A GENERAL LINEARIZED THEORY FOR CAVITATING HYDROFOILS IN NONSTEADY FLOW

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1. INTRODUCTION

Since the exact theory of cavitating hydrofoils is rather complicated for fluid bodies, a linear theory will be developed which is assumed to be applicable to the bodies of small thickness ratio, such as propeller blades or thin hydrofoil, which prevail in most applications in naval architecture.

In particular the case of a nonuniform motion is considered here, which is important for vibrating motion as well as for the motion of a hydrofoil in a nonuniform field of flow.

The underlying assumptions can be stated as follows:

(a) The motion is two-dimensional, the hydrofoils move with velocity U in the direction of the negative x-axis.

(b) The fluid is considered incompressible and nonviscous as long as the pressure exceeds the vapor pressure p_v. If it reaches this value, the density abruptly falls off to zero, thus causing a cavity, on the surface of which the pressure has the value p_v, corresponding to vapor.

(c) The motion of the hydrofoil causes a small disturbance. Here both the thickness ratio of the hydrofoil as well as the amplitude of its nonsteady motion are of the same (small) order of magnitude.

(d) The motion of the fluid is irrotational outside the hydrofoil and its wake, which extends along the part of the x-axis which lies behind the hydrofoil.

It is further assumed that the disturbance created by the hydrofoil dies out at infinity, except eventually in the wake. This means that the pressure, which is continuous in the wake, must vanish at infinity.

Based on these assumptions the mathematical problem can now be formulated. The hydrofoil is assumed to extend along the segment $-\ell \leqq x \leqq +\ell$ of the x-axis.

Its motion is given by

$$\begin{cases} y = h^{+}(x,t) & -\ell \leqq x \leqq +\ell \qquad \text{upper side} \\ y = h^{-}(x,t)\,. & \qquad \text{lower side} \end{cases}$$

The cavity is assumed to extend only over a part of upper side of the hydrofoil, extending from $x = -\ell$ to $x = +c$. Along this part the pressure has the constant value p_v. On the remaining part of the hydrofoil contour the fluid wets the surface. Since viscosity is neglected, the boundary condition is simply that the total normal velocity of the fluid must be equal to the normal velocity of the hydrofoil.

The moving hydrofoil creates a disturbance field with velocity components (u, v). The linearized condition then gives on the wetted part of the hydrofoil

$$v = h_t + Uh_x = w(x,t), \tag{1.1}$$

where $w(x,t)$ is a known function of x and t. (Subscripts denote partial differentiation.)

In order to derive the boundary condition on the cavity in terms of the velocity, we remark that the flow is assumed to be irrotational outside the hydrofoil and the wake. Thence a velocity potential Φ exists which is related to the pressure by Bernoulli's law.

$$\Phi_t + \frac{1}{2}\left\{(U+u)^2 + v^2\right\} + \frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p_\infty}{\rho}\,, \tag{1.2}$$

which, in the linearized theory takes the form

$$\Phi_t + Uu = \frac{p_\infty - p}{\rho}\,. \tag{1.3}$$

The velocity potential satisfies in an incompressible fluid Laplace's equation

$$\Delta\Phi = 0 \tag{1.4}$$

and, since Eq. (1.3) is linear, we can introduce a pressure or acceleration potential φ.

$$\varphi = \frac{p_\infty - p}{\rho}\,, \tag{1.5}$$

equally satisfying

$$\Delta\varphi = 0\,. \tag{1.6}$$

Now the boundary value problem can be posed for the velocity potential Φ or for the acceleration potential φ. Since the latter is regular in the complete plane outside the hydrofoil (the pressure is continuous in the wake), it is somewhat easier in the first stage of the problem to formulate the problem for the acceleration potential. It is, however, to be remarked here, that for an unsteady motion the length of the cavitation bubble depends on the time. The acceleration potential φ is related to the velocity potential Φ by

$$\varphi = \Phi_t + U\,\Phi_x\,. \tag{1.7}$$

From the condition for the velocity $v = \Phi_y$ we find on the wetted part of the hydrofoil:

$$c < x < \ell, \quad y = +0,$$

$$-\ell < x < \ell, \quad y = -0,$$

$$\varphi_y = \Phi_{ty} + U\,\Phi_{xy} = w_t + U w_x. \tag{1.8}$$

On the cavity $-\ell < x < c$, $y = +0$, the pressure is the vapor pressure and hence

$$\varphi = \frac{p_\infty - p_v}{\rho} = \sigma, \tag{1.9}$$

where σ has a constant positive value.

Since φ must be a harmonic function, there are different methods available for the solution of this problem, e.g., Mushkelishvili's method of singular integral equations. In this paper a function of Green is introduced, which can be used to find a solution to this boundary value problem.

2. REGULAR SOLUTION TO THE BOUNDARY VALUE PROBLEM FOR THE PRESSURE POTENTIAL

Recapitulating the boundary value problem for the pressure potential:

$$\varphi = \sigma \qquad -\ell < x < c, \qquad y = +0,$$

$$\varphi_y = w(x,t) \qquad c < x < \ell, \quad -\ell < x < +\ell, \quad y = -0, \tag{2.1}$$

and

$$\Delta\varphi = 0.$$

We can solve this problem by the introduction of a Green's function

$$G_p\ (x,y;\ x_p,y_p).$$

This function is a solution of the nonhomogeneous equation

$$\Delta\, G_p = \delta(x_p,y_p) \tag{2.2}$$

(where $\delta(x_p,y_p)$ is Dirac's δ-function for the plane), which satisfies the boundary conditions

$$G_p = 0, \quad -\ell < x < c, \quad y = +0,$$

$$\frac{\partial G_p}{\partial y} = 0, \quad c < x < \ell, \quad y = +0,$$

$$-\ell < x < \ell, \quad y = -0. \tag{2.3}$$

G is regular at infinity.

Then, applying Green's theorem to the outer region of the hydrofoil, we obtain

$$\varphi_p = \frac{1}{2\pi}\int\left(\varphi\,\frac{\partial G_p}{\partial n} - G_p\,\frac{\partial\varphi}{\partial n}\right)ds$$

$$= \frac{1}{2\pi}\left\{\int_{-\ell}^{c}\left\{\sigma\,\frac{\partial G_p}{\partial y}\right\}dx + \int_{c}^{\ell}\left(\frac{\partial\varphi}{\partial n}\right)^{+} G_p\,dx + \int_{+\ell}^{-\ell}\left(\frac{\partial\varphi}{\partial n}\right)^{-} G_p\,dx\right\}. \tag{2.4}$$

If we assume that both G, $\partial G/\partial n$, φ and $\partial\varphi/\partial n$ vanish at infinity of sufficient high order, the point at infinity gives no contribution. Hence, once the Green's function is known, the solution is obtained. This expression for the pressure potential, however, is not unique since, at the singular points $-\ell$, $+\ell$, and $+c$ the boundary conditions are not specified. If we put the point P of the Green's function in one of these points, we obtain a solution of the homogeneous equation, which is regular outside the hydrofoil and does not alter the boundary conditions. Thus we may assume for the total solution

$$\varphi_p = \varphi_{reg} + A_1 G_1 + A_2 G_2 + A_3 G_3, \tag{2.5}$$

where A_1, A_2, and A_3 are indeterminate constants which must be determined by additional conditions. Since the differential equation is Laplace's equation, Green's function can easily be found by conformal mapping (Fig. 1). At first the physical plane $z = x+iy$ is mapped on a $Z = X+iY$ plane.

$$Z = \sqrt{\frac{\ell - z}{\ell + z}}\,. \tag{2.6}$$

Then the hydrofoil $-\ell < x < +\ell$ passes into the X-axis, and the outer region passes into the lower half plane. The outer region passes into the lower half plane $Y < 0$ and point $z = \infty$ into $Z = i$. On the hydrofoil we write $z = x = -\ell\cos\theta$, the interval $0 < \theta < \pi$ corresponding to the upper side, and $\pi < \theta < 2\pi$ to the lower side. If $c = -\ell\cos\gamma$, this point passes into $c = \cot\gamma/2$. For $x > c$ the boundary condition is $G = 0$, for $x < c$ it is $\partial G/\partial n = 0$. By a second mapping

$$\zeta = \xi + i\eta = \sqrt{Z - \cot\frac{1}{2}\gamma}\,. \tag{2.7}$$

The Z-plane is mapped into a ζ-plane. Where the outer region now occupies the fourth quadrant, c passes into the origin and B into the point $\eta = -\sqrt{\cot 1/2\,\gamma}$. Green's function is constructed by a reflection of the pole P with respect to the axis $\xi > 0$, $\eta = 0$ and $\eta < 0$, $\xi = 0$. Since on the first axis the boundary condition is $G = 0$, we must add a pole with negative sign in $\bar{P} = \xi_p - i\eta_p$ and since on the second the condition is $\partial G/\partial\xi = 0$ we add here a pole with positive sign. This gives the resulting complex potential

$$G_p + i\,H_p = \ell n\,\frac{(\zeta-\zeta_p)(\zeta+\bar{\zeta}_p)}{(\zeta-\bar{\zeta}_p)(\zeta+\zeta_p)}\,. \tag{2.8}$$

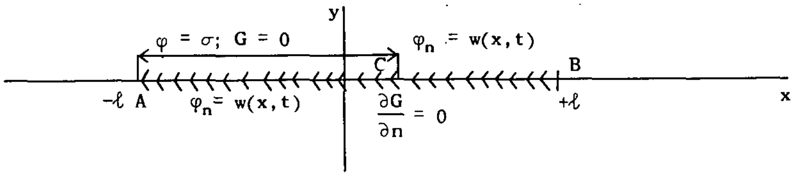


Figure 1

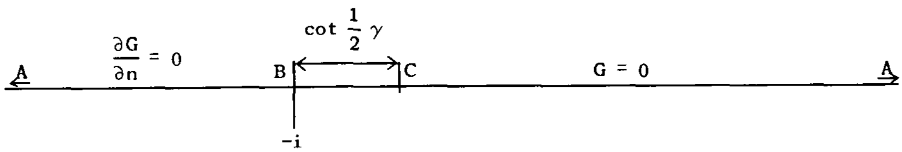


Figure 2

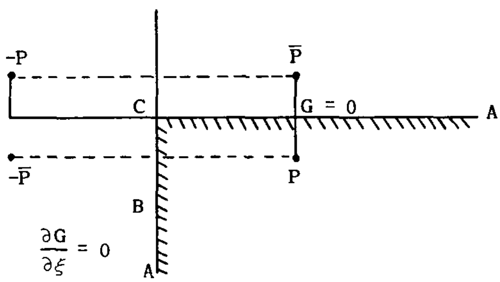


Figure 3

In the formula for the regular solution we need only the value for Green's function on the hydrofoil contour

$$x = -\ell \cos \theta, \qquad x_p = -\ell \cos \theta_p,$$

$$Z = \cot \frac{1}{2}\theta, \qquad Z_p = \cot \frac{1}{2}\theta_p,$$

$$\zeta = \sqrt{\cot \frac{\theta}{2} - \cot \frac{\gamma}{2}}, \qquad \zeta_p = \sqrt{\cot \frac{\theta_p}{2} - \cot \frac{\gamma}{2}}.$$

The contribution of the first part in the integral in Eq. (2.4) can be evaluated easily by the use of the Cauchy-Riemann equations

$$\frac{\partial G_p}{\partial y} = -\frac{\partial H_p}{\partial x} \tag{2.9}$$

since G_p and H_p are conjugate harmonic functions. This gives

$$\frac{1}{2\pi}\int_{-\ell}^{c} \sigma \cdot \frac{\partial G_p}{\partial y} \cdot dx = -\frac{\sigma}{2\pi}\int_{-\ell}^{c} \frac{\partial H_p}{\partial x}\, dx = \frac{\sigma}{2\pi}\left[H(-\ell)-H(c)\right] = \sigma. \tag{2.10}$$

Hence, we obtain the resulting regular solution

$$\varphi_{p\ reg} = \sigma + \ell \int_{\gamma}^{2\pi} (w_t + U w_x) \cdot G_p \cdot \sin\theta\, d\theta. \tag{2.11}$$

The additional singular solutions can be found from the expressions for G_p in the neighborhood of the singular points C $\zeta = 0$, A $\zeta = \infty$, B $= \zeta = i\sqrt{\cot \gamma/2}$. Near $\zeta = 0$, we put $\zeta = i\varepsilon$, i.e., we let P $\to$ A along the upper side of the contour.

Then

$$G_p + iH_p = \ell n \frac{(i\varepsilon-\zeta_p)(i\varepsilon+\overline{\zeta}_p)}{(i\varepsilon-\overline{\zeta}_p)(i\varepsilon+\zeta_p)} = \ell n \frac{(1-i\varepsilon/\zeta_p)(1+i\varepsilon/\overline{\zeta}_p)}{(1-i\varepsilon/\overline{\zeta}_p)(1+i\varepsilon/\zeta_p)}$$

$$= 2i\varepsilon\left(\frac{1}{\overline{\zeta}_p} - \frac{1}{\zeta_p}\right) - \frac{2i\varepsilon^3}{3}\left(\frac{1}{\zeta_p^3} - \frac{1}{\overline{\zeta}_p^3}\right) + \cdots . \tag{2.12}$$

This gives an infinite number of singular solutions of increasing order, each of which has a zero value on the cavity and a zero normal derivative on the wetted part of the hydrofoil. The higher singularities must be excluded by another physical condition, viz., the condition that the pressure must be integrable over the hydrofoil. On the segment CB we have $\zeta_p = -i\eta_p$ and

$$\frac{dx}{d\eta} = i\frac{dz}{dZ} \cdot \frac{dZ}{d\zeta} = -i \cdot \frac{\delta\, \ell\, \eta_p\left(\cot\frac{1}{2}\gamma - \eta_p^2\right)}{\left\{1+\left(\cot\frac{1}{2}\gamma-\eta_p^2\right)^2\right\}^2} .$$

Further, the first of the potentials is

$$\varphi_1 = \frac{2}{\eta_p},$$

and the integral

$$\int_{-\ell}^{+\ell} \varphi_1\, dx = \int_0^{+i\infty} \varphi_1 \frac{\delta\, \ell\eta_p\left(\cot\frac{1}{2}\gamma-\eta_p^2\right)}{\left\{1+\left(\cot\frac{1}{2}\gamma-\eta_p^2\right)^2\right\}^2}\, d\eta_p$$

will exist, while for the higher solutions it becomes divergent. Therefore the singular solution corresponding to the point C is

$$\varphi_1 = \frac{i}{2}\left(\frac{1}{\zeta_p} - \frac{1}{\overline{\zeta}_p}\right) = \frac{\eta_p}{\xi_p^2 + \eta_p^2} . \tag{2.13}$$

In order to derive the second singular solution we put $\eta_p = -\, i/\varepsilon$.

$$G_p + iH_p = \ln \frac{(-i/\varepsilon-\zeta_p)(-i/\varepsilon+\overline{\zeta}_p)}{(-i/\varepsilon-\overline{\zeta}_p)(-i/\varepsilon+\zeta_p)} \approx -\, 2i\varepsilon\left[\zeta_p-\overline{\zeta}_p\right] - \frac{2i\varepsilon^3}{3}\left[\zeta_p^3-\overline{\zeta}_p^3\right] + \cdots .$$

Here the integrability condition does not give a unique singular solution, since both η_p and η_p^3 give an integrable expression. It is shown by Wu, however, that only the first term will give a pressure field, which is higher than the vapor pressure. Hence the second singular solution is

$$\varphi_2 = \frac{(\zeta_p-\overline{\zeta}_p)i}{2} = \eta_p, \tag{2.14}$$

and the third is

$$\varphi_3 = \frac{\zeta_p^3 - \overline{\zeta}_p^3}{2} = \eta[3\ \xi^2-\eta^2].$$

This singular solution, corresponding to the trailing edge B will give a pressure which is infinite at this point. This is excluded by the Kutta condition which requires the pressure to be finite at this point. Thus we are left with the following expression for the pressure potential:

$$\varphi_p = \sigma + \ell \int_\gamma^{2\pi} \{w_t+Uw_x\}\, G_p \cdot \sin\theta\, d\theta$$

$$+ A_1\, i\left(\frac{1}{\zeta_p} - \frac{1}{\overline{\zeta}_p}\right) + A_2\, i\, (\zeta_p-\overline{\zeta}_p). \tag{2.15}$$

We further remark that the disturbance pressure must vanish at infinity. Since in the ζ-plane the point at infinity corresponds to

$$\zeta_p = \sqrt{-i - \cot\frac{1}{2}\gamma} = \frac{e^{i(\gamma/4-\pi/2)}}{\sqrt{\cos\frac{1}{2}\gamma}} = -\frac{i\, e^{i\gamma/4}}{\sqrt{\cos\frac{1}{2}\gamma}},$$

we obtain the condition

$$\sigma + \ell\int_\gamma^{2\pi} (w_t+U\,w_x)G_\infty \sin\theta\, d\theta + A_1\, 2\cos(\gamma/4)\sqrt{\cos\frac{1}{2}\gamma} + A_2\frac{\cos(\gamma/4)}{\sqrt{\cos(\gamma/2)}} = 0. \tag{2.16}$$

Near the trailing edge B we put $\zeta = -i\sqrt{\cot 1/2\,\gamma} - i\delta$ which gives

$$G_p + iH_p = \ln \frac{(-i\sqrt{\cot \gamma/2}-i\delta-\zeta_p)(-i\sqrt{\cot \gamma/2}-i\delta+\overline{\zeta}_p)}{(-i\sqrt{\cot \gamma/2}-i\delta-\overline{\zeta}_p)(-i\sqrt{\cot \gamma/2}-i\delta+\zeta_p)} =$$

$$= \ln(\text{---}) + i\delta\left[\frac{1}{-i\sqrt{\cot \gamma/2}-\zeta_p} + \frac{1}{-i\sqrt{\cot \gamma/2}+\overline{\zeta}_p} - \right.$$

$$\left. - \frac{1}{-i\sqrt{\cot \gamma/2}-\overline{\zeta}_p} - \frac{1}{-i\sqrt{\cot \gamma/2}+\zeta_p}\right] =$$

$$= \ln(\text{- -}) + i\delta\left[\frac{-2\zeta_p}{\cot \gamma/2+\zeta_p^2} + \frac{+2\overline{\zeta}_p}{+\cot \gamma/2+\overline{\zeta}_p^2}\right].$$

The first singular solution is

$$\varphi_4 = i\left\{\frac{\zeta_p}{Z_p} - \frac{\overline{\zeta}_p}{\overline{Z}_p}\right\} = i\left[\frac{\xi_p + i\eta_p}{(C+\xi^2-\eta^2) + 2i\xi\eta} - \frac{\xi - i\eta}{(C+\xi^2-\eta^2) - 2i\xi\eta}\right]$$

$$= i\,\frac{-4i\,\xi^2\eta + 2i\eta(C+\xi^2-\eta^2)}{(C+\xi_p^2-\eta_p^2)^2 + 4\xi_p^2\,\eta_p^2} = \frac{4\xi^2\eta - 2\eta(C+\xi^2-\eta^2)}{C^2 + 2C(\xi^2-\eta^2) + (\xi^2+\eta^2)^2}$$

$$= \frac{-2C\eta + 2\eta(\xi^2+\eta^2)}{C^2 + 2C(\xi^2-\eta^2) + (\xi^2+\eta^2)^2}.$$

3. DETERMINATION OF THE COMPLETE SOLUTION

For the determination of the unknown functions of the time $A_1(t)$, $A_2(t)$, and $\gamma(t)$, two additional conditions are necessary. These are found by first remarking that up to now the problem has only been solved for the acceleration potential, whereby only the differentiated form of the boundary condition (1.8) is used. In the original problem, however, not only the normal value of the acceleration, but also the velocity is given which contains more information. In fact, once more it can be seen that any solution, satisfying the velocity condition (1.1) satisfies also (1.8), but the singular solutions, which do not affect (1.8) must be added in such a way that (1.1) is satisfied. Another condition is that the cavitation bubble must be closed.

In order to express these two conditions in a mathematical formulation, it is necessary to derive an expression for the velocity potential Φ in terms of the acceleration potential φ. This can be found by solution of the differential equation

$$\Phi_t + U\,\Phi_x = \varphi. \tag{1.7}$$

Considering that the velocity potential must vanish at infinity upstream to the hydrofoil, this expression is

$$\Phi(x,y,t) = \frac{1}{U}\int_{-\infty}^{t} \varphi\left(x',y,t - \frac{x-x'}{U}\right)dx' = \int_{-\infty}^{t} \varphi(x-U(t-t'),y,t')dt'. \tag{3.1}$$

Substitution of (2.15) and (2.16) gives

$$\Phi(x,y,t) = \frac{1}{U}\int_{-\infty}^{x} dx' \cdot \int_{\gamma}^{2\pi} w_t\left\{-\ell \cos\theta, t - \frac{x-x'}{U}\right\} \sin\theta(G_p - G_\infty)d\theta$$

$$+ U\, w_t\left\{-\ell\cos\theta, t - \frac{x-x'}{U}\right\} +$$

$$+ \int_{-\infty}^{x} A_1\left(t - \frac{x-x'}{U}\right) \cdot \left\{i\left(\frac{1}{\zeta_p} - \frac{1}{\bar{\zeta}_p}\right) - 2\cos(\gamma/4)\sqrt{\cos\frac{1}{2}\gamma}\right\}, dx' +$$

$$+ \int_{-\infty}^{x} A_2\left(t - \frac{x-x'}{U}\right)\left\{i(\zeta_p - \bar{\zeta}_p) - \frac{\cos\gamma/4}{\sqrt{\cos(\gamma/4)}}\right\}dx'. \tag{3.2}$$

After complicated calculations, which are given in the Appendix, the expression for the velocity on the hydrofoil takes the form given in Eq. (A10) of the Appendix. Applying this result to a point of the wetted surface, we derive

$$0 = -\frac{1}{4}U\int_{-\infty}^{x} dx' \cdot \int_{\gamma}^{2\pi} w\left(\theta, t - \frac{x-x'}{U}\right)\left[3C\varphi_1 \cdot \varphi_{1y}^p + (\varphi_1\varphi_{3y}^p + \varphi_3\varphi_{1y}^p) - \frac{(1+C^2)^2}{C}\varphi_2\varphi_{2y}^p\right.$$

$$\left. + \frac{1}{C}\varphi_4\varphi_{4y}^p\right]\sin\theta\, d\theta - \int_{-\infty}^{x} A_1 \cdot \left(t - \frac{x-x'}{U}\right) \cdot \frac{\partial\varphi_1}{\partial y}dx' - \int_{-\infty}^{x} A_2\left(t - \frac{x-x'}{U}\right)\frac{\partial\varphi_2}{\partial y}dx'$$

$$- U\int_{-\infty}^{x} dx' \cdot \int_{\gamma}^{2\pi} w\left(\theta, t - \frac{x-x'}{U}\right)\varphi_2\frac{\partial\varphi_2^p}{\partial y} \cdot \frac{1}{\sin(2\gamma/2)} \cdot \frac{\partial\gamma}{\partial t}\sin\theta\, d\theta. \tag{3.3}$$

Since the first term gives the solution to the boundary value problem where the velocity on the wetted part has the given value w, this first term cancels against the left-hand member. Considering now the integrals with respect to x', they again can be simplified, remarking that the singular potentials φ_i are all imaginary parts of complex functions. Applying the Cauchy-Riemann equations,

$$\frac{\partial\varphi_i}{\partial y} = \frac{\partial\psi_i}{\partial x}$$

where

$$\psi_1 = \mathrm{Re}(1/\zeta) = \frac{\xi}{\xi^2 + \eta^2},$$

$$\psi_2 = \mathrm{Re}\ \zeta = \xi,$$

$$\psi_3 = \mathrm{Re}\ \zeta^3 = \xi(\xi^2 - 3\eta^2),$$

$$\psi_4 = \mathrm{Re}(\zeta/Z) = \frac{\xi(C+\xi^2+\eta^2)}{(C+\xi^2-\eta^2)^2 + 4\xi^2\eta^2}.$$

We can reduce (3.3) by partial integration, remarking that the wetted surface corresponds to the line $\xi = 0$ in the ζ plane. Introducing

$$\int_\gamma^{2\pi} w_t\left(\theta, t - \frac{x-x'}{U}\right) \cdot \varphi_i \cdot (\theta) \cdot \sin\theta\, d\theta = w_i(t, x'),$$

we obtain

$$0 = \frac{1}{4}\int_{-\infty}^{-\ell} \left\{3C\cdot\psi_1\cdot w_1 + \psi_3 w_1 + \psi_1 w_3 - \frac{(1+C^2)^2}{C}\psi_2\, w_2 + \frac{1}{C}\psi_4\, w_4\right\} dx'$$

$$+ \int_{-\infty}^{-\ell} A_{1t} \cdot \psi_1\, dx' + \int_{-\infty}^{-\ell} A_{2t} \cdot \psi_2\, dx' - \int_{-\infty}^{x} dx' \cdot \cdot \cdot \cdot .$$

It should be noted that $\psi_i = 0$ for $\xi = 0$. Therefore the only part of the integration which contributes to the result is the part of the x-axis extending from $-\infty$ to $-\ell$. The integration along the lower part of the hydrofoil, which corresponds to the η-axis in the ζ-plane, gives a vanishing contribution. A point on the wetted upper surface between B and C is reached from the trailing edge. Taking the contour along the upper edge of the hydrofoil, i.e., along the real axis in the $\zeta = \xi + i\eta$ plane, we get an additional contribution from the cavitation bubble. Hence, it is obvious that behind this bubble extends a line of discontinuity in v.

The last equation for A_1 and A_2 is furnished by the closure condition. If the contour of the hydrofoil and the cavitation bubble is represented by

$$y = h(x, t),$$

where h is only given on the wetted part. The normal velocity is

$$v = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}.$$

In order to derive the contour of the bubble, we consider this as a partial differential equation

$$h = h_{-\ell} = \frac{1}{U}\int_{-\ell}^{x} v\left(x', t - \frac{x-x'}{U}\right) dx'.$$

Then the closure condition is

$$h_c - h_{-\ell} = \frac{1}{U}\int_{-\ell}^{c} v\left(x', t - \frac{c-x'}{U}\right) dx'.$$

The velocity v satisfies a similar partial differential equation

$$a = v_t + Uv_x,$$

where a is the normal acceleration. This gives in a similar way

$$v(x,t) - v(-\ell,t) = \frac{1}{U}\int_{-\ell}^{x} v\left(x', t - \frac{x-x'}{U}\right) dx'.$$

Here $v(-\ell)$ is the given normal velocity, corresponding to the motion of the leading edge. Substitution gives

$$h_c - h_{-\ell} = \frac{1}{U}\int_{-\ell}^{c} v\left(-\ell, t - \frac{c-x'}{U}\right) dx' + \frac{1}{U^2}\int_{-\ell}^{c} dx \int_{-\ell}^{x} a\left(x', t - \frac{c-x}{U} - \frac{x-x'}{U}\right) dx' =$$

$$= \frac{1}{U}\int_{-\ell}^{c} v\left(-\ell, t - \frac{c-x}{U}\right) dx + \frac{1}{U^2}\int_{-\ell}^{c} dx \int_{-\ell}^{x} a\left(x', t - \frac{c-x'}{U}\right) dx' =$$

$$= \frac{1}{U}\int_{-\ell}^{c} v\left(-\ell, t - \frac{c-x}{U}\right) dx + \frac{1}{U^2}\left[x \int_{-\ell}^{x} a\left(x', t - \frac{c-x'}{U}\right) dx'\right]_{-\ell}^{c} -$$

$$- \frac{1}{U^2}\int_{-\ell}^{c} x \; dx \; a\left(x, t - \frac{c-x}{U}\right) dx =$$

$$= \frac{1}{U}\int_{-\ell}^{c} v\left(-\ell, t - \frac{c-x}{U}\right) dx + \frac{1}{U^2}\left[c\int_{-\ell}^{c} a\left(x', t - \frac{c-x'}{U}\right) dx' - \int_{-\ell}^{c} x \; a\left(x, t - \frac{c-x}{U}\right) dx\right] =$$

$$= \frac{1}{U}\int_{-\ell}^{c} v\left(-\ell, t - \frac{c-x}{U}\right) dx + \frac{1}{U^2}\int_{-\ell}^{c} (c-x) a\left(x, t - \frac{c-x}{U}\right) dx = h_c - h_{-\ell}.$$

Substitution gives the third relation:

$$\frac{h}{U^2}\int_{-\ell}^{c}(\ell-x)\cdot\int_{\gamma}^{2\pi}\{w_t+Uw_x\}\cdot\frac{\partial G}{\partial y_p}\cdot\sin\theta\,d\theta\cdot dx+\frac{1}{U^2}\int_{-\ell}^{c}(c-x)A_1\left(t-\frac{c-x}{U}\right)\varphi_1(x,t)\,dx+$$

$$+\frac{1}{U^2}\int_{-\ell}^{c}(c-x)A_2\left(t-\frac{c-x}{U}\right)\varphi_2\left(x,t-\frac{c-x}{U}\right)dx=h_c-h_{-\ell}-\frac{1}{U}\int_{-\ell}^{c}v\left(-\ell,t-\frac{c-x}{U}\right)dx.$$

4. SOLUTIONS FOR SPECIAL CASES

Owing to the time-dependence of γ on t, the equations for A_1, A_2, and γ are difficult to solve. For this reason only special cases can be considered.

The Steady Case

As a check on this theory we consider the steady case. The pressure potential is

$$\varphi_p=\sigma+\ell\int_{\gamma}^{2\pi}(Uw_x)G_p\cdot\sin\theta\cdot d\theta+A_1\,i\left(\frac{1}{\zeta_p}-\frac{1}{\bar{\zeta}_p}\right)+A_2\,i(\zeta_p-\bar{\zeta}_p),$$

where A_1 and A_2 are constants. The first condition now reads:

$$\sigma+\ell\,U\int_{\gamma}^{2\pi}w_x\cdot G_\infty\sin\theta\cdot d\theta+A_1\,2\cos(\gamma/4)\sqrt{\cos(\gamma/2)}+A_2\cdot\frac{\cos(\gamma/4)}{\sqrt{\cos(\gamma/2)}}=0.$$

In the second condition all the quantities w_i vanish and we obtain

$$0=A_1\cdot\int_{-\infty}^{-\ell}\psi_1\cdot dx'+A_2\cdot\int_{-\infty}^{-\ell}\psi_2\cdot dx'.$$

The third condition gives:

$$\frac{1}{U}\int_{-\ell}^{c}(\ell-x)\int_{\gamma}^{2\pi}w_x\cdot\frac{\partial G}{\partial y_p}\cdot\sin\theta\,d\theta+\frac{A_1}{U^2}\int_{-\ell}^{c}(c-x)\,\varphi_1(x)\,dx$$

$$+\frac{A_2}{U^2}\int_{-\ell}^{c}(c-x)\cdot\varphi_2(x)\,dx=h_c-h_{-\ell}-\left(\frac{c+\ell}{U}\right)\cdot v(-\ell).$$

For a flat plate the formulas simplify considerably. The pressure potential is

$$\varphi_p = \sigma + A_1\, i\left(\frac{1}{\zeta_p} - \frac{1}{\bar{\zeta}_p}\right) + A_2\, i\, (\zeta_p - \bar{\zeta}_p),$$

and the three conditions are

$$\sigma + A_1\, 2\cos(\gamma/4) \cdot \sqrt{\cos(\gamma/2)} + A_2 \cdot \frac{\cos(\gamma/4)}{\sqrt{\cos(\gamma/4)}} = 0,$$

$$0 = A_1 \cdot \int_{-\infty}^{-\ell} \psi_1 \cdot dx + A_2 \cdot \int_{-\infty}^{-\ell} \psi_2 \cdot dx,$$

$$A_1 \int_{-\ell}^{c} (c-x)\, \varphi_1(x)\, dx + A_2 \int_{-\ell}^{c} (c-x)\, \varphi_2(x)\, dx = U^2\left\{h_c - h_{-\ell} - \frac{c+\ell}{U} v(-\ell)\right\}.$$

Harmonic Oscillations About a Certain Steady Motion

We consider further the case that the hydrofoil performs harmonic oscillations about a certain steady motion. Owing to the dependence of the formula on the parameter γ we assume that the amplitude is so small, that the time dependence of γ is also small.

$$\gamma = \gamma_0 + \gamma_1^* \cdot e^{i\nu t}.$$

Then we introduce a "second linearization," expanding all quantities, containing γ into powers of γ_1. This gives rise to a set of equations for the determination of the amplitudes A_1, A_2, and γ_1. The pressure potential takes the form, (if $w = \bar{w}^* e^{i\nu t} + w_0$)

$$\varphi_p = \sigma + \ell\, U \int_{\gamma_0}^{2\pi} w_x \cdot G_p \sin\theta\, d\theta + \ell \int_{\gamma_0}^{2\pi} (\nu\, \bar{w}^* + U w_x^*) G_p \sin\theta\, d\theta +$$

$$+ \ell \int_{\gamma_0}^{2\pi} w_x \cdot \frac{\partial G_p}{\partial \gamma} \cdot \sin\theta\, d\theta\, \gamma_0^*\, e^{i\nu t} +$$

$$+ A_1 \cdot \frac{\partial \varphi_2}{\partial \gamma} \cdot \gamma_0\, e^{i\nu t} + A_2^*\, e^{i\nu t} \cdot \varphi_2 + A_1 \frac{\partial \varphi_2}{\partial \gamma} \cdot \gamma_0\, e^{i\nu t}.$$

The conditions for A_1, A_2, and γ_1 are derived in a similar way. This gives three nonhomogeneous linear equations for the three amplitudes. It would be of interest to study the case, where the determinant of this system vanishes. This would mean that for the corresponding steady-state, spontaneous oscillations of the cavity can occur.

4. CONCLUSIONS

The general theory, laid down in this paper, is based on rather crude assumptions on the behavior of cavitating flow. Considering the underlying hypothesis, the assumption of potential flow is not sufficient to represent details of formation and decay of cavities. On the other hand, it is well known that for steady-flow, results from potential flow agree reasonably well with experiments. The further simplification, introduced by the linearization, seems, in the case of a free hydrofoil, sufficiently accurate for thin hydrofoils at small incidence. In a channel, however, this simplication might be the cause of considerable error, owing to blockage effects. This phenomenon is analogous to the behavior of linearized theory for transonic flow in wind tunnels, where the nonlinear approximation is necessary.

As regards the special additional assumptions, introduced to describe the nonsteady behavior, the condition is that in the cavitation bubble evaporation occurs instantly, as soon as the pressure has reached the cavitation pressure, which is constant. However, every other hypothesis which must be based on more detailed physical considerations will greatly involve the calculations. As regards the introduction of free sources and sinks, somewhat more explanation is necessary.*

The concept is completely analogous to the concept of free vortices in nonsteady airfoil theory, which already goes back to Birnbaum (1923).

Calculation of forces, based on this assumption, yields reasonably good agreement with experiments. In linearized theory the circulation around the airfoil is defined by

$$\int \underline{v} \cdot ds = \int U \cdot dx,$$

where u is related to the pressure potential by

$$\varphi = \Phi_t + U\,\Phi_x \quad \text{or} \quad \varphi_x = u_t + U\,u_x,$$

which means that

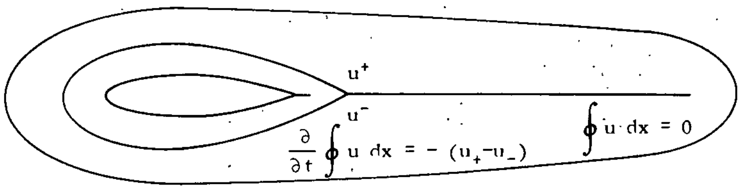
$$\oint \varphi_x \, dx = \frac{\partial}{\partial t} \oint u \, dx + U \oint u_x \, dx + U \oint \Phi_x \, dx.$$

If we suppose that φ is a regular function outside the airfoil, including infinity, $\oint \varphi_x \, dx = 0$ for any contour enclosing the airfoil, and we find that

$$\frac{\partial}{\partial t} \oint u \, dx = \oint u_x \, dx = -(u_+ - u_-).$$

If in the wake the velocity u is discontinuous, we find that the time rate of the circulation around the airfoil is equal to the strength of the discontinuity at the trailing edge, which is the vorticity in the wake. For a motion started at a definite moment

*This explanation was found in a discussion between the author and Mr. J. A. Geurst.



in the past, the wake is finite and the total circulation around a contour enclosing the airfoil and the wake vanishes. A similar consideration can be applied to the integral

$$V = \int h\ dx$$

if the contour is given by $y = h(x)$, which represents the total volume enclosed by the contour. Since

$$v = h_t + U\, h_x,$$

we have, for a closed contour,

$$\oint v\ dx = \frac{\partial}{\partial t} \int h\ dx,$$

which expresses that the time rate of change of the volume by the contour is equal to the contour integral of the function v.

For incompressible flow we have simply

$$\oint v\ dx = -\Psi_+ + \Psi_-,$$

where Ψ is the stream function and $v = -\Psi_x$ for incompressible flow. If Ψ is regular outside the hydrofoil, we have $\Psi_+ = \Psi_-$ and the volume enclosed by the contour is a constant, as was posed by Wu.

We have seen previously that from our calculation it follows that a sheet of discontinuity in v extends behind the cavitation bubble. This gives rise to a sheet of discontinuity in the Ψ. Hence, considering a contour which lies along the bubble and the wetted part of the hydrofoil, there is a discontinuity at the end of the bubble. The strength of the discontinuity is given by the change in volume of the bubble. Since, in linearized theory the pressure is a continuous function outside the hydrofoil, the discontinuity $\Delta\Psi$ in the wake satisfies

$$(\Delta\Psi)_t + U(\Delta\Psi)_x = 0,$$

which means that this discontinuity is propagated with the velocity U.

Since a discontinuity in v corresponds to a source distribution, the singularities are denoted as "free sources" in analogy to the free vortices considered before. The theory opens a possibility of an explanation of the oscillations, which are observed for a steady cavity of finite extent. The entrainment process at the end of the cavity during which small volumes of the gas phase are carried downstream with the fluid, as the cavity volume decreases, might correspond to the creation of the free sources.

Only quantitative calculations of the occurring frequencies and length of the steady state bubble would confirm whether or not even this potentially theoretical approach can account for this complicated phenomenon.

NOTE ADDED IN PROOF: From discussion of the author with Messrs. Geurst, Tulin, and Eisenberg (October 1959) a slightly different explanation of the "free sources" concept arose. In this concept they are localized in the reentrant jet. Hence, for a closed contour, including the stagnation point behind the reentrant jet, the volume is constant. The volume enclosed by the free surface varies and is compensated by "free sources" on the second blade of the Riemann surface. This theory gives consistent conditions and is worked out by Mr. Geurst. (To be published in Archiv for Rational Mechanics.)

* * * * *

APPENDIX

REDUCTION OF THE FORMULA FOR THE VERTICAL VELOCITY ON THE HYDROFOIL

The formula for the velocity component v is:

$$v(x,0,t) = \frac{1}{U}\int_{-\infty}^{x} dx' \int_{\gamma\left(t - \frac{x-x'}{U}\right)}^{2\pi} w_t \left\{-\ell \cos\theta,\ t - \frac{x-x'}{U}\right\} \frac{\partial G}{\partial y}\left\{x', -\ell\cos\theta;\ t - \frac{x-x'}{U}\right\}$$

$$\sin\theta\, d\theta + \ell \int_{-\infty}^{x} dx' \cdot \int_{\gamma\left(t - \frac{x-x'}{U}\right)}^{2\pi} w_\theta\left(-\ell\cos\theta,\ t - \frac{x-x'}{U}\right)\cdot \frac{\partial G}{\partial y}\left(x', -\ell\cos\theta;\ t - \frac{x-x'}{U}\right) d\theta$$

$$+ \int_{-\infty}^{x} A_1\left(t - \frac{x-x'}{U}\right)\frac{\partial \varphi_1}{\partial y}\left(t - \frac{x-x'}{U}\right) dx' + \int_{-\infty}^{x} A_2\left(t - \frac{x-x'}{U}\right)\cdot \frac{\partial \varphi_2}{\partial y}\cdot dx'. \qquad \text{(A1)}$$

We can reduce the first two terms by partial integration

$$\int_{-\infty}^{x} dx' \int_{\gamma}^{2\pi} w_t\left(\theta, t - \frac{x-x'}{U}\right)\frac{\partial G}{\partial y} \sin\theta \, d\theta = U\int_{-\infty}^{x} dx' \int_{\gamma}^{2\pi} w_{x'}\left(\theta, t - \frac{x-x'}{U}\right)\cdot \frac{\partial G(\gamma)}{\partial y} \sin\theta \, d\theta$$

$$= U\int_{\gamma}^{2\pi} w(\theta, t)\frac{\partial G}{\partial y}\cdot \sin\theta \, d\theta - U\int_{-\infty}^{x} dx' \int_{\gamma}^{2\pi} w\left(\theta, t - \frac{x-x'}{U}\right)\cdot \frac{\partial^2 G}{\partial y \partial x'} \sin\theta \, d\theta$$

$$- \int_{-\infty}^{x} dx' \frac{\partial \gamma}{\partial t}\cdot w\left(\gamma, t - \frac{x-x'}{U}\right)\frac{\partial G}{\partial y} \gamma - U\int_{-\infty}^{x} dx' \int_{\gamma}^{2\pi} w\left(\theta, t - \frac{x-x'}{U}\right)\frac{\partial^2 G}{\partial y \partial \gamma}\frac{\partial \gamma}{\partial t} \sin\theta \, d\theta$$

and

$$\int_{-\infty}^{x} dx' \int_{\gamma}^{2\pi} w_\theta\left(\theta, t - \frac{x-x'}{U}\right)\frac{\partial G}{\partial y} d\theta = \int_{-\infty}^{x} dx' \left[w\left(\theta, t - \frac{x-x'}{U}\right)\frac{\partial G}{\partial y}\right]_{\gamma}^{2\pi}$$

$$- \int_{-\infty}^{x} dx' \int_{\gamma}^{2\pi} w\left(\theta, t - \frac{x-x'}{U}\right)\frac{\partial^2 G}{\partial y \partial \theta}\cdot d\theta.$$

Apparently we have to calculate the expression

$$\frac{\partial^2 G}{\partial y \partial x'} \sin\theta + \frac{\partial^2 G}{\partial y \partial \theta} = \sin\theta \frac{\partial}{\partial y}\left[\frac{\partial^2 G}{\partial x} + \frac{\partial G}{\partial x'}\right]. \tag{A2}$$

From the definition

$$G + iH = \ln \frac{(\zeta-\zeta')(\zeta+\overline{\zeta'})}{(\zeta-\overline{\zeta'})(\zeta+\zeta')} \tag{A3}$$

we see that

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial x'} = \mathrm{Re}\left\{\frac{d}{dz}(G+iH) + \frac{\partial}{\partial z'}(G+iH) + \frac{\partial}{\partial \overline{z'}}(G+iH)\right\}$$

$$= \mathrm{Re}\left\{\frac{\partial(G+iH)}{\partial \zeta}\cdot\frac{d\zeta}{dz} + \frac{\partial(G+iH)}{\partial \zeta'}\cdot\frac{d\zeta'}{dz'} + \frac{\partial(G+iH)}{\partial \overline{\zeta'}}\cdot\frac{d\overline{\zeta'}}{d\overline{z'}}\right\}$$

$$= \mathrm{Re}\left\{\left(\frac{1}{\zeta-\zeta'} - \frac{1}{\zeta+\zeta'} + \frac{1}{\zeta+\overline{\zeta'}} - \frac{1}{\zeta-\overline{\zeta'}}\right)\frac{d\zeta}{dz} + \left(\frac{-1}{\zeta-\zeta'} - \frac{1}{\zeta+\zeta'}\right)\frac{d\zeta'}{dz'}\right.$$

$$\left. + \left(\frac{1}{\zeta+\overline{\zeta'}} + \frac{1}{\zeta-\overline{\zeta'}}\right)\cdot\frac{d\zeta'}{d\overline{z'}}\right\}. \tag{A4}$$

Since $\zeta = \sqrt{Z - C}$, where $C = \cot(\gamma/2)$ and $Z = \sqrt{\ell - z/\ell + z}$

we have

$$\frac{d\zeta}{dz} = \frac{d\zeta}{dZ}\cdot\frac{dZ}{dz} = \frac{1}{2\zeta}\cdot\frac{d}{dz}\sqrt{\frac{\ell - z}{\ell + z}} = -\frac{(\ell+z^2)^2}{\zeta Z\ell}.$$

Substitution gives

$$-\frac{1}{4\ell}\,\mathrm{Re}\left\{\frac{1}{\zeta^2-\zeta'^2}\left(\frac{\zeta'}{\zeta}\,\frac{(1+Z^2)^2}{Z} - \frac{\zeta}{\zeta'}\,\frac{(1+Z'^2)^2}{Z_1}\right)\right.$$

$$\left. - \frac{1}{\zeta^2-\overline{\xi_1^2}}\left(\frac{\zeta_1}{\zeta}\cdot\frac{(1+Z^2)^2}{Z} - \frac{\zeta}{\overline{\zeta}_1}\cdot\frac{(1+\overline{Z}_1^2)^2}{\overline{Z}_1}\right)\right\}$$

$$= -\frac{1}{4\ell}\,\mathrm{Re}\left[\frac{1}{\zeta\xi_1}\cdot\frac{Z_1(Z_1-C)(1+Z^2)^2 - (Z-C)(1+Z_1^2)Z^2}{(Z-Z_1)ZZ_1}\right.$$

$$\left. - \frac{1}{\zeta\overline{\xi}_1}\cdot\frac{\overline{Z}_1(\overline{Z}_1-C)(1+Z^2)^2 - (Z-C)(1+\overline{Z}_1^2)Z^2}{(Z-Z_1)Z\overline{Z}_1}\right]$$

$$= -\frac{1}{4\ell}\,\mathrm{Re}\left[\frac{1}{\zeta\zeta_1}\cdot\frac{+(Z+Z_1)(Z_1^2Z^2-1) - C\,(-1+2ZZ_1+Z_1Z^3+Z^2Z_1^2+ZZ_1^3)}{ZZ_1}\right.$$

$$- \frac{1}{\zeta\bar{\zeta}_1} \cdot \frac{(Z+\bar{Z}_1)(Z^2\bar{Z}_1^2-1) - C\,(-1+2Z\bar{Z}_1+\bar{Z}_1 Z^3+Z^2\bar{Z}_1^2+Z\bar{Z}_1^3)}{Z\bar{Z}_1}\Bigg]$$

$$= - \frac{\mathrm{Re}}{4\ell} - \frac{1}{\zeta}\left(\frac{1}{Z_1\zeta_1} - \frac{1}{\bar{Z}_1\bar{\zeta}_1}\right) - \frac{1}{\zeta Z}\left(\frac{1}{\zeta_1} - \frac{1}{\bar{\zeta}_1}\right) + \frac{Z^2}{\zeta}\left(\frac{Z_1}{\zeta_1} - \frac{\bar{Z}_1}{\zeta_1}\right) + \frac{Z}{\zeta}\left(\frac{Z_1^2}{\zeta_1} - \frac{Z_1^2}{\bar{\zeta}_1}\right)$$

$$- C\,\frac{1}{\zeta Z}\left(\frac{1}{\zeta_1 Z_1} - \frac{1}{\bar{\zeta}_1\bar{Z}_1}\right) - 2C\,\frac{1}{\zeta}\left(\frac{1}{\zeta_1} - \frac{1}{\bar{\zeta}_1}\right)$$

$$- C\,\frac{Z^2}{\zeta}\left(\frac{1}{\zeta_1} - \frac{1}{\bar{\zeta}_1}\right) - \frac{C}{\zeta}\left(\frac{Z_1^2}{\zeta_1} - \frac{\bar{Z}_1^2}{\bar{\zeta}_1}\right) - C\,\frac{Z}{\zeta}\left(\frac{Z_1}{\zeta_1} - \frac{\bar{Z}_1}{\bar{\zeta}_1}\right). \tag{A5}$$

We can express this into the four singular potentials

$$\varphi_1 = i\,(\zeta-\bar{\zeta}),$$

$$\varphi_2 = \left(\frac{1}{\zeta} - \frac{1}{\bar{\zeta}}\right),$$

$$\varphi_3 = (\zeta^3-\bar{\zeta}^3),$$

$$\varphi_4 = i\left(\frac{\zeta}{Z} - \frac{\bar{\zeta}}{\bar{Z}}\right), \tag{A6}$$

and the corresponding quantities for P.

At first we remark that

$$-i\left(\frac{1}{\zeta Z} - \frac{1}{\bar{\zeta}\bar{Z}}\right) = \frac{\varphi_2-\varphi_4}{C},$$

$$-i\left(\frac{Z}{\zeta} - \frac{\bar{Z}}{\bar{\zeta}}\right) = \varphi_1 + C\,\varphi_2,$$

$$-i\left(\frac{Z^2}{\zeta} - \frac{\bar{Z}^2}{\bar{\zeta}}\right) = \varphi_3 + 2C\varphi_1 + C^2\varphi_2.$$

This gives:

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial x^p} = \frac{1}{4\ell}\left[-\varphi_2 \cdot \frac{\varphi_2^p-\varphi_4^p}{C} - \frac{\varphi_2-\varphi_4}{C} \cdot \varphi_2^p + (\varphi_3+2C\varphi_1+C^2\varphi_2)(\varphi_1^p+C\varphi_2^p)\right.$$

$$+ (\varphi_1+C\varphi_2)(\varphi_3^p+2C\varphi_1^p+C^2\varphi_2^p) + \frac{(\varphi_2-\varphi_4)(\varphi_2^p-\varphi_4^p)}{C} - 2C\varphi_2\varphi_2^p$$

$$\frac{\partial G}{\partial x} - C(\varphi_3+2C\varphi_1+C^2\varphi_2)\varphi_2^p - C(\varphi_3^p+2C\varphi_1^p+C^2\varphi_2^p)\varphi_2 - C(\varphi_1+C\varphi_2)(\varphi_1^p+C\varphi_2^p)\Big]$$

$$= \frac{1}{4\ell}\left[\frac{\varphi_4\varphi_4^p-\varphi_2\varphi_2^p}{C} + (\varphi_3\varphi_1^p+\varphi_1\varphi_3^p) + C(3\varphi_1\varphi_1^p-2\varphi_2\varphi_2^p) - C^3\varphi_2\varphi_2^p\right]$$

$$= \frac{1}{4\ell}\left[3C\varphi_1\varphi_1^p + (\varphi_1\varphi_3^p+\varphi_3\varphi_1^p) - \left(\frac{1}{C}+2C+C^3\right)\varphi_2\varphi_2^p + \frac{1}{C}\varphi_4\varphi_4^p\right]. \tag{A7}$$

Further

$$\frac{\partial G}{\partial \gamma} = \frac{\partial G}{\partial C}\cdot\frac{dC}{d\gamma} = -\frac{1}{2\sin^2\gamma/2}\cdot\frac{\partial G}{\partial C}. \tag{A8}$$

$$\frac{\partial(G+iH)}{\partial C} = \frac{\partial}{\partial C}\,\ell n\,\frac{(\sqrt{Z-C}-\sqrt{Z_1-C})(\sqrt{Z-C}+\sqrt{Z_1-C})}{(\sqrt{Z-C}-\sqrt{\overline{Z_1-c}})(\sqrt{Z-C}+\sqrt{\overline{Z_1-c}})}$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{Z-C}-\sqrt{Z_1-C}}\left(\frac{1}{\sqrt{Z-C}}-\frac{1}{\sqrt{Z_1-C}}\right)+\frac{1}{\sqrt{Z-C}+\sqrt{\overline{Z_1-C}}}\left(\frac{1}{\sqrt{Z-C}}+\frac{1}{\sqrt{\overline{Z_1-C}}}\right)\right.$$

$$\left.-\frac{1}{\sqrt{Z-C}-\sqrt{\overline{Z_1-C}}}\left(\frac{1}{\sqrt{Z-C}}-\frac{1}{\sqrt{\overline{Z_1-C}}}\right)-\frac{1}{\sqrt{Z-C}+\sqrt{Z_1-C}}\left(\frac{1}{\sqrt{Z-C}}+\frac{1}{\sqrt{Z_1-C}}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{\zeta-\zeta_1}\left(\frac{1}{\zeta}-\frac{1}{\zeta_1}\right)+\frac{1}{\zeta+\overline{\zeta}_1}\left(\frac{1}{\zeta}+\frac{1}{\overline{\zeta}_1}\right)-\frac{1}{\zeta-\overline{\zeta}_1}\left(\frac{1}{\zeta}-\frac{1}{\overline{\zeta}_1}\right)-\frac{1}{\zeta+\zeta_1}\left(\frac{1}{\zeta}+\frac{1}{\zeta_1}\right)\right]$$

$$= +\frac{1}{2}\left[\frac{1}{\zeta\zeta_1}-\frac{1}{\zeta\overline{\zeta}_1}-\frac{1}{\zeta\overline{\zeta}_1}+\frac{1}{\zeta\zeta_1}\right] = \frac{1}{\zeta}\left(\frac{1}{\zeta_1}-\frac{1}{\overline{\zeta}_1}\right).$$

Hence

$$\frac{\partial G}{\partial C} = \frac{1}{2}\left(\frac{1}{\zeta}-\frac{1}{\overline{\zeta}}\right)\left(\frac{1}{\zeta_1}-\frac{1}{\overline{\zeta}_1}\right) = -2\,\varphi_2\,\varphi_2^p. \tag{A9}$$

Substitution of these results into Eq. (A1) gives:

$$v(x,0,t) = \ell\int_\gamma^{2\pi} w(\theta,t)\cdot\frac{\partial G}{\partial y}\sin\theta\, d\theta - \frac{1}{U}\int_{-\infty}^{x} dx'\cdot\frac{\partial\gamma}{\partial t}\cdot w\!\left(\gamma, t-\frac{x-x'}{U}\right)\frac{\partial G}{\partial y}\sin\gamma$$

$$-\int_{-\infty}^{x} dx'\int_\gamma^{2\pi} w\left(\theta, t-\frac{x-x'}{U}\right)\cdot\frac{\sin}{4}\theta\,\frac{\partial}{\partial y}\left[3C\varphi_1\varphi_1^p+(\varphi_1\varphi_3^p+\varphi_3\varphi_1^p)\right.$$

$$\left.-\frac{(1+C^2)^2}{C}\varphi_2\varphi_2^p+\frac{1}{C}\varphi_4\varphi_4^p\right]$$

$$- U \int_{-\infty}^{x} dx' \int_{\gamma}^{2\pi} w \left(\theta, t - \frac{x-x'}{U}\right) \cdot \frac{\partial}{\partial y} (\varphi_2 \varphi_2^{\xi}) \cdot \frac{1}{\sin^2 \gamma/2} \cdot \frac{\partial \gamma}{\partial t} \sin \theta \, d\theta$$

$$+ \ell \int_{-\infty}^{x} dx' \left[w \left(\theta, t - \frac{x-x'}{U}\right) \cdot \frac{\partial G}{\partial y} \right]_{\gamma}^{2\pi}$$

$$+ \int_{-\infty}^{x} A_1 \left(t - \frac{x-x'}{U}\right) \cdot \frac{\partial \varphi_1}{\partial y} \cdot dx' + \int_{-\infty}^{x} A_2 \left(t - \frac{x-x'}{U}\right) \cdot \frac{\partial \varphi_2}{\partial y} \cdot dx'.$$

* * * * *

DISCUSSION

L. Landweber

One usually thinks of Green's functions in connection with existence theorems or numerical methods in which the Green's function is obtained as the solution of an integral equation. In Professor Timman's paper, however, a Green's function which yields the solution of a time-dependent problem in potential theory is explicitly displayed. I believe that the subject of cavitating hydrofoils, which has yielded a remarkable succession of elegant papers, has attracted so many mathematicians as much for its aesthetic appeal as for its practical fruitfulness.

I would urge the author to assume less mathematical erudition on the part of the reader and to include more details of his mathematical developments. For example, I would have liked a demonstration that the expression for the acceleration potential in terms of the Green's function, Eq. (5), satisfies the boundary condition on the acceleration potential.

A condition not considered in the present paper is that the pressure should be a minimum within the cavitation bubble $-\ell < x < c$, $y = +0$. It is known in the steady case, for example, that this latter condition is necessary in order to determine a unique solution. This raises the question concerning the uniqueness of the solution in the unsteady case.

M. Tulin (Office of Naval Research)

First I would like to comment on the question of the leading edge singularity which occurs in the linearized theory of lifting cavity flow. When I first did the linearized theory for zero cavitation number flows past lifting foils, I compared the result for the flat plate with the expansion (in powers of the angle of attack) of Rayleigh's exact solution and I verified that the linearized theory solution, which had a -1/4 singularity at the leading edge, was identical with the first term in that expansion. That was, of course, sufficient verification that the linearized solution was correct. Then, when I worked out the case of the finite cavity flow past the lifting flat plate I noticed that it was possible to select more than one solution to the boundary value problem; only one of these solutions had a -1/4 singularity at the leading edge and I naturally assumed that that one was the proper solution. I never thought that it could be any other way–in view of the information provided by the infinite cavity case. In order, however, to "prove" the validity of the present linearized theory solution for finite cavities, may I suggest that some interested persons carry out an expansion (in powers of the angle of attack) of the Gilbarg-Serrin, re-entrant jet, exact theory solution of the flat plate problem. The result would also confirm the almost certain conjecture that the linearized theory for finite cavities, as I first formulated it, is in fact a linearized version of re-entrant jet theory.

Now may I comment on Professor Timman's model for the unsteady cavity flow–which includes some sources or sinks in the flow field trailing behind the cavity. First, I don't understand the physical basis for this model, I don't see how the flow field in the wake of the cavity can contain the postulated sources or sinks. Second, I don't see the mathematical necessity for the model. It is certainly true that a cavity of changing volume would imply, according to incompressible theory, that the pressures become unbounded at infinity, but it is also true that pressure waves are propagated with finite speed, even in water, and that as a result the effects of compressibility alter these pressure waves at large distances from a source (or cavity of changing volume) and cause the pressures at infinity to remain unchanged. I think that even acoustic theory applied to the problem of the flow field at some distance from a cavity of changing volume would reveal the true nature of the pressure field there and would dispel the need for both Professor Timman's model with distributed sources and sinks outside the cavity, and Professor Wu's model–which calls for a cavity of unchanging volume. I believe that the solution of the unsteady cavity problem which properly takes into account the boundary and dynamical conditions on the body and cavity will produce, in general, a cavity of changing volume without sources and sinks distributed in the wake.

W. G. Cornell (General Electric Company)

I certainly hope that some of the techniques that have been shown can be applied to the problems of unsteady, separated aerodynamic flows which, so far, have been treated by what one may call quasi-steady-state methods. I refer to problems such as rotating stall and propagating stall in cascades.

I would like to hear some comments about the use of the Kutta condition when one has an unsteady flow.

B. Parkin (California Institute of Technology)

I would appreciate it very much if Professor Wu would indicate the bearing of the pressure gradient at infinity on the values of the coefficients which he quoted from Karman's work.

R. Timman

Professor Landweber asked whether I did verify that a constant pressure on the cavitation bubble was actually the minimum pressure in the field. I must again confess I was a bit careless and did not verify that.

With respect to the Kutta condition, if the cavity extends nearly the whole of the upper surface, I hardly believe that a Kutta condition could be valid. On the other hand, as long as the cavity is small I don't see why the Kutta condition should be dangerous. Also, there is a lot of experimental evidence in the use of the nonsteady flow theory. Of course, they all gave imperfect agreement with theory; they don't even agree completely in the steady case. In the steady case you know the Kutta condition, yet you predict 110 percent of the actual values, so why should you expect in a nonsteady theory the agreement to be better? So as far as I can see, as long as the cavity does not extend too close to the trailing edge, the Kutta condition is just as bad or as good as it is in all the hydrodynamical theories.

T. Wu (California Institute of Technology)

In reply to Dr. Parkin's question, it is true that the wake flow generated by a flat plate accelerating broadwise through the fluid which is otherwise at rest in an inertial frame is a problem different from that treated by Karman. In this case you may take a coordinate system fixed with respect to the fluid at infinity so that at infinity the velocity is zero and the pressure is equal to P_∞, a constant. Then the plate will be moving with acceleration a.

If the shape of the free boundary is still a constant in time, then when the condition of constant pressure is applied on the moving boundary, you will have a different expression for the pressure equation.

Next, I wish to supply a uniqueness proof with respect to the leading edge singularity, $Z^{-3/4}$ or $Z^{-1/4}$. The first time I looked at the 3/4 singularity I thought this singularity may be ruled out by the argument that the energy is not integrable at the leading edge. Though this statement is true, I wasn't too happy with this answer. Then I looked for other physical requirements and found a satisfactory one. I imposed another condition, namely that the pressure outside of the solid body and the cavity wake must not be less than the pressure in the cavity. That is, $u = \mathcal{R}\ell\, w = -1/2\, C_p \le 0$ in the flow field. Suppose the solution of w has in the neighborhood of the leading edge the expansion

$$w = iA\, Z^{-3/4} + iB\, Z^{-1/4} + 0(Z^{1/4}),$$

where A, B are two real coefficients so that $u = 0$ on the cavity. Then, with $Z = r e^{i\theta}$, we have near $\gamma = 0$

$$u = A\, \gamma^{-3/4} \sin \frac{3\theta}{4} + B\, \gamma^{-1/4} \sin \frac{\theta}{4} + 0(\gamma^{1/4}).$$

From this result, we notice that the term $\sin(3\theta/4)$ will change sign, whereas the term $\sin(\theta/4)$ will not, as θ changes from 0 to 2π. Since the first term violates this physical condition on the minimum pressure, we must therefore have $A = 0$.

My own viewpoint with respect to the existence of the flow source is as follows. Suppose we take a two-dimensional cylinder, and let it pulsate with its radius R as a function of t in a uniform stream of velocity U, then the velocity potential is

$$\varphi(\gamma, \theta, t) = U\left(\gamma + \frac{R^2}{\gamma}\right)\cos\theta + R\dot{R}\log\gamma,$$

so that we have a source of strength $2\pi R\dot{R}$ which depends on the rate of change of the cross section or the radius. In the problem of unsteady cavity flows, however, it seems that the physical requirement must be imposed that the pressure be finite at infinity. Otherwise, we would require an infinite amount of energy, and hence an infinite time, to create such a flow.

Thus the question arises: why don't we let the cavity volume grow and have an infinite pressure at infinity? At the first sight it seems to me that the affirmative is not the case. One flow model which avoids the flow source at infinity is that, when the volume of the cavity near the body changes, there will be a wake which becomes thinner or fatter in the opposite sense.

In regard to the philosophical question about the physical background of these flow models and their agreement with experiments, I have the following point of view: We realize that all the wake flow is the end product of the real fluid effect. But in order to solve the problem in an easy way we want to keep the potential problem as a possible approximation by making some mathematical assumptions, which we call mathematical models. If any mathematical model gives a good approximation of the flow quantities near the solid body, so that we can predict very accurately the total hydrodynamic forces on the body, then, as far as I am concerned, the model should be quite acceptable. However, we should not expect that the simple model is also capable to provide a good description of the complicated wake flow downstream. The problem of the wake flow in the wake is entirely different from those considered here and I believe one cannot obtain a good result without considering the viscous effect, vortex shedding, the turbulent mixing, and so forth.

Several experimental results have been available for a few special cases of cavity flow past a flat plate inclined at a small angle. These results give substantial support to these mathematical models. With respect to the present linearized model, there are certain features which are different from the nonlinear cases. Here, again, the philosophical way to answer this question is to examine if it gives a fairly accurate description of the flow near the body, so as to determine if it is acceptable.

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