NEW DEVELOPMENTS IN THE THEORY OF SUPERCAVITATING FLOWS

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INTRODUCTION

Cavity flows in fluids occur in nature under a variety of circumstances. The least sophisticated of these are the cases where the flowing fluid is a heavy liquid such as water or a volatile fuel, and when the cavity is filled with a light gas such as air or steam or volatilized fuel. The gas may be supplied externally as in the case of ventilated flows such as occur when a surface-piercing strut proceeds at such high speeds through water that a cavity forms behind it into which rushes air from the atmosphere, or the gas may be supplied by vaporization of the flowing fluid due to flow-induced reduction of pressure. This is called cavitation and commonly occurs when hydrofoils, ships' screws, hydraulic machines, or fuel pumps are driven too fast, or when the pressure of the approaching fluid is originally too low. These cases are least complicated because the pressure of the gas in the cavity is nearly constant and known in advance. Most of the recent interest in cavity flows has been motivated by problems of cavitation.

It is, of course, fortunate that the external flow may so often be considered non-viscous; when this is not the case complications naturally ensue. In fact very little is known about the effect of viscosity on cavity flows. As an interesting extreme case, very viscous flows involving gas cavities in fluids occur sometimes in oil bearings into which atmospheric air is sucked, and when gas is forced up through oil-filled soil. Such interesting problems have recently been studied by Sir Geoffrey Taylor and Phillip Saffman at the Cavendish Laboratory and will be discussed at this symposium by the latter.

Perhaps most sophisticated of all cavity flows are those involving homogeneous fluids which "separate" from the obstacle around which they flow. It will be remembered that the original infinite-wake theory of Kirchoff was intended to deal with such a flow past a bluff obstacle. The difficulty in dealing with these flows lies in the fact that although the pressure in the cavity may be roughly constant it is not known in advance. But that difficulty is not insuperable, as an interesting illustration demonstrates. Professor M. J. Lighthill (1) has treated the problem of a bluff two-dimensional obstacle placed in the path of the flow along a flat plate. It is known that a "dead-water" region forms ahead of such an obstacle. Postulating constancy of

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pressure in the dead-water, it maybe shown that forward extent of that region depends on the pressure therein, as does the pressure distribution ahead of it. Lighthill reasoned that the boundary layer on the plate must just separate åt the beginning of the dead-water, and he was thus able, by matching boundary layer behavior and dead-water pressure, to determine the latter, and show how it depends on the extent of the plate ahead of the obstacle and the state of the boundary layer on the plate. Separated flows of all kinds abound in nature, and it is probably not too much to hope that cavity flow theory will soon be put to further use in providing understanding of some of them. Particularly tempting and important problems involve bluff-body flame holders and stalled cascades, about which Mr. Cornell will speak at this symposium.

Most of the recently devised theory of supercavitating flows applies directly to problems of truly cavitating or ventilating flows and it is those problems that have most benefited by timely recent progress. Indeed, as will be revealed here by Professor Lerbs, and by Mr. Tachmindji and Mr. Morgan the application of low-drag hydrofoil sections to the design of supercavitating propellers was carried out successfully at the David Taylor Model Basin immediately the theory was available in early 1954, with the consequence that the possibility of operating ships' screws efficiently at such high speeds that considerable cavitation occurs was for the first time conclusively proved. At the same time the mental barrier that had existed in designers' minds as regards the safe and efficient use of hydrofoils at very high speeds was also lowered. Indeed, many of us now take for granted the eventual existence of ships' screws, hydrofoil boats, and hydrofoil-equipped aircraft, all operating with supercavitating or ventilated sections. For this change in atmosphere we are in great part indebted to theoretical developments.

In retrospect, the most significant feature of these recent developments has been their concern with lifting hydrofoil sections, a complete departure from the wartime trend which emphasized bodies of revolution such as torpedoes. In fact there now exist reasonably adequate mathematical tools to deal with lifting problems of considerable complexity. The fact that so much use has been made of linearized theory requires no apology. It is very fortunate that so many practical engineering configurations are reasonably slender. It should also be noted that experimental studies have provided much valuable information concerning lifting hydrofoils under cavitating conditions; the tests performed between 1955 and 1957 in the Hydrodynamics Laboratory of the California Institute of Technology and in the Langley Laboratory of the NACA are particularly to be noted.

In the papers to follow, the behavior of supercavitating hydrofoils in a variety of situations will be discussed. I have on different occasions in the past discussed my own results for blunt struts and lifting foils, including the particular possibility of designing for low drag (2-4). For my part now I would like to give some new results for several different and varied cavity flow problems. The topics involved are the cavity flow past smooth two-dimensional struts with unspecified cavity detachment points; effects of high speed on cavity flows past blunt struts; and the three-dimensional cavity flow past slender delta wings.

I am sure that the speakers to follow me will provide both a thorough review of their recent advances in the general field of cavity flows and a thorough discussion of practical applications. It was my interpretation of my role here that I try to say something about the future. I have thus chosen these diverse and what I hope are provocative topics. I hope that I will be excused if they have not been dealt with as conclusively as one might wish.

THE SUPERCAVITATING FLOW PAST SLENDER STRUTS

In the first application of linearized theory to problems of supercavitating flows, I considered the case of slender blunt-based struts of such shape that the point of detachment of the cavity was necessarily at the very end of the strut. In that way the problem of determining an unknown detachment position was avoided. The case considered was of practical interest, however, and it was possible to reduce the problem of calculating cavity shapes and drags to one of quadratures. Later the problem of minimum drag struts was considered; I gave some reasons for preferring a simple parabolic shape.

Now I would like to show how linearized theory may be used in the more general case of slender struts whose cavity detachment point is unknown at the outset, and to present some results for struts of simple shape.

The pertinent two-dimensional steady flow, symmetric about the x-axis, is shown schematically in Fig. 1. The wetted boundary of the smooth strut and the cavity wall are continuous streamlines, the cavity wall being in addition a surface of constant pressure.

The point of detachment is not known in advance. However, something can be said about the behavior of the flow there in the case where the flow is truly cavitating. The curvature of the cavity just at the detachment point cannot be infinite and curved towards the body, lest it intersect the body; nor can it be infinite and curved away from the body, lest the pressure in the stream just away from that point be less than the cavity pressure. Nor can the curvature of the streamline at the point of detachment undergo a finite jump, for it is well known that the pressure cannot be constant along a streamline in the neighborhood of such a discontinuity. As has long been known then, the cavity in a cavitating flow must have just the curvature of the smooth body at the detachment point. It can in addition be stated that since the cavity must be convex (from the outside), the body itself must be convex at the point of detachment.

The condition of continuous curvature is just sufficient to allow solution of the present problem, when used together with the conditions imposed in the earlier

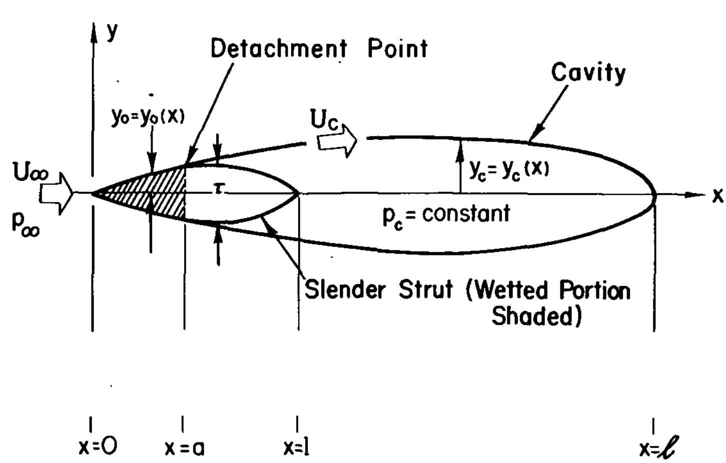
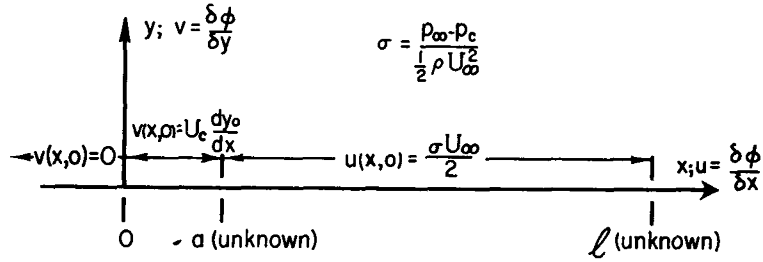


Fig. 1 - Supercavitating flow past a slender strut

linearized problem in which the detachment point was known in advance. For struts considered here (third-degree polynominals), only one convex solution is found for each strut. It seems clear that the smoothness and convexity condition, while necessary, are not sufficient to yield a physically meaningful solution representing the infinite cavity past a truly cavitating body, for it need only be supposed that a very small but very sharp bump be placed near the nose of such a body. Surely a convex cavity with continuous curvature at detachment will still exist with a detachment point close to its previous position, despite the fact that the local pressures near the sharp bump will fall far below the cavity pressure. The struts considered here do not, of course, have small sharp bumps on them. When slender enough they have in fact monotonic curvature, increasing in some cases, decreasing in others. For those cases where the wetted body is convex and the curvature is nondecreasing (i.e., the convexity is nondecreasing) it seems likely that with smooth detachment the flow is a truly cavitating one, for it has been shown (5) that such is the case for bodies of nondecreasing convexity (accolades). Some of the struts considered here are not of that type however, and it must only be hoped that the flows produced are meaningful. No verification is attempted through computation of the pressures on the forebody. Interestingly enough it is found that for some of these struts which have monotonically decreasing curvature two smooth cavities exist: one detaching forward and initially convex, the other detaching aft and concave at detachment. The latter is clearly meaningless with regard to cavitation problems.

The pertinent linearized problem here involves the determination, subject to the conditions illustrated in Fig. 2, of a harmonic function φ, the potential of the perturbation velocity, and at the same time, of the proper cavity detachment point and cavity length, these being unknown in advance.

The determination of the cavity shape for arbitrary detachment point was, in fact, made in the earlier work on blunt-based struts. The method employed involved the determination of the unknown source distribution for the cavity through inversion of the well known integral equation of incompressible thin airfoil theory.



ϕ is harmonic and symmetric about $y=0$.

Cavity Closure Condition: $\int_0^{\ell} v(x,o)\,dx = 0.$

Continuous Slope and Curvature at Detachment: $\begin{cases} \dfrac{dy_o(a)}{dx}=\dfrac{dy_c(a)}{dx} \\[2ex] \dfrac{d^2y_o(a)}{dx^2}=\dfrac{d^2y_c(a)}{dx^2}. \end{cases}$

Fig. 2 - Linearized boundary value problem

The results are given below:

$$2U_c \frac{dy_c}{dx} = -\sigma U_\infty \frac{\sqrt{x-a}}{\sqrt{\ell - x}} + \frac{\sqrt{x-a}}{\pi\sqrt{\ell - x}} \int_0^a 2U_c \frac{dy_0}{dt} \frac{\sqrt{\ell - t}}{(x-t)} \frac{dt}{\sqrt{a-t}} \text{ and } \frac{\sigma U_\infty}{2U_c}\frac{\ell\pi}{2} = \int_0^a \frac{dy_0}{dt}\frac{\sqrt{\ell - t}}{\sqrt{a-t}}\,dt$$

where

$$\frac{U_c}{U_\infty} = 1 + \frac{\sigma}{2}.$$

For a given strut shape $y_0 = y_0(t)$. The first of these expressions may be differentiated to yield the cavity curvature; and thus may the detachment point a be determined by imposing the condition of smooth curvature there. For cavitation numbers other than zero, the second expression, stating that the cavity is closed, must simultaneously be used.

The actual exploitation of these relations may not be simple because of complicated quadratures, but if only very long cavities ($\ell/a \gg 1$) and simple struts (polynomial shapes) are considered, results are readily obtained.

For very long cavities:

$$2U_c \frac{dy_c}{dx} = \sqrt{x-a}\left\{-\sigma \frac{U_\infty}{\sqrt{\ell}} + \frac{2U_c}{\pi}\int_0^a \frac{dy_0}{dt}\frac{dt}{(x-t)}\frac{1}{\sqrt{a-t}}\right. \text{ and } \frac{\sigma U_\infty}{U_c}\frac{\sqrt{\ell\pi}}{4} = \int_0^a \frac{dy_0}{dt}\frac{dt}{\sqrt{a-t}}\,.$$

For struts of the shape, $y_0/b = a_1 t + a_2 t^2 + a_3 t^3$, then,

$$2U_c \frac{dy_c}{dx} = \left\{-\sigma U_\infty \frac{\sqrt{x-a}}{\sqrt{\ell}} + \frac{2U_c b}{\pi}\left[(2a_1 + 4a_2 x + 6a_3 x^2)\tan^{-1}\sqrt{\frac{a}{x-a}}\right.\right.$$
$$\left.\left. + \sqrt{x-a}\,(-4a_2\sqrt{a} - 6a_3\sqrt{ax} - 4a_3 a^{3/2})\right]\right\}$$

and, differentiating to obtain the curvature,

$$2U_c \frac{dy_c^2}{dx^2} = \frac{1}{\sqrt{x-a}}\left[-\frac{\sigma U_\infty}{2\sqrt{\ell}} + \frac{2U_c b}{\pi}\left\{-\frac{\sqrt{a}}{2x}(2a_1 + 4a_2 x + 6a_3 x^2) - (2a_2\sqrt{a} + 3a_3\sqrt{a}\,x + 2a_3 a^{3/2})\right\}\right]$$
$$+ \frac{2U_c b}{\pi}\sqrt{x-a}\,(-6a_3\sqrt{a}) + \frac{2U_c b}{\pi}\tan^{-1}\sqrt{\frac{a}{x-a}}\,(4a_2 + 12a_3 x)\,.$$

The smooth curvature condition is consequently satisfied only when a satisfies the relation:

$$\frac{\sigma\pi}{(1+\sigma/2)\,4\sqrt{\ell}\,b} = -\frac{1}{\sqrt{a}}(a_1 + 4a_2 a + 8a_3 a^2)\,.$$

The other necessary relation which follows from the cavity closure condition is

$$\frac{\sigma\pi\sqrt{\ell}}{(1+\sigma/2)\,4b} = \sqrt{a}\,(2a_1 + 8/3\, a_2 a + 16/5\, a_3 a^2)\,.$$

These two relations may fortunately be combined to give simple formulas relating (a) detachment point to strut shape and cavitation number, and (b) cavity length to the same quantities. These are:

$$\left(\frac{\sigma\pi}{(1+\sigma/2)\ 4b}\right)^2 = -\left(a_1 + 4a_2a + 8/3\ a_3a^2\right)\left(2a_1 + 8/3\ a_2a + 16/5\ a_3a^2\right)$$

$$\ell = -a\,\frac{(2a_1 + 8/3\ a_2a + 16/5\ a_3a^2)}{(a_1 + 4\,a_2a + 8\,a_3a^2)}\ .$$

In the case of zero cavitation number and infinite cavity length, the detachment point is given by the vanishing of the quadratic, $a_1 + 4a_2a + 8a_3a^2$, and is independent of the strut thickness. The coefficients a_1, a_2, a_3, which define a strut, cannot be chosen independently of one another. If a_1 is zero, then a_2 may be taken as unity; in order for the strut ordinate to vanish at $t = 0,\ 1$ then $a_3 = -1$. If a_1 is not zero then it may be chosen as unity and again for a strut spanning the interval $0, 1$ it must be that $a_2 = -(a_3 + 1)$; a one parameter family of struts results. It may be shown that these struts are segments of the third-degree polynomial whose zeros are at $(0,\ 1,\ 1/a_3)$; the coefficient a_3 must then be less than unity.

The position of the detachment point is plotted against the parameter a_3 in Fig. 3. For values of a_3 between 1 and 3/4, two detachment points are found. The forward point, falling where the body is convex, may have physical meaning, but the after one clearly does not as the body is concave there; only the forward point is plotted in Fig. 3. The fact is illustrated, however, that at least for very slender bodies, the continuous curvature condition is not sufficient in itself to guarantee truly cavitating flows.

Calculations have been made to determine the movements of detachment point with σ. These indicate that the point moves aft very slowly with increasing cavitation number, and that its position depends on the strut thickness. If τ is the ratio of strut maximum thickness to strut chord, and δ is the displacement of the detachment point forward of its position for zero σ, the approximate results hold that δ is proportional to the square of the ratio of σ to τ and that the cavity length ℓ, is inversely proportional to δ. The calculated constants of proportionality for three different struts are shown in Fig. 4.

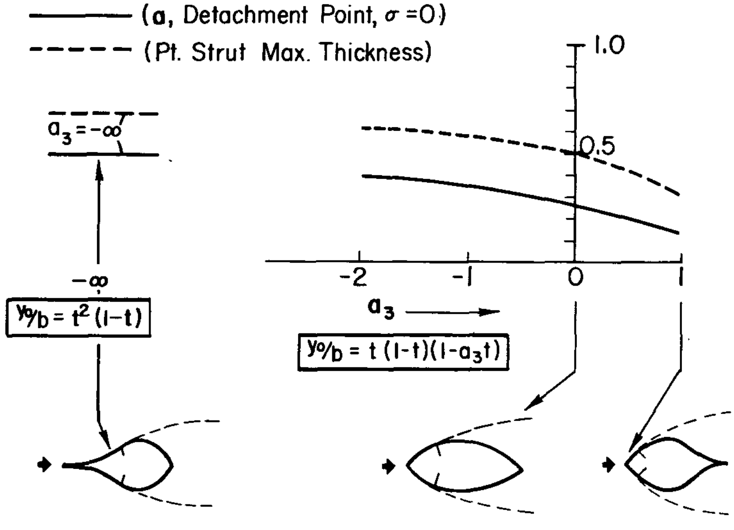
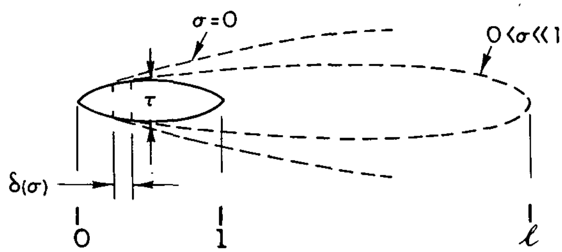


Fig. 3 - Cavity detachment point, $\sigma = 0$



$$\sigma \ll 1: \quad \delta = k_1\left(\frac{\sigma}{\tau}\right)^2 \quad ; \quad \ell = \frac{k_2}{\delta}.$$

a_3	k_1	k_2
$-\infty$	0.025	0.067
0	0.029	0.083
+1	0.008	0.034

Fig. 4 - Cavity length and detachment point, $\sigma \ll 1$

The resistance of any strut, the detachment point having been determined, may be obtained using the result of previous work. Neglecting terms of order σ^2, the drag coefficient based on maximum strut thickness T is

$$\frac{D}{1/2\,\rho U_\infty^2 T} = C_D = \frac{2(1+\sigma)}{\pi T}\left[\int_0^{a(\sigma)} \frac{dy_0}{dt}\,\frac{dt}{\sqrt{a-t}}\right]^2 .$$

The theory indicates that the very slow forward movement of the detachment point contribute only terms of order higher than σ to the drag integral, so that just as for blunt-based struts:

$$C_D(\sigma) = C_D(0)\ [1+\sigma] .$$

The drag coefficients ($\sigma = 0$) of the polynomial struts considered here are shown in Fig. 5. The conclusion might be drawn that blunt noses are harmful to resistance; this is certainly the case for the family considered here. However, the superiority of parabolic struts has been shown before. In order to put the present results in proper perspective, the shapes and drags of three struts which cover the entire range of those considered here are compared with three other struts from outside the family in Fig. 6. These are a symmetric diamond strut, a symmetric strut with parabolic nose and tail, and a blunt parabolic strut.

All of these results concerning drag and position of detachment point may be boldly presented as no experimental results for two-dimensional struts seem to be available (although some results for bodies of revolution are available; see Eisenberg (6)). It would clearly be worthwhile for someone to carry out pertinent tests.

Finally, implications of some of the results obtained here with regard to the uniqueness of smooth cavity flows past closed struts may be stated. First: as mentioned before, it was found that for certain of those struts whose curvature is monotonically decreasing two smooth cavities were found, one concave at detachment, the other convex. Second: struts given by higher degree polynomials (than three)

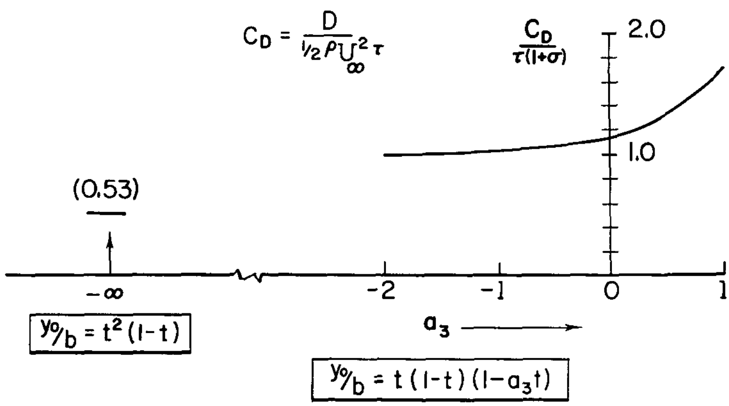


Fig. 5 - Drag of polynomial struts

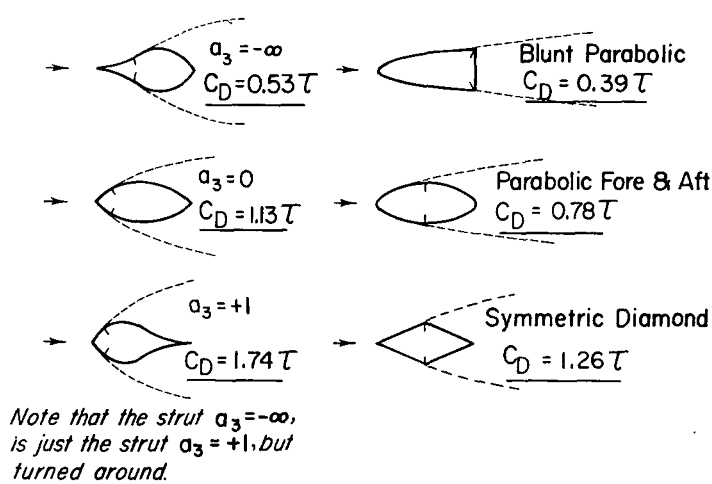


Fig. 6 - Comparison of strut drags

may very well yield more than two smooth cavity flows, for it is clear that the position of detachment satisfies an algebraic equation of one degree less than the degree of the polynomial defining the strut. It would be interesting to see whether a strut could be found for which more than one smooth and convex cavity flow presents itself.

HIGH-SPEED EFFECTS ON SYMMETRIC CAVITY FLOWS

It is probably not too frivolous an undertaking to try to say something about cavity flows in compressible fluids, for the successful and general application of cavity flow information to the study of gas flows with "dead-water" wakes such as may exist in the case of bluff-body flame holders and stalled cascades will certainly require some understanding of the effects of high speed. In addition, problems of symmetric cavity flows should be tempting to the students of compressible fluids for not only are they relatively simple flows, but they are flows with finite drag at subsonic speeds. The subject is, indeed, not a new one. For certain polygonal obstacles including single wedges, the infinite-wake flow for zero cavitation number may be determined exactly (for Mach numbers at least as great as unity) through the use of a suitable hodograph method, for then the transformed boundaries of the flow are known, the boundary conditions imposed thereon are linear, as are the transformed gas-dynamic equations. Students of transonics have been particularly interested in this possibility and attempts at making calculations for wedge forebodies have been made by Imai (7) and Mackie and Pack (8). Most recently, Helliwell and Machie (9) have treated the flow past slender wedges trailing free streamlines with sonic velocity thereon; they used von Karman's transonic approximations and Roshko's free streamline model.

In the absence of exact results of sufficient detail or even of the pertinent transonic solution, the use of a suitably modified Prandtl-Glauert rule to extrapolate incompressible results for low cavitation number flows past suitably slender forebodies, the cavity detachment point being known, seems particularly inviting.

For suitably low cavitation numbers, even an unslender body will possess a very slender cavity, since the thickness ratio of the latter is, at least in an incompressible flow, proportional to σ. The Mach number of the flow in the vicinity of these slender cavities will be very close to the Mach number corresponding to the cavity pressure coefficient. This cavity Mach number, M_c, will generally be greater than free stream Mach number, M_∞, except in the case of an infinite cavity when they will be equal. No matter how slender the forebody, therefore, the streamwise perturbation of the free stream does not tend to vanish, but, instead, approaches on the body and cavity a constant value equal to $(M_c - M_\infty)$. The proper linearization of the nonlinear gas-dynamic equations is, then,

$$(1 - M_c^2)\,\phi_{xx} + \phi_{yy} = 0,$$

and the Prandtl-Glauert rule must be suitably modified.

As implied above, this linearized equation must be a very good approximation for slender bodies because of the cavity enforced uniformity of the near flow field. For still another reason, the Prandtl-Glauert rule seems especially useful: according to linearized theory (2), it is a consequence of D'Alembert's paradox that the cavitation drag of the forebody must be identical with the "nose drag" of the end of the cavity (which in turn depends only on the radius of curvature of that nose). Jones and van Dyke (10) have recently shown (but not rigorously) that the Prandtl-Glauert rule is particularly applicable to the estimation of nose drag. In fact they have shown that even though it may fail to predict exactly the distribution of pressures, the P-G rule does extrapolate the overall drag of a parabolic half body exactly for Mach numbers up to one. All of which implies that were the shape of the cavity behind a given slender forebody known for all Mach numbers, the drag could be estimated without error for zero σ and with small error for suitably small σ. Unfortunately the changing shape of the cavity is not known in advance, but must be estimated; hence the use of the P-G rule involves unknown errors. However, as remarked previously, the cavity enforced uniformity of the perturbation flow gives reason to hope that nonlinear distortions of the cavity shape which are not predicted by the P-G rule will be limited in importance to very high subsonic Mach numbers and that this completely linear theory has a large range of applicability. Confirmation of this speculation awaits the solution of the pertinent transonic problem.

Figure 7 shows the cavity flow in the real physical plane (on the left) and in the affinely transformed incompressible flow plane (on the right). The primed quantities refer to the incompressible transformed plane. The application of the P-G rule is well discussed by Sears (11) and only results for the case of cavity flows are presented here. Certain results for incompressible cavity flows presented by Tulin (2) are also utilized without derivation.

Important results are that the drag increases with Mach number as

$$\frac{1 + \sigma(1 - M_c^2)}{\sqrt{1 - M_c^2}\,(1 + \sigma)},$$

and that the cavity becomes more slender as $\sqrt{(1 - M_c^2)}$, the length increasing as $(1 - M_c^2)^{-1}$ and the thickness as $(1 - M_c^2)^{-1/2}$. All of these quantities may be readily determined for a given forebody utilizing the available incompressible theory. Some results are summarized below:

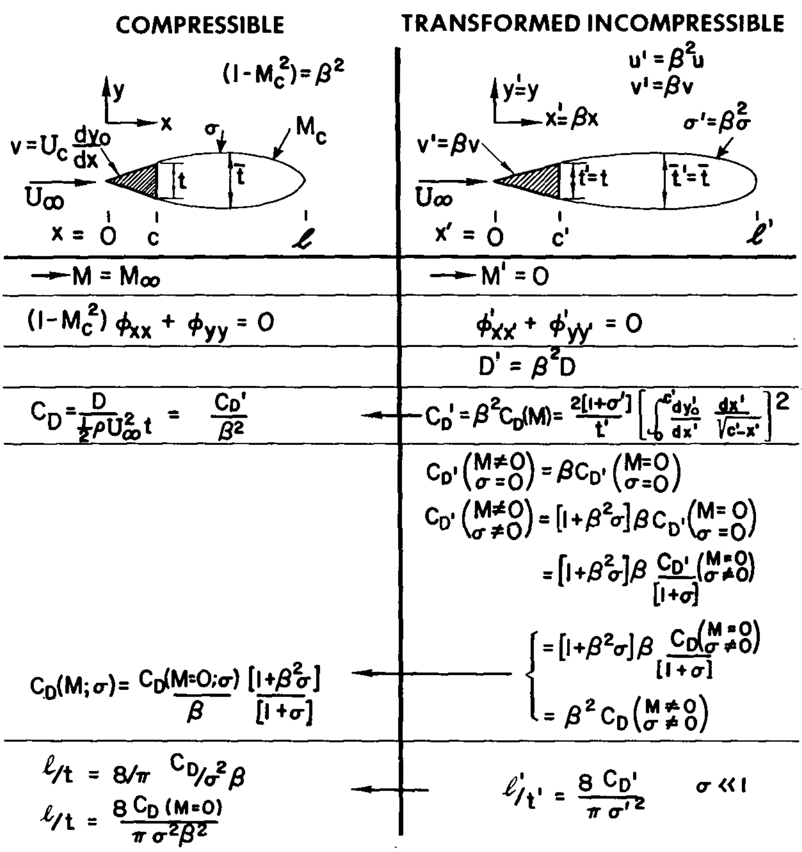


Fig. 7 - The Prandtl-Glauert transformation, cavity flows

$$\sigma << 1: \quad C_D(M,\sigma) = \frac{2}{\pi} \frac{[1 + \sigma(1 - M_c^2)]}{\sqrt{1 - M_c^2}\ T} \left[\int_0^1 \frac{dy_0}{dx} \frac{dx}{\sqrt{1-x}} \right]^2$$

where C_D is based on the strut maximum thickness, T, and where y_0 is the ordinate of the strut, which is itself of unit length.

$$\sigma << 1: \quad \ell = \frac{8T}{\pi\sigma^2} \frac{C_D(0,0)}{(1 - M_c^2)}$$

where ℓ is the cavity length.

$$\sigma << 1: \quad \bar{T} = \frac{4T}{\pi\sigma^2} \frac{C_D(0,0)}{\sqrt{1 - M_c^2}}$$

where $\bar{T}$ is the maximum thickness of the cavity.

The P-G rule predicts infinite drag for sonic cavity Mach numbers and transonic theory must be invoked before such speeds are reached. As previously noted, Helliwell and Mackie (9) have considered the case of slender wedges trailing free streamlines with sonic velocity thereon. According to their results, the drag coefficient, C_D, of a wedge of semi-apex angle equal to θ is, when C_D is based on wedge total thickness and when the base pressure is taken into account:

$$C_D = \frac{1.89\ \theta^{2/3}}{(\gamma + 1)^{1/3}}$$

which result implies that the Mach number distribution on the wetted portion of the wedge is insensitive to changes in free stream Mach number; this effect is not really surprising as, in the case considered, the velocity at the apex and shoulder of the wedge are fixed and independent of M_∞. If the Mach number on the free streamline is not fixed as unity but is instead taken as equal to M_∞ ($\sigma = 0$), then the reason for the insensitivity of the wedge Mach number distribution to M_∞ is partially removed. It is conjectured that, in fact, the wedge Mach number distribution would in that case be closely proportional to M_∞. The drag coefficient of the wedge would then be relatively insensitive to M_∞ for subsonic Mach numbers close to unity (since $C_p = 4/\gamma + 1\ [1 - M/M_\infty]$). It would be interesting to verify this conjecture by actual calculation using transonic theory.

The estimated drag of slender wedges of semi-apex angle, 2-1/2, 5, and 10 degrees with infinite trailing cavities, $\sigma = 0$, are presented in Fig. 8; the ratio of specific heats, γ, is assumed to be 1.4.

These brief considerations have been undertaken mainly to discover some effects of speed on simple cavity flows, but it is well that they serve at the same time to indicate for the symposium record the existence of the interesting transonic theory for cavity flows. It is worthwhile to note that a very recent pertinent paper by Nonweiler (12) contains the interesting result that a half-body whose ordinates vary as $x^{2/5}$ has a sonic surface velocity everywhere, except at the nose, when the free stream speed is sonic. This gives hint that the asymptotic form of infinite cavities at sonic speed will be of the form $x^{2/5}$; the result of Jones and van Dyke (10) shows, of course, that the asymptotic form of cavities for all subsonic speeds must be $x^{1/2}$, and that a different form must obtain at sonic speed.

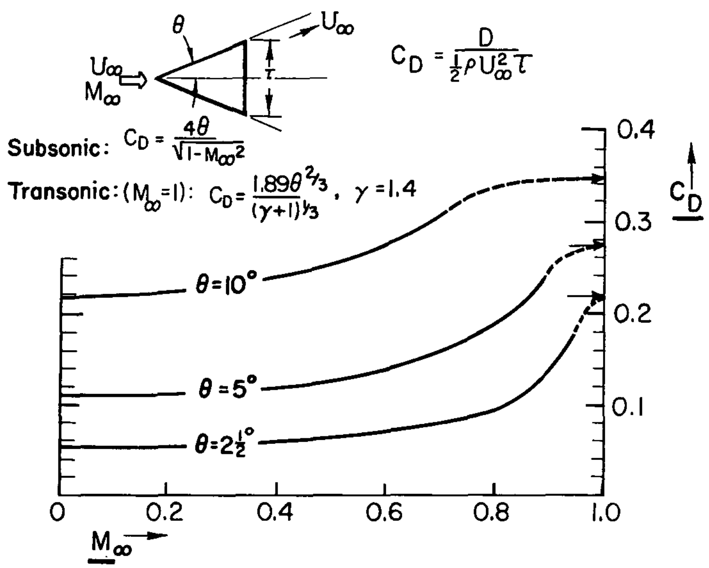


Fig. 8 - The drag of slender wedges, $\sigma = 0$

SUPERCAVITATING FLOW PAST SLENDER DELTA WINGS

Without implying that interesting and important problems* do not remain to be investigated, it may be said that the behavior of two-dimensional, thin, lifting foils with full cavitation in steady unbounded flow is now fairly well understood. Despite the very considerable work that has been devoted to two-dimensional problems, however, there do not seem to exist any theoretical results pertaining particularly to the behavior of three-dimensional lifting wings with full cavitation. It is true that the direct application in practice of standard aspect ratio corrections to the case of lightly loaded propellers has so far given satisfaction, but no attempt to rationalize the procedure, even for the simple case of a high or moderate aspect ratio plane wing, has been made.

When the wing cavity length is somewhat shorter than the span of the wing, no difficulties would seem to present themselves in the direct extension of lifting line results, but when the cavity is longer than the span then even the forces on the high aspect ratio foil may very well be affected not only by the decreased effective angle of attack due to the trailing vortex system, but also by the difference, due to the finite span of the cavity, between the two-dimensional and actual cavity shapes. For a given cavitation number, clearly the cavity at mid-span must become fatter for decreasing cavity aspect ratio. It is not clear now to what order this fattening of the trailing cavity will affect the lift on the foil, and it is this very point which requires future clarification.

In the extreme case of very low aspect ratio (or slender) supercavitating wings, the flow pattern becomes primarily dependent on the "tip" effects, and different

*For instance, the behavior of airfoils of conventional shape (rounded nose, sharp tail) for which the cavity detachment point varies with σ and is not known in advance.

considerations apply than in the case of moderate aspect ratio wings. This extreme case is, however, very interesting in its own right and seems to offer a special opportunity to study three-dimensional cavities. Some considerations of the flow past slender supercavitating wings, in particular past the mathematically tractable flat delta wing, are now presented.

As is well known, it is possible to visualize the flow past lightly loaded, low aspect ratio wings having thin cross-sections and fully wetted, as a changing two-dimensional flow in the transverse plane being convected with the main stream. The theory for such purely lifting slender wings allows, even for incompressible flows, no downstream effects to be propagated upstream. If the slender body has significant thickness, however, the flow at a particular point on the body does depend on the shape of the body everywhere else. If the body is smooth and slender and its shape known in advance, the slender body theory reduces the problem of finding the flow to one of quadrature (for the thickness effects) plus a set of two-dimensional potential problems, the normal derivatives of the potential being specified. Thus is the problem solved.

In the present case, and characteristic of cavity flow problems, the cavity thickness is not known in advance. Instead the pressure on that part of the top of the wings just under the cavity is specified. This pressure has, in general, contributions from both the thickness effects and the transverse flows. Unfortunately, adequate boundary conditions in the transverse flow planes are not really known in advance, but must be determined through consideration of the interaction between thickness effects and transverse flows and their additive effect on the wing pressure distribution. This involves a complicated procedure.

In the case of flat, slender, deltas the possibility exists that the supercavitating flow past such wings is essentially conical. The determination of the transverse flow and longitudinal thickness distribution may then be simplified. In the present analysis it is shown that an almost conical flow may be found for all cavitation numbers; the flow being more conical at the apex of the delta than at the trailing edge. Let it be stated in advance that the cavity, which is itself conical in shape, does not cover all of the wing top in any case but as the cavitation number decreases, grows from the leading (or side) edge region until at $\sigma = 0$ it covers all of the wing top except for a wedge-shaped region along the wing centerline which region is wetted by a sort of re-entrant jet flow. The theory, in fact, imposes a very special relationship between the cavity shape and the amount of fluid in the impinging jet. The cavities carried off downstream are assumed to take up such a shape that the pressure within them has the proper value, and they are assumed to have an important influence on the flow over the wing only at the trailing edge where, of course, the flow cannot be truly conical. The effect of these cavities, diminishing toward the apex, will at any rate only be such as to change the ambient pressure field equally on the top and bottom of the wing and it is here neglected.

The flow about a flat, slender delta wing of apex angle β, inclined at an angle of attack α, operating at cavitation numbers both very large ($\sigma/\alpha\beta >> 1$) and very small ($\sigma/\alpha\beta << 1$) is shown schematically in Fig. 9. It is these extreme cases for which numerical results are provided in the present paper. Some assumptions about the flow are already introduced into the figure. These are that (a) the flow over the wing is assumed to be similar in the various transverse planes - and thus conical, and (b) the conical cavity envelops only part of the upper surface of the wing.

Some of the nomenclature to be used is illustrated in Fig. 10.

For simplicity, a delta wing of unit length has been chosen.

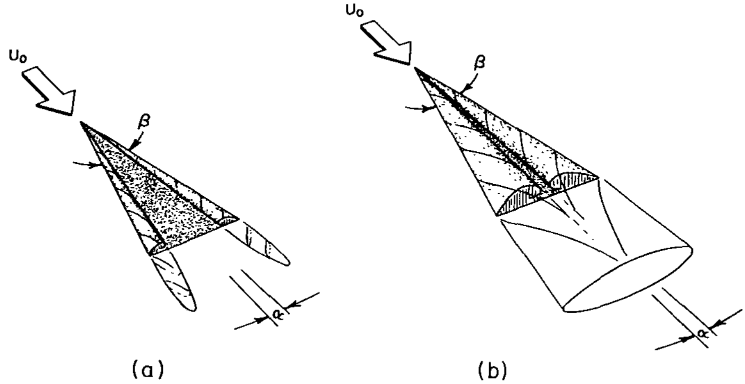


Fig. 9 - (a) Leading edge cavities on Delta wing ($\sigma/\alpha\beta >> 1$), (b) heavy cavitation on delta wing ($\sigma/\alpha\beta << 1$)

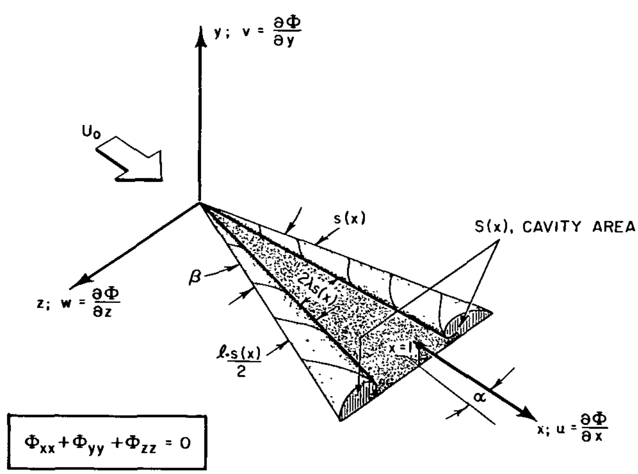


Fig. 10 - Cavity flow past a delta wing. Nomenclature

The pressure at infinity is taken to be zero. For small angles of attack, the pressure then becomes

$$\mathrm{P}(x, y, z) = -\rho U_o \frac{\partial\Phi}{\partial x} .$$

On the cavity, wherein the pressure p_c is constant, $\partial\Phi/\partial x$ is thus constant.

For small angles of attack, the vertical velocity on the bottom of the wing is constant and equal to $-U_o\alpha$. Thus

$$\frac{\partial\Phi}{\partial y} = -U_o\alpha$$

everywhere on the bottom of the wing. The available boundary conditions have now been stated.

For sufficiently slender bodies (13):

$$\Phi(x, y, z) = \varphi(y, z; x) + f(x)$$

where

$$\varphi_{yy} + \varphi_{zz} = 0$$

and

$$f(x) = -\frac{U_o}{4\pi} \frac{\partial}{\partial x} \int_0^\infty S'(x_1) \frac{x - x_1}{|x - x_1|} \ln 2|x - x_1| dx_1 .$$

The term $\varphi(y, z; x)$ will be recognized as the two-dimensional flow in the transverse $(y - z)$ plane while the $f(x)$ is a flow due to thickness.

The two-dimensional potential may be expressed in terms of a contour integral around the boundary of the body in the transverse plane (Green's Theorem). For small angles of attack, the cavity thickness is also small and the integration may thus be made around the projection of the wing and cavity on the horizontal plane (i.e., along the top and bottom of a symmetric slit in the z axis). Then

$$\varphi(y, z; x) = \frac{1}{2\pi} \oint \left(\frac{\partial \varphi}{\partial n} - \varphi \frac{\partial}{\partial n}\right) \ln \sqrt{(z - z')^2 + y^2}\, dz' .$$

The assumption that the flow is conical over the wing leads, through consideration of the above integral and the definitions $\bar{y} = 2y/s$, $\bar{z} = 2z/s$, to the following:

$$\varphi(y, z; x) = g(x) + \frac{s(x)}{2} \bar{\varphi}(\bar{y}, \bar{z})$$

where

$$g(x) = \frac{U_o s(x)}{4\pi} \cdot \ln \frac{s(x)}{2} \cdot \int_{-1}^{+1} \frac{\partial h}{\partial x} (\bar{z}) d\bar{z}$$

and $\bar{\varphi}(\bar{y}, \bar{z})$ is harmonic. The latter represents the two-dimensional flow in the non-dimensionalized cross-flow planes.

The fact that the two-dimensional flow φ makes a contribution to the total flow which is only a function of x (the term $g(x)$) is especially to be noted.

The boundary conditions now become:

On the bottom of the wing,

$$\frac{\partial \Phi}{\partial y} = -U_o \alpha = \frac{\partial \bar{\varphi}}{\partial \bar{y}}$$

and, on the part of the wing top covered by the cavity ($\lambda < |\bar{z}| < 1$),

$$\frac{\partial \Phi}{\partial x} = + \frac{U_o \sigma}{2} = f'(x) + g'(x) + \frac{1}{2}\beta\left(\bar{\varphi} - \bar{z}\frac{\partial \bar{\varphi}}{\partial \bar{z}}\right).$$

In order that this condition be satisfied,

$$\bar{\varphi} - \bar{z}\frac{\partial \bar{\varphi}}{\partial \bar{z}} = \text{constant}$$

and

$$f'(x) + g'(x) = \text{constant}.$$

However it may readily be shown, utilizing the definitions of f and g given earlier, that near the apex of the delta ($x \longrightarrow 0$),

$$f'(x) + g'(x) \sim \frac{U_o k \beta \alpha}{2\pi} \ln x$$

where k is defined such that $S''(x) = 2k\,\alpha\beta$. ($S(x)$ is the contour integral of $\partial\Phi/\partial n$ in a transverse plane.)

Thus, in order for the flow to be conical as assumed and still to contain a constant pressure region, the contour integral of $\partial\Phi/\partial n$ (or $\partial\bar{\varphi}/\partial\bar{n}$) in the transverse plane must be identically zero. If a growing cavity exists at all then, the outflow in the transverse plane that it will cause must be accompanied by an inflow just sufficient to make the net source flow in the transverse plane null. The inflow may be imagined to go into two symmetrically placed re-entrant jets. It is just this situation which was anticipated in Figs. 9 and 10. The present theory does not, of course, allow a detailed discussion of the behaviour of the flow in the vicinity of the juncture of cavity and wetted region on the top of the wing. It is believed nevertheless that the theory will reproduce the general features of the real flow.

The new condition has been obtained that the flow in the transverse plane cannot be source-like, or

$$\int_{-1}^{+1} \frac{\partial \bar{\varphi}}{\partial \bar{y}}\, d\bar{z} = 0 .$$

Since f and g are then identically zero,

$$\bar{\varphi} - \bar{z}\frac{\partial \bar{\varphi}}{\partial \bar{z}} = + \frac{U_o \sigma}{\beta} \qquad \text{for} \quad \bar{y} > 0, \qquad \lambda < |\bar{z}| < 1 .$$

Since $\bar{\varphi}$ must be symmetric with respect to $\bar{z}$,

$$\bar{\varphi} = + \frac{U_o \sigma}{\beta} - U_o a|\bar{z}| ,$$

or

$$\frac{\partial \bar{\varphi}}{\partial \bar{z}} = -aU_o , \qquad \bar{z} > 0$$

$$\frac{\partial \bar{\varphi}}{\partial \bar{z}} = +aU_o , \qquad \bar{z} < 0 .$$

Two unknown parameters exist as part of the boundary value problem. They are (a) λ, the nondimensional width of the cavity, and (b) $U_o a$ the magnitude of the transverse velocity on the cavity. Two conditions exist for their simultaneous determination. They are (a) that

$$\int_{-1}^{+1} \frac{\partial \bar{\varphi}}{\partial \bar{y}} d\bar{z} = 0$$

and (b) that

$$\bar{\varphi}(0,1) = +U_o\left(\frac{\sigma}{\beta} - a\right).$$

The boundary value problem in the transverse plane is illustrated in Fig. 11.

The lift, L, on the delta wing may readily be determined in terms of the solution in the transverse plane.

$$\frac{dL}{dx} = \int_{-S/2}^{+S/2} [P(-0,z) - P(+0,z)]\, dz$$

$$= \int_{-S/2}^{+S/2} -\rho U_o[\Phi_x(-0,z) - \Phi_x(+0,z)]\, dz = -\rho U_o \frac{\partial}{\partial x} \int_{-S/2}^{+S/2} [\Phi(-0,z) - \Phi(+0,z)]\, dz$$

$$= -\rho U_o \frac{\partial}{\partial x} \frac{S^2}{4} \int_{-1}^{+1} [\bar{\varphi}(-0,\bar{z}) - \bar{\varphi}(+0,\bar{z})]\, d\bar{z}.$$

Therefore

$$L = -\rho \frac{U_o}{4} S^2(1) \int_{-1}^{+1} [\bar{\varphi}(-0,\bar{z}) - \bar{\varphi}(+0,\bar{z})]\, d\bar{z}.$$

If $\bar{\Psi} = \bar{\varphi} - i\bar{\psi}$ is the complex potential in the transverse plane, then

$$\bar{\Psi} = \frac{A_1}{Z} + \frac{A_2}{Z^2} + \cdots$$

where $Z = \bar{z} + i\bar{y}$. The ℓnZ term is omitted since the flow must be non source-like.

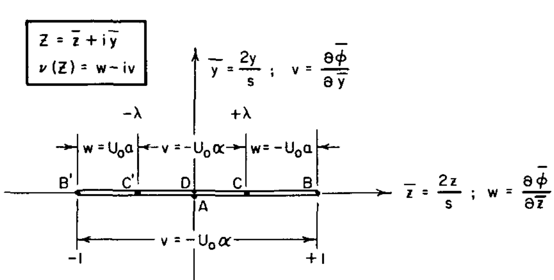


Fig. 11 - Boundary value problem in transverse (Z) plane

It is easily shown that A_1 is pure imaginary, and that

$$\frac{L}{1/2\,\rho U_o\, S^2(1)} = -\pi i A_1 .$$

If $\nu = \bar{\omega} - i\bar{v}$ is the complex velocity in the transverse plane, and $\tau = 1/Z$, then

$$\frac{L}{1/2\,\rho U_o\, S^2(1)} = \frac{\pi i}{2}\, \frac{d^2\nu}{d\tau^2}(0) .$$

Incidentally, the condition that

$$\int_{-1}^{+1} \frac{\partial\bar{\phi}}{\partial\bar{y}}\, d\bar{z} = 0$$

becomes

$$\frac{d\nu}{d\tau}(0) = 0 .$$

The boundary value problem illustrated in Fig. 11 is more easily solved by considering the complex velocity $\nu(Z)$ in a transformed plane,

$$\zeta = \sqrt{1 - Z^2} = \xi + i\eta .$$

The reason for this is that the problem in the Z plane has no symmetry about the plane on part of which the boundary conditions are stated, whereas the ζ plane has been chosen in order that v be symmetric about the ξ plane on the whole of which either w or v are specified. The problem for $\nu(\zeta)$ is illustrated in Fig. 12.

The proper solution to the problem posed in Fig. 12 may be found by inspection. It is

$$\nu(\zeta) = iU_o\alpha - \frac{iU_o\alpha\sqrt{1-\zeta^2}}{\sqrt{\zeta(c-\zeta)}} + \frac{iU_o a}{\pi}\, \frac{\sqrt{1-\zeta^2}}{\sqrt{\zeta(c-\zeta)}} \int_0^c \frac{\sqrt{\zeta_o}\,\sqrt{c-\zeta_o}}{\sqrt{1-\zeta_o^2}\,(\zeta-\zeta_o)}\, d\zeta_o .$$

The first two terms on the right represent the appropriate flow past the two plates A - BB′ and CC′ - D having zero horizontal velocity on that part of the ξ axis outside of the plates. The third and most complicated term on the right represents a distribution between BB′ and CC′ of elemental vortices of strength $2U_o a$, each in the presence of the two plates (i.e., having null vertical velocities in that part of the ξ axis where the plates exist).

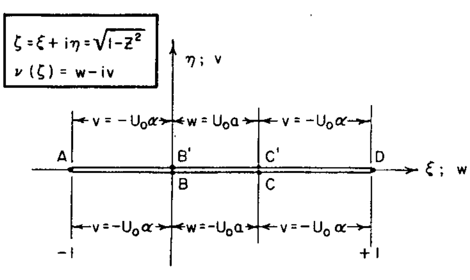


Fig. 12 - Boundary value problem in transformed (ζ) plane

The two parameters a and c yet remain to be simultaneously determined. Very complicated quadratures unfortunately appear, especially in the evaluation of $\bar{\varphi}$ at the wing leading edge. For the purpose of this paper we shall be satisfied with having described the appearance of these conical flows and with certain approximate results for the cases of very high ($\sigma/\alpha\beta >> 1$) and very low ($\sigma/\alpha\beta << 1$) cavitation numbers. These cases correspond, respectively, to the cases of very short leading edge cavities and cavities covering the entire delta upper surface except in the immediate neighborhood of the wing centerline.

($\sigma/\alpha\beta >> 1$). The following asymptotic results for high cavitation number regime may be derived making use of various conditions and results previously given herein. The tedious calculations are omitted.

The nondimensional width, ℓ, of the cavity measured in the transverse plane from the wing leading edge to the inboard end of the cavity is

$$\ell \sim 8 \left(\frac{\beta\alpha}{\sigma}\right)^2 .$$

The lift coefficient for a delta is increasingly reduced by cavitation and equals

$$\frac{L}{1/2\ \rho U_o^2 S^2(1)} = \frac{\pi\alpha}{2}\ (1 - \ell) = \frac{\pi\alpha}{2} \left(1 - \frac{8\beta^2 \alpha^2}{\sigma^2}\right).$$

As the region of cavitation vanishes ($\beta\alpha/\sigma \rightarrow 0$) the lift approaches, as it should, that of a fully wetted delta wing according to linear slender body theory.

It is to be noted that the effect of cavitation is to reduce the lift, but that the effect only appeared in terms of the order of α^3. It is well known that other nonlinear effects associated with separated vortices (which may exist above the top of the wing) will alter the lift to the order of α^2; these effects serve to increase rather than decrease the lift on the wing. Their nature will certainly be affected by the possibility of cavitation, but their existence can probably not be ignored in any proper theory. This fact should be kept in mind when comparing the present theory (without separated vortices) with actual experiments. In particular, the reduction of linear lift through cavitation may not be nearly as severe for small angles of attack as the present theory predicts - due to the compensating effect of separated vortices which tend to increase the lift above the linear prediction.

According to present theory, the transverse velocity on short bubbles is, as might have been otherwise deduced, simply related to α and β:

$$a = \frac{\sigma}{\beta} \qquad \left(\frac{\sigma}{\alpha\beta} >> 1\right).$$

($\sigma/\alpha\beta << 1$). As the cavitation number is decreased the cavities cover more and more of the wing, but always with a re-entrant flow along the centerline. Whether this re-entrant jet actually wets the wing or is somehow dissipated before doing so is problematical, but, at any rate, almost all of the top surface of the wing will be at the pressure in the cavity. The transverse velocity on the very wide bubbles depends only on the wing angle of attack:

$$a = \pi\alpha \qquad \left(\frac{\sigma}{\alpha\beta} << 1\right).$$

The lift for this case of asymptotically small cavitation numbers is related to the cavity width:

$$C_L = \frac{L}{1/2\ \rho U_o^2 S^2(1)} \approx \frac{1}{5}\pi\alpha\ (1 + 3\lambda^2)$$

where, it will be remembered, 2λ is the width of the wetted portion of the wing top - or to look at it in another way, the width of the re-entrant jet. The actual relationship between λ and $\sigma/\alpha\beta$ is of great interest, but has not been obtained here. Presumably λ becomes very small as $\sigma/\alpha\beta \rightarrow 0$, with the implication that at $\sigma = 0$ a wing of given geometry develops approximately 4/10 of its fully wetted linear lift. Full cavitation is then just a little more effective in reducing lift than is planing; it may be recalled (14) that the lift of a slender delta surface planing on smooth water is

$$\frac{\pi \rho U_o^2 \beta^2 \alpha}{8}$$

just one-half of its fully wetted linear lift.

The drag, D, of the cavitating delta is simply $L \cdot \alpha$, i.e.,

$$D(\sigma = 0) \approx \frac{\pi \rho U_o^2}{10} \cdot \beta^2 \alpha^2 .$$

It may be shown that one-half of this drag is associated with the kinetic energy in the vortex wake (induced drag), the other half, then, being due to cavitation.

Generally speaking, the cavitation drag increases with cavity volume; thus the problem of minimizing cavitation drag for a given lift is equivalent to the problem of minimizing the volume of the cavity. Even without deriving a theory for the supercavitating flow past a more general slender wing it is clear that the same sort of planforms and cambers which reduce spray drag for slender planing surfaces will be effective in the present case. Thus, reduced cavity sizes and drags (relative to the uncambered delta) should be obtained with surfaces whose planform curvature is concentrated on the forward portion of the surface, and whose (positive) camber is concentrated aft; for example, a surface whose planform shape is of the form $s = ax^{1/2}$ and whose bottom surface shape is $y = bx^3$.

The results described above which relate the lift on the delta wing to the cavity width are displayed in Fig. 13. Except in the case of very high cavitation numbers ($\sigma/\alpha\beta >> 1$) the relation between cavity width and other physical parameters has not been given. In place of an exact theoretical prediction of the variation of lift on a given delta, only a rough estimate based on the asymptotic results obtained here is presented in Fig. 14. The highly nonlinear behavior of the lift as a function of angle of attack is especially to be noted; it is, of course, a consequence of the widening of the cavities as the angle of attack increases, the cavitation number being fixed.

It is to be hoped that the near future will provide experimental results for slender supercavitating wings such as the deltas considered here, with the consequence that our understanding of these complex and interesting flows may be very much broadened.

SUMMARY

The three separate topics discussed here concern the cavity flow past smooth two-dimensional struts with unspecified cavity detachment points; effects of high speed on cavity flows past blunt struts; and the three-dimensional cavity flow past slender delta wings. The use of linearized theory is emphasized.

The dependence of cavity detachment point on strut shape and cavitation number (σ) is studied. It is shown that for small σ the detachment point motion is rearward

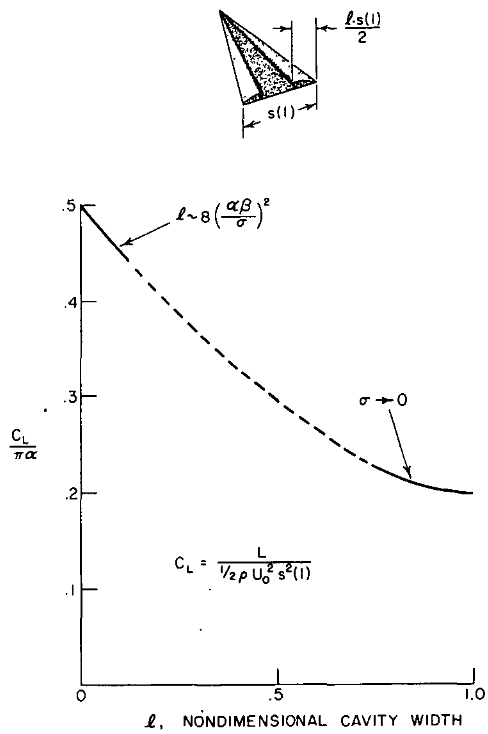


Fig. 13 - Lift on cavitating delta wing as a function of cavity width

and of the order σ^2. The drag of a family of struts is calculated and the dependence of resistance on strut shape discussed. Some interesting indications concerning the non-uniqueness of the Helmholtz flow past closed struts are revealed.

A Prandtl-Glauert rule for the estimation of Mach number effects on cavity flows past blunt struts is derived. For infinite cavities ($\sigma = 0$) it is shown that the cavity drag varies inversely as $\sqrt{1 - M^2}$. For finite cavities the length is shown to vary inversely as $(1 - M^2)$, while the maximum cavity breadth varies inversely as $\sqrt{1 - M^2}$. The situation for transonic speeds is very briefly discussed.

The flow past slender delta wings is studied under the assumption that it may be considered conical. Theory is developed for a flow involving cavities which spring from the leading edges and cover a part of the top of the wing. Results are obtained for the two separate asymptotic cases in which the cavitation number is either very small or very large. The width of the cavities on the wing upper surface increases with decreasing σ and the lift decreases. It is shown that the upper surface never becomes completely enveloped in a cavity, even for $\sigma = 0$. Finally, the lift of a fully cavitated wing ($\sigma = 0$) is estimated to be approximately 4/10 of its fully wetted lift.

$$C_L = \frac{L}{\frac{1}{2}\rho U_0^2 s^2(1)}$$

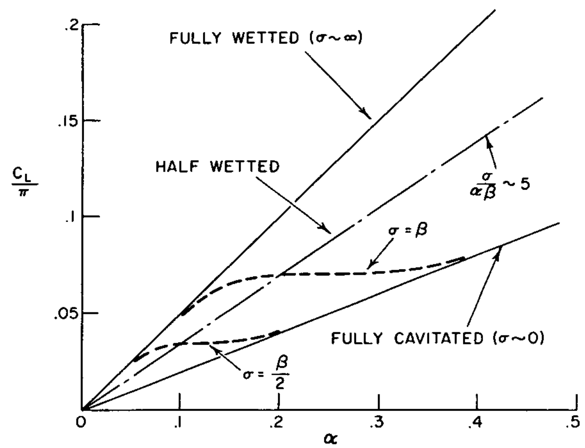


Fig. 14 - Estimated lift on cavitating delta

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DISCUSSION

E. H. Handler (U. S. N. Bureau of Aeronautics)

As a consequence of the work by Marshall Tulin of the Office of Naval Research and Virgil Johnson of the National Advisory Committee for Aeronautics, the Bureau of Aeronautics has awarded a contract for the design, construction, installation, and evaluation of a Gruenberg, supercavitating, hydrofoil system mounted on a JRF-5 seaplane. While the hydrofoil boat operates at essentially a fixed trim angle, the hydrofoil airplane must trim up many degrees during the actual process of take-off or alighting. The Gruenberg system, with the foil just aft of the center of gravity, is ideal for seaplane use. The forward planing, or feeler, skis are a low-speed stabilizing device and a high-speed safety device.

The JRF-5 airplane will fly within a year and the results will be published by the Bureau of Aeronautics.

E. Müller (Max-Plank Institüt für Strömungsforschung)

The Max-Plank Institüt für Strömungsforschung is currently performing experiments with delta wings for cavitation numbers ranging from 0.02 to 0.13. The flat delta wing with an apex angle of 60 degrees has been tested, and drag and lift were measured as functions of the angle of attack and cavitation number. The shape of the wing is quite similar to the shape of the bodies assumed by Mr. Tulin but, as he has said, his work is for an apex angle of not more than 30 degrees, while our work is for an apex angle of 60 degrees. Thus, the results cannot be compared, but experiments with the same bodies as were considered by Mr. Tulin will be made in the near future.

L. M. Milne-Thomson (Brown University)

As Mr. Tulin observes, no account is taken in his paper of viscosity or gravity. In regard to the effect of gravity, I would like to indicate a method of approach which, as far as I know, has been overlooked in the literature.

Consider an inviscid liquid in steady irrotational flow under gravity. Fritz John* has given the following equation for a surface of constant pressure, assuming the y-axis to be vertically upwards:

$$z''(\beta) + ig = iS(\beta)\, z'(\beta).$$

Here β is a certain parameter and $S(\beta)$ is an arbitrary real function. To each specific problem there belongs a specific function $S(\beta)$ which, if we could divine its form, would solve the problem. The integration of the above equation can be effected by finding a complementary function and a particular integral. This observation would be completely trivial, were it not for the fact that in many problems we know the complementary function z_0 in advance. Take the case of a jet from an orifice in a flat plate in the absence of gravity—a well-known Helmholtz problem. Here

*Communs. Pure and Appl. Math. 6:497-503 (1953).

$$z_0''(\beta) = iS(\beta)\, z_0'(\beta)\,,$$

which determines $S(\beta)$ insofar as $S(\beta)$ is independent of g in the gravity problem. This is a matter for investigation in each case.

If we now regard β as a function of an auxiliary parameter, θ, we can integrate the full equation in the form

$$z(\theta) = z_0(\theta) + g[f(\theta) - f(\alpha)]\,,$$

where $f(\theta)$ is a determinate function and $\theta = \alpha$ at the edge of the orifice. This equation reduces to $z_0(\theta)$ when $g = 0$, and gives

$$z(\alpha) = z_0(\alpha)$$

at the edge of the orifice. It remains to see whether the condition at infinity is satisfied.

Thus, in the case of an orifice of breadth $2a$ and a vertically downwards jet, this condition is satisfied when

$$U^2 = \frac{2\pi a g}{\pi + 2}$$

where U is the skin velocity on the jet in the Helmholtz problem.

W. A. Clayden (Armament Research and Development Establishment)

Concerning Mr. Tulin's work on cavity detachment, we have noticed in our work with cavitating spheres at ARDE that the flow does not separate tangentially to the surface, but separates with a non-zero angle which may be well over 90 degrees. I think that this effect has also been studied by other workers in more detail.

We have also performed some tests on yawed cavitating cones which may be considered to behave in a manner similar to slender delta wings, and the cavities have been kidney-shaped, or distorted ellipses, in cross section before rolling up into a pair of vortices as predicted by Mr. Tulin's calculations.

I should like to take a few minutes to tell you of the work on cavitating wedges that has been done at ARDE by my colleague, A. D. Cox. He considered the following two cases:

1. An inclined wedge with non-zero cavitation number. The mathematical model used is the familiar re-entrant jet at the rear of the cavity and a subsidiary leeward cavity. This work is an extension of the zero cavitation number case.

2. A generalized image-model for non-zero cavitation number and zero incidence in which the cavity is closed on another wedge of arbitrary angle. It is shown that the standard image-model of Plesset and Shaffer and the dissipative model of Roshko are special cases of the generalized model. The model may also be applied to include a representation of the familiar jet flow at the rear of the cavity.

W. G. Cornell (General Electric Company)

In regard to the first problem wherein, for a given strut, one is able to find, by Mr. Tulin's method, the detachment point of the cavity and the cavity itself, I have a slight philosophical problem. It seems to me that one is getting something for nothing by obtaining all this information without considering boundary layers and turbulence.

Also, I wish to express great pleasure in seeing this consideration of compressibility effect in cavity flows.

B. R. Parkin (California Institute of Technology)

In connection with Mr. Tulin's two-dimensional solution, I wish to cite some experimental results*† which are quite old, and which are familiar to Mr. Tulin and perhaps to other members of this symposium also. These measurements concern the role of surface tension and viscosity in the inception of cavitation on a smooth symmetrical body.

About five years ago, R. W. Kermeen and I investigated the nature of the separation zone on a blunt, axially-symmetric body just after the cavity flow had become established. Although we did not investigate the flow in detail we did obtain some magnified motion pictures and still photographs which showed the shape of the small cavities near their point of attachment on the body. These photographs indicate that such cavities originate in the boundary layer. They are shaped somewhat as shown in the schematic diagram of Fig. D1.

The boundary-layer and displacement thicknesses are shown in Fig. D1 so that we may judge the effect of the changing velocity of the water in the boundary layer upon the shape of the liquid-vapor interface. The water has zero velocity on the body and its velocity increases continuously as one progresses from the surface of the body to the main flow. From the shape of the free streamline in the neighborhood of the cavity leading edge it is evident that the flow does not separate from the body in accordance with the assumption commonly employed in potential problems which involve flow separation. In view of the fact that this difference is found in the boundary layer flow on the body, even for those cases in which very large cavities are attached to the body, its effect upon the overall flow should be small, once the location of the separation point is established. Therefore it does seem that Mr. Tulin's solution should give a valid picture of the flow in the large when the point of separation corresponds closely to that which one would observe in practice. Because of the physical evidence discussed above one might ask if Mr. Tulin's theoretical solution, although mathematically determinate, must always give results which correspond to those obtained in an actual flow; might not there still be some ambiguities remaining to be resolved? If one were to adopt the most pessimistic view, as a believer in cause and effect, he might even suppose that it would be necessary to trace the whole history of the inception process in order to fix the position of a separation point on the body. One may certainly hope that less-stringent measures would be required.

*B. Parkin and R. Kermeen, "Incipient Cavitation and Boundary Layer Interaction on a Streamlined Body," California Institute of Technology Hydrodynamics Laboratory Report E-35.2, Dec. 1953.

†Kermeen, McGraw and Parkin, "Mechanism of Cavitation Inception and the Related Scale-Effects Problem," Trans. ASME 77(No. 4):533 (May 1955).

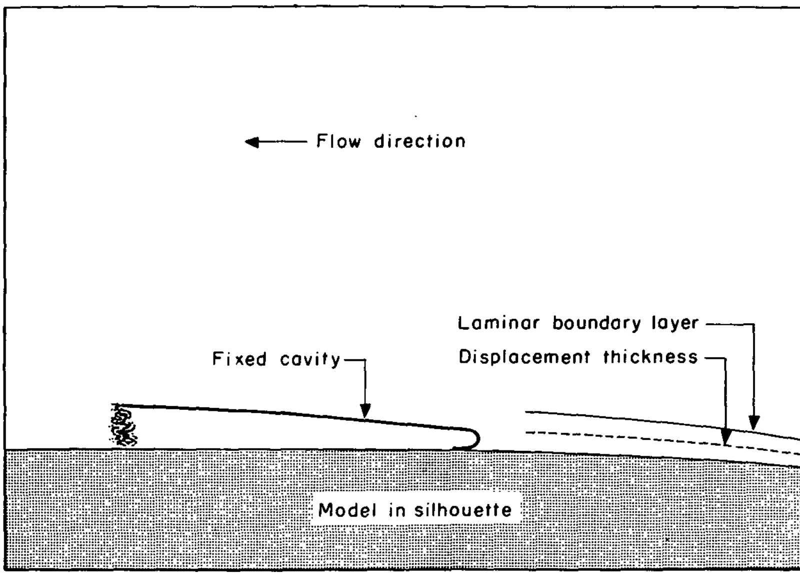

Fig. D1 - Vapor cavity attached to the surface of a smooth body

M. Tulin

Mr. Cornell's philosophical problem is old as far as potential flow theory is concerned. One can read about previous attacks on the problem within the framework of potential flow theory, ignoring any effects of viscosity and surface tension. The problem has been formulated, more-or-less, in the same way as in my linearized version, and in a few cases it has been solved. There is a very good discussion of the problem by Birkhoff and Zarantonello. I did touch on the question of the uniqueness of these flows. The sufficiency of the condition of smooth detachment is also discussed. There is no conclusive statement to be made about it, but I think this condition of smooth detachment is sufficient for the bodies that have been tried. It gives one a meaningful problem; I have only linearized its formulation.

Now, I should have said a few words about the possibility that this may be fantasy, because we just don't know enough about the role played by viscosity and surface tension in disturbing the detachment point. This is a very good problem. The way I might wiggle out of it is this: I try to show how, using linearized theory and the usual linear condition for an ideal fluid, one can solve the problem for slender bodies.

When we know enough about the effects of viscosity or surface tension, then we still presumably must solve some potential flow problems. Perhaps they will be something like Lighthill's problem, where he achieves a matching between the boundary layer behavior ahead of the plate on the body and the constant pressure flow. If so, then one still needs to solve a potential flow problem to calculate the pressures on the body and determine where the boundary layer separates, or where viscosity or surface tension is involved, and include its effects.

This is definitely a subject for future research, both experimental and theoretical.

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