_0}$ at infinity, where $\overline{q_0} \geqq q_0$. Let D and $\overline{D}$ be bounded by the streamlines γ and $\overline{\gamma}$, extending to infinity in both directions. If $D \subset \overline{D}$, and if γ and $\overline{\gamma}$ have a point P in common (Fig. 6), then *the respective flow speeds* q *and* $\overline{\text{q}}$ *satisfy at* P *the inequality* $\overline{\text{q}}(\text{P}) \geqq \text{q}(\text{P})$, *and furthermore, when* $\overline{\text{q}}(\text{P}) \neq 0$ *the equality holds if and only if* $\text{D} = \overline{\text{D}}$ *and the two flows are identical.*

As illustrative consequences that are useful in cavitation we mention the following:

1. In plane symmetric or axially symmetric infinite cavity flow, let T_1 and T_2 be obstacles in the upper half plane having the same endpoint A, and suppose that T_1 lies above (or touches) T_2; then the relative positions of the corresponding free streamlines detaching from A are reversed. This implies that the shape factor C in (18) is larger for T_2 than for T_1 and hence the cavity drag of T_2 exceeds that of T_1. Because of the formula (16) we may infer a similar inequality for small positive σ. The same idea can be used to obtain simple bounds on the drag of an obstacle. Indeed, let the latter be a curve denoted by C extending from (0, 0) to (a, b). It is assumed that the infinite cavity flow, which may detach from any point on C, is physically acceptable in the sense of the previous discussion. Then the drag on C is bounded from below by the drag on E_1, and from above by that on E_2, where E_1 and E_2 are the following



(A)



(B)

Figure 8.

curves: E_1 consists of a vertical segment L erected at the origin and of an arc of the free streamline in the cavity behind it, the length of L being such that the endpoint of the arc is at (a, b); E_2 is the vertical segment $x = a$, $0 \leq y \leq b$.

2. In plane or axially symmetric Riabouchinsky flow past a convex obstacle, the length of the cavity and the ratio of length to width are monotonically decreasing functions of the cavitation number. This result is a simple consequence of the above comparison theorem, and for sake of illustration of the technique we carry out the details of proof.

Let Fig. 7a indicate a typical configuration of two Riabouchinsky flows with the same incident velocity past the obstacle C, in which C_1, C_2 and Γ_1 Γ_2 are the respective image obstacles and free streamlines. If Γ_1 lies below Γ_2 then direct application of the comparison theorem at the separation point P shows that the smaller of the two cavities has the larger cavitation number, as required. On the other hand, if Γ_1 intersects Γ_2, as in Fig. 7a, a suitable similarity contraction takes $C + \Gamma_1 + C_1$ into a curve $C' + \Gamma'_1 + C'_1$ lying within the cavity $C + \Gamma_2 + C_2$ and such that Γ'_1 has a point of contact with Γ_2 (Fig. 7b). Under this contraction the flow speed at corresponding points can be preserved. It follows by the comparison theorem that the speed on Γ'_1, and hence on Γ_1, exceeds that on Γ_2. This proves the monotonicity of the cavity length. The monotonicity of the length-width ratio is proved similarly. Let the cavity with the smaller cavitation number be contracted to have the same length as the other. It suffices now to prove its width smaller. Suppose this were not so; then a typical configuration of the two flows would be as in Fig. 8a. Another contraction of the flow with the smaller cavitation number would bring it into a position where its free streamline lies below the other, and at one point is in contact with it (Fig. 8b). The comparison theorem is now applicable, and provides an evident contradiction, thereby establishing the required monotonicity property.

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DISCUSSION

J. D. Nicolaides

I wonder if the speaker would care to comment on the efforts that I mentioned yesterday, of attempting to determine the transient development of the cavity, using the NORC computer?

D. Gilbarg

No. I think—well, I will say something. Because of the difficulties around the singular point, I would say the finite difference method would give a useful answer only if you are not too concerned with accuracy.

However, if they are pressed to their limits of capability, these machines may give pretty good answers. The likelihood is, from what we know, you will get useful information, which is not too accurate, whatever that happens to mean.

C. J. Cohen

I would just comment that one could do this matter by systematically reducing the size of the mesh, so one would at least get a bound-under answer and you would know where you stand.

D. Gilbarg

Yes, I think that should be done, definitely.

C. J. Cohen and L. D. Gates, Jr.

Professor Gilbarg stated in his paper that the relaxation solution of the Riabouchinsky problem at Dahlgren indicated that the method was ill-suited for cavitational flow problems because although the spread in the computed velocities along the free boundary was of the order of one per cent, the error in the maximum cavity width was of the order of 10 per cent.

During the Symposium, one of us, (C. J. Cohen) commented from the floor that to his recollection the 10 per cent error in maximum boundary width had been mostly eliminated by applying corrections at the outer boundaries of the flow. (These outer boundaries had been introduced to allow a difference system solution, and the outer boundary values for the stream function were at first taken to be those for undisturbed flow.)

Dr. Cohen's recollection was in error although we feel that the large error in cavity width noted by Professor Gilbarg was in fact due to the original crude conditions imposed on the outer boundaries.

Perhaps a more serious failure of the difference method was revealed when a trial boundary for plane flow computed from the analytical plane flow solution was tested using corrections to the outer boundary conditions. The observed velocity variation was of the order of 5 per cent, which was felt to be rather large and due to truncation error. The average velocity was about 2 per cent above the true velocity. This latter figure represented an improvement over previous data obtained without the outer boundary correction. This was what prompted Dr. Cohen's remark.

M. D. Van Dyke

I should like to ask whether there is any doubt that locally, just at the edge of the disk, the singularity has the same nature as in the two-dimensional free streamline flow.

D. Gilbarg

No, there is no doubt about that.

T. Y. Wu

In Professor Gilbarg's lecture the re-entrant jet model seems to be preferred as the "only non-artificial model" for the irrotational cavity flows. Actually there are many other mathematical models, e.g., Riabouchinsky's image model and Roshko's dissipation model; and they are all, at least in my opinion, more or less artificial. I like to discuss these models by pointing out their physical basis and certain limitations, hoping that a better understanding of these models may help to clarify some unsolved problems.

In the real fluid flow past a bluff body with a separated flow region in the wake, experimental observations indicate that the discontinuous surfaces in the flow, or free streamlines, are actually thin shear layers, into which the vorticity is fed from the boundary layer in front of the separation point. The shear layers in general do not continue smoothly far downstream, but roll up to form vortices which in turn diffuse and eventually are dissipated in the wake. Consequently, when the hydrodynamic force exerted on the body is calculated by integrating the local pressure over the body, the flow may be assumed to be irrotational provided the dissipative wake can be represented by an equivalent, properly chosen, potential flow model. This argument, I believe, should provide the physical justification of any workable mathematical model for cavity flows. A model is satisfactory as long as it gives a good description of the flow near the body; and it is artificial in the sense that it fails to describe the wake far downstream.

Consider a two-dimensional flat plate set normal to a uniform stream of velocity U. The image model introduces an image plate, of the same size as the real plate, to be put at a certain distance downstream of the real plate, the pair of plates being joined by free streamlines. From the potential theory, the force exerted on the image is then equal and opposite to the drag D on the real plate. In a coordinate system where the fluid at infinity is at rest, the force holding the moving image plate does a negative work equal to $(-DU)$, which should be removed from the flow field if the resulting flow is assumed to be potential. An estimate of this amount of work corresponds approximately to the energy dissipated in the wake. Along the same line of reasoning, the jet momentum of the reentrant jet model carried away from the first mathematical sheet should be associated physically with the wake dissipation. For even though the jet can often be observed, it is quite weakened by the turbulent mixing; certainly its observed width is much smaller than its theoretical value $(0.22\ (1 + \sigma) \times$ plate width$)$. In the dissipation model the viscous dissipation is assumed to take place within a strip in the wake, aligned parallel to the free stream; the detailed mechanism as to how the flow is dissipated is rather immaterial. All these models have in their mathematical formulations one free parameter (such as the location of the image, the jet width, or the starting point of the dissipation zone), which can be expressed in terms of the cavitation number σ. Perhaps for this reason they yield about the same value of C_D. For small values of σ, the first two models give

$$C_D(\sigma) \approx \frac{2\pi}{\pi + 4}\left\{1 + \sigma + \frac{\sigma^2}{8(\pi + 4)}\right\} \tag{21}$$

whereas the coefficient of σ^2 is $[6(\pi + 4)]^{-1}$ for the dissipation model. Other than this one parameter, no freedom is left elsewhere to adjust the models. For example, there is no solution of the potential flow when the image plate is different in size from the real one.

Furthermore, the application of these models to more general cases, such as a cavitated lifting surface, must be carried out with care. For instance, the image of a lifting surface could be made symmetric with respect to either a central plane or only a certain point. To make sure that the formulation of any model is correct and compatible with the potential theory, I would believe that nothing else could serve as a better guide than the physical significance underlying the model itself. Compared with this point, the relative simplicity of the mathematical details involved in the analysis of each model is rather a minor matter. To name one example, the cavitated lifting surface with the cavity length equal approximately to the body length presents a difficult mathematical problem; apparently the difficulties could not be removed without a thorough understanding of the physical features of the flow near the rear of the cavity. Therefore it seems fruitful to devote some effort to the investigation of the physical basis of the cavity models; such an investigation would undoubtedly benefit future research in this field.

M. P. Tulin

I know that time prevented Dr. Gilbarg from discussing aspects of this rich subject which now have important implications for the design engineer, so permit me to supplement his fine presentation with a brief discussion of certain currently important cavity flow problems.

Flows involving lifting surfaces of one configuration or another are now especially of interest. For two-dimensional foils with sharp edges (which fix the cavity separation), adequate, although approximate, theory for the prediction of lift, drag, pitching moment, and cavity shape exist. In practice, however, supercavitating or ventilating foils will often be operating with time varying angles of attack, as wings of finite aspect ratio, and often in the proximity of free surfaces; implied is the possible use of supercavitating foils for propellers, for alighting gear on water based aircraft, and for sustention of high speed hydrofoil boats.

In an excellent earlier talk, Mr. John Parkinson of the NACA has given us some idea of trends in water based aircraft alighting gear development. We can surmize from his remarks that designers of water based aircraft in addition to hydrofoil boat designers would like to know how the loads on a ventilated (or supercavitating) foil fluctuate during operations in a seaway, and whether, even in smooth water, these foils might flutter; they would also like to be able to predict effects of finite span and free surface proximity. Ability to design high speed foils which won't suffer large changes in lift due to free surface effects is obviously essential.

Water tunnel research people, working to provide needed experimental information, would like to know how the presence of solid walls or a free jet influence the flow over a supercavitating foil, particularly for low cavitation numbers when the cavity becomes very large. The usefulness of their information depends upon adequate knowledge of interference effects.

The current problems indicated above together with those mentioned by Dr. Maccoll, Mr. Parkinson and Mr. Nicolaides in their earlier talks and discussions are few in number compared to those which designers will have to face in the future as speeds of ships, aircraft, and underwater ordnance vehicles increase. The demands of technology in the field of cavity flows deserve to be met and influenced by pertinent mathematical treatments.

I. Imai

I would like to point out one thing about the determination of the free-streamline separation point on the body. Professor Gilbarg told us that there are three possibilities for the free streamline curvature at the separation point, that is, negative (inward) curvature, positive (outward) infinite curvature, and finite curvature. He ruled out the negative curvature from the geometrical viewpoint and the positive infinite curvature by an intuitive argument to avoid "infinity", concluding that the finite curvature is the only possibility.

However, I think that the same conclusion could be drawn most naturally by considering the separation phenomenon on the basis of the boundary layer theory. If we assume that the streamline leaves the body at a certain point with infinite curvature, it can be proved by the inviscid potential flow theory that the pressure gradient there should be infinite. This in turn means on the boundary layer theory, that the separation should have occurred earlier somewhere upstream of that assumed separation point. This is a contradiction. Therefore, on the basis of the boundary layer theory we can determine the separation point as such a point that the free streamline leaves the body with finite curvature. It can also be shown by the inviscid flow theory that the condition of finite streamline curvature necessarily leads to the conclusion that the finite curvature of the free streamline should be equal to the curvature of the body at the separation point.