PRESSURE WAVES FROM COLLAPSING CAVITIES

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The topic of this paper relates to the practical problem of noise generated in the collapse phase of cavitation. The account is mainly theoretical; but a brief reference is made to some recent experiments on isolated vapour bubbles. The compressibility of water is pointed out to be an essential consideration in this aspect of cavitation; and a certain property of observed cavitation noise spectra is interpreted as evidence of the presence of shock waves. A method is demonstrated whereby a theory of the collapse of cavities in compressible liquids can in principle be developed to an arbitrary degree of accuracy; and the second-order approximation is presented as illustration. This approximation is definite in the sense that all second-order effects are correctly represented, and it reveals an error in the Kirkwood-Bethe hypothesis on which much previous work on this problem has been based. The second-order theory is shown to admit the possibility that, as a consequence of compressibility of the liquid, the collapse of an empty cavity may be concluded at a finite radial velocity. However, even though this possibility remains when higher approximations are worked out, the author now considers it to be unlikely. Some very recent ideas on this matter, which have been developed since the first draft of this paper was written, are summarized in an appendix.

A feature of the present theory is that it accounts in a straightforward manner for nonlinear distortion of the transmitted pressure wave. The conditions under which the wave radiated from a pulsating gas-filled cavity will develop a shock front are examined; and a simple practical criterion for shock formation is derived.

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1. INTRODUCTION

Notwithstanding the variety of means by which transient cavitation may be generated in liquids, one can in general identify three distinct phases in the history of any individual cavity. The sequence begins with its initiation from a microscopic nucleus and a brief period during which the influence of the nucleus contents remains significant. The cavity then grows to observable size, and behaves almost exactly as if it were completely empty and in an incompressible inviscid liquid. Finally, as the formative agency is removed, it collapses back to microscopic size and may then disappear or perhaps "rebound." It is in this third phase that the more remarkable effects of cavitation arise. If the cavity contains only vapour or a very small quantity of gas, the collapse even under a very moderate hydrostatic pressure takes place with great violence: in the concluding stages enormous velocities and accelerations are attained by the inrushing wall, and a large transient pressure is transmitted to distant parts of the liquid. The abruptness of such pressure pulses accounts for the sharp crackling sound which is often audible from severe cavitation, as may for instance occur in the draught-tube of a hydraulic turbine; and also much of the noise energy is contained in the ultrasonic range of the frequency spectrum. The practical importance of cavitation noise in the context of Naval Hydrodynamics need scarcely be pointed out here: it is sufficient to recall that the spectral properties of noise from severe propeller cavitation make it quite distinct from other forms of underwater noise. We may also note that the cavitation erosion is a consequence of the large pressures developed in the collapse phase.

These remarks merely rephrase some familiar ideas. To anyone interested in cavitation, the violent action of a contracting cavity is likely to form a vivid intuitive concept; and it is easy to appreciate in a qualitative way how cavitation noise pulses arise. However, this third phase of the cavitation process is by far the least amenable to precise experimental or theoretical study. On the experimental side, the extreme rapidity of the vital events makes adequate observation difficult, and a further source of difficulty is the minute size of cavitation bubbles at the stage having most interest. The theoretical side of the problem is even more formidable. Towards the end of the collapse many factors become important which can safely be assumed to be insignificant when the cavity is fully grown; and to account for them comprehensively would seem a hopeless task. Nevertheless, as in many other physical problems, a lot may be learned from even severely simplified theoretical models. The effect of compressibility of the liquid is the factor probably worth closest study, and the theoretical model of an empty spherical cavity contracting under constant pressure in a compressible liquid appears to be a primary key towards physical understanding of cavitation collapse. Other factors arising in the final stages include viscosity, in stability of the spherical form, and compression of the enclosed vapour due to delayed heat exchange with the liquid.

This paper reports some current researches on the collapse of cavities and the resulting pressure waves. The main topic considered is the effect of compressibility on the radial flow, and some new results will be summarized concerning the motion of an empty cavity and the formation of shock waves. No attempt is made to survey all that is known about this phase of cavitation; several excellent reviews of the literature are available (for instance that by Fitzpatrick and Strasberg (1)), and we may turn to these for background. Rather, the aim of this paper is to explain the objects of work still in progress and to offer some ideas about aspects of the problem which remain debatable.

Although a passing reference will be made later to some recent results from the author's experiments on isolated cavities (2), the material of this paper is mainly theoretical. This choice of subject matter reflects a belief that in the present state of the subject the greatest need is for further clarification on the theoretical side. The general physical characteristics of cavitation noise appear now to be fairly well established: for instance, there is plentiful experimental evidence that shock waves are always present when the cavitation is reasonably severe, and many measurements of cavitation noise spectra appear to comply with a well-defined general picture. This state of affairs owes notably to the careful experimental work of Mellen (3,4). On the other hand, a good deal of existing theoretical work seems open to doubt. It is not of course disputed that perfectly sound qualitative descriptions have been given of such effects as shock-wave formation (e.g., see (1)). Further, a certain measure of success has undoubtedly been achieved in predicting the motion of a cavity over certain ranges of the radial velocity. But it remains that the theory of collapsing cavities, taking account of the compressibility of the liquid, has yet to be put on a secure foundation, so that results of a definite character can be calculated.

The valuable theoretical work of Gilmore (5), Mellen (4), and Flynn (6) has in the past gone furthest towards the solution of the present problem. All three of these authors developed their theories from the assumption, which recalls the Kirkwood-Bethe hypothesis in the theory of blast waves, that in radial flow the quantity $r\,\partial\phi/\partial t$ (where r is the radius and ϕ the velocity potential) is propagated approximately unchanged along a characteristic. The Kirkwood-Bethe hypothesis is accurate for spherical disturbances whose wavelength is small compared with radius (e.g., a blast wave in its initial stages), owing mainly to the fact that the motion is then approximately the same as in a plane wave. Again, it is accurate for weak spherical disturbances far from the centre. These facts provide some justification for the use of this assumption in the cavitation problem, when even the fluid velocities are comparable with the velocity of sound; and some further justification may be found by comparing results calculated on this basis with numerical solutions of the equations of motion (5). However, a more cautious assessment shows the assumption to be far from as well founded as one might wish. It would seem in general to represent merely a correction to acoustic theory and does not appear even to be an accurate second-order approximation (in the sense that incompressible-fluid theory comprises a zeroth-order and acoustic theory a first-order approximation). Nevertheless, although errors in the Kirkwood-Bethe hypothesis are easily detected, it is a quite different matter to assess the accuracy of complicated calculations based on it. The principal aim here is only to emphasize the need for further study, and to suggest alternative ways of dealing with the problem.

In this paper a method is given by means of which the theory of collapsing cavities in a compressible liquid can be worked out without recourse to any special hypothesis. The method is one of successive approximations, and can in principle be carried through to an indefinitely high degree of accuracy—although of course the work becomes more laborious in successive stages. The second-order approximation will be given explicitly: this is a formally correct approximation in that all possible second-order effects are properly included; and there appears to be a discrepancy between this and the theory based on the Kirkwood-Bethe hypothesis. The method is an example of Lighthill's celebrated "technique for rendering approximate solutions to physical problems uniformly valid" (7), and has previously been applied by Whitham (8,9,10) to several problems of spherically symmetric flow. Readers will recall that Lighthill's technique obviates a type of difficulty which often arises when the solution of a physical problem is attempted by a conventional perturbation method, whereby successive approximations are obtained in powers of a suitable small parameter. If the zeroth-order solution has a singularity within the domain of interest, progressively worse singularities appear in the higher-order solutions. The principle of the technique is to expand the independent variables as well as the dependent variable in terms of the small parameter, the expansion of the independent variables being evaluated term by term in the process of determining the dependent variable.

In the present application, successive approximations are made to the "characteristic" variable for outgoing spherical waves. This closely follows the work of Whitham cited above. Although the method provides a uniformly valid approximation over the whole flow field, this particular advantage may not be specially worth while in certain cases. Dr. Ian Proudman at Cambridge has recently investigated the collapse of an empty cavity by using a Lighthill-Whitham type of approximation for the "outfield," that is the region far from the cavity, yet using a more conventional perturbation method for the "infield" adjoining the cavity wall. This course apparently leads to a considerable simplification of the work in getting higher approximations. Dr. Proudman has carried his calculations to terms involving the fourth power of the Mach number at the cavity wall, and has accordingly found fairly strong evidence that the method of approximation is convergent even when a substantial part of the flow is supersonic (although the condition of unit Mach number does not appear to have any particular physical significance in this problem).

Now, the first object of most theoretical work on the present problem is the derivation of an ordinary differential equation for the radius R of a spherical cavity as a function of time. The equation will of course be only approximate, and if the cavity is taken to be empty so that the collapse proceeds to completion, there is necessarily some uncertainty in deciding from the equation how the velocity dR/dt behaves in the limit as $R \rightarrow 0$. This uncertainty derives from the fact that although the behaviour $dR/dt \rightarrow -\infty$ is always indicated as a possibility, the method of approximation leading to the equation for R breaks down for very large velocities. (We recall that the terminal velocity is definitely infinite according to Rayleigh's theory (11) which neglected the compressibility of the liquid.) However, on the basis of the second-order theory presented below, it will be shown that another possibility, which is perfectly consistent with the overall approximation, is that dR/dt has a finite value as $R \rightarrow 0$, a result which apparently has not been noted before. This remains a consistent interpretation on the basis of the more accurate calculations worked out by Dr. Proudman—which seem to give by far the most complete analytical results yet available; and since the apparent terminal velocity is not especially large, being not much greater than the velocity of sound in the undisturbed liquid, there would appear to be reasonably strong evidence in favour of this conclusion. (It may be noted that previously obtained approximations to the equation for R, such as Gilmore's (5), also admit this conclusion when the solution is appropriately interpreted.) At the time of writing the first draft of this paper, the author was strongly inclined to the view that an empty cavity does have a finite velocity of collapse; but upon further reflection this conclusion now appears doubtful. The main reason for this change of mind was a numerical calculation performed with the EDSAC II computer. The results indicate that the velocity of collapse of an empty cavity increases indefinitely; and although this indication is not yet conclusive, since there is still some slight uncertainty about the control of errors during the computation, the need has obviously arisen for thinking again. Some recent considerations concerning the asymptotic behaviour of a collapsing empty cavity are summarized in the Appendix to this paper; work now in progress along these lines seems to be leading to a final clarification of the interesting theoretical question of how the collapse is concluded.

Nevertheless, though this question may at first sight seem crucial and the idea of a finite terminal velocity may now seem a possible serious misinterpretation from the point of view of the physical application of the theory, the matter is after all not particularly important physically. The more important conclusion is that in any case the energy of the motion near the centre towards the end of the collapse becomes vanishingly small; the greater part of the overall energy has already been stored in a compressive wave well before the collapse is terminated, and so what happens in practice very close to the centre may well have an insignificant effect on the wave as observed from a distance. This conclusion is important since it implies that there is often justification for neglecting the additional physical factors arising near the end of a collapse, such as viscosity and vapour compression.

An advantage of the theory to be presented is that it provides a very simple description of shock-wave formation by rebounding gas-filled cavitation bubbles, and leads easily to an estimate of how severe a collapse need be for a shock wave to develop (mild "gassy" cavitation may of course still give rise to brief noise pulses;

but a typical pressure wave will not have a shock front). The generation of shock waves is a feature of cavitation which has practical importance in several ways, notably in relation to cavitation damage. In the next section of the paper we digress slightly from the main discussion to comment on another effect attributable to this cause.

2. THE EFFECT OF THE PRESENCE OF SHOCK WAVES ON CAVITATION NOISE SPECTRA

Discounting experiments where isolated cavities are generated by special means (4,12,13), cavitation is in most circumstances an erratic process during which a large number of transient cavities appear in scattered positions throughout the region under reduced pressure. The behaviour of individual cavities with an irregular assembly can sometimes be observed, as was done by Knapp and Hollander (14); but in measurements of the radiated noise, the contribution from any particular cavity is generally impossible to identify. Determination of the frequency spectrum is then the most useful object of noise measurements, and reasons can be found for supposing that the spectrum of the aggregate of the received pressure waves is not widely discrepant from the spectrum of a typical component pulse (we note that the spectrum of a random sequence of similar pulses has the same form as the spectrum of a single pulse). Thus, theoretical spectra calculated on the basis of single-cavity models may be relevant to observed cavitation noise (3); and indeed a fair measure of agreement has established (1).

A usual feature of measured energy spectra is a slope of approximately -6 db/octave at high frequencies, i.e., the spectral density varies as f^{-2}, where f is frequency. It has often been pointed out that this feature can be explained by the presence of shock waves in the received pressure signals. A shock front in water would appear an abrupt discontinuity in pressure unless it could be observed on a time scale much less than the order of 1 microsecond; and this apparent discontinuity would therefore exert a dominant influence on the high-frequency end of the spectrum well into the megacycle range. The discontinuity need not have the familiar form of a fully developed shock (e.g., the blast wave from an explosion) for it to affect the spectrum in the required way: the only essential property is that there should be a finite discontinuity in some part of the wave form.

Although this explanation of the asymptotic form of the spectrum appears to be fairly widely known, a mathematical demonstration of it seems worth giving here since this has apparently not been done before. It is instructive to consider the special case of f^{-2} dependence as part of a general classification, and indeed this is the only way of showing that the presence of shocks is necessary, not merely sufficient, to account for this case in practice. We restrict the argument to a single noise pulse possessing only one singularity: the extension of the argument to a random sequence of identical pulses is quite straightforward; and although an extension to a more general type of random noise would be difficult to make with complete rigour, it seems clear intuitively that the spectral properties due to an isolated singularity, as considered below, are common to any noise in which singularities of the sort in question occur at random.

Suppose that $p(t)$ is a typical noise pulse with finite duration, as for instance detected by a hydrophone. Its energy spectrum may be defined as

$$S(f) = |g(f)|^2 ,$$

where

$$g(f) = \int_{-\infty}^{\infty} p(t)e^{-2\pi i f t}\, dt. \tag{2.1}$$

If desired, some convenient normalization factor can be introduced into the definition; but this is unnecessary for the purpose of the present argument.

We suppose $p(t)$ to be always finite, although there is a singularity of its derivatives at, say, $t = 0$. A large class of such singularities is characterized by the behaviour $p(t) = O(|t|^{\nu})$ as $t \to 0$, where $\nu > 0$ but is not an integer; thus, if $(n - 1) < \nu < n$ where $n = 1, 2, 3 \ldots$, then the nth and higher derivatives of $p(t)$ tend to infinity as $t \to 0$. (For complete generality we would need also to consider cases where $p(t)$ is an odd function in the neighbourhood of $t = 0$, and where the singularity is logarithmic, but this would be a pointless digression here.) To find the asymptotic form of $S(f)$ in this case, the most satisfactory course is to make use of a theorem given by Bromwich (15). It is sufficient here to quote the final result: we find that $S(f) \sim O\{f^{-2(\nu+1)}\}$ as $|f| \to \infty$. Thus, all possible asymptotic slopes of the spectrum (on a logarithmic plot) are covered except the cases where the dependence of $g(f)$ is $f^{-1}, f^{-2}, f^{-3} \ldots$.

The latter cases are more interesting to us than the intermediate ones. They are given by functions of t which undergo abrupt discontinuities or whose derivatives do. To deal with singularities of this type, the following method is quite lucid though to an extent only intuitive. For further detail we may refer to various treatises dealing with asymptotic expansions.

The case which directly concerns us, as it represents a shock front, is where $p(t)$ changes abruptly at $t = 0$ from a value $p(0^-)$ to another value $p(0^+)$: here 0^- and 0^+ indicate arbitrarily small negative and positive value of t, respectively. All the derivatives of $p(t)$ are infinite—or rather, meaningless—at $t = 0$; but it can be assumed that the derivatives are integrable over the whole range $(-\infty, \infty)$. Hence, integration of Eq. (2.1) by parts gives directly

$$g(f) = \frac{p(0^+) - p(0^-)}{2\pi i f} + \frac{1}{2\pi i f} \int_{-\infty}^{\infty} \frac{dp}{dt} e^{-2\pi i f t} \, dt. \tag{2.2}$$

According to the Riemann-Lebesque theorem, the integral in Eq. (2.2) tends to zero as $|f| \to \infty$. Therefore, if the step in $p(t)$ is of finite height (i.e., $p(0^-) \neq p(0^+)$), we have that

$$|g(f)| \sim O(f^{-1}) \quad \text{as} \quad |f| \to \infty. \tag{2.3}$$

Hence, $S(f)$ decreases asymptotically at the rate of $10 \log_{10}(2^{-2}) = -6.02$ db/octave. It is now perhaps obvious that no function other than one with a finite discontinuity like a shock wave has a spectrum with this asymptotic property.

Incidentally, an extension of the above line of argument, wherein the integral on the right-hand side of Eq. (2.2) is reduced by successive integrations by parts, leads to the following rule: if $p, p', p'' \ldots p^{(n-1)}$ are all finite and continuous, and if $p^{(n)}$ is finite yet discontinuous, then $|g(f)| \sim O(f^{-n-1})$ as $|f| \to \infty$. It follows from this that if $p(t)$ and all its derivatives are continuous, $S(f)$ must decrease asymptotically more rapidly than any negative power of f, i.e., it must decrease at least in exponential fashion. Thus, even when shocks are present in cavitation noise, the spectrum must ultimately decrease exponentially when the frequency is so high that its reciprocal is comparable with the time of passage of a shock front, so that on such a time scale the shock no longer appears discontinuous.

We also note that if the signal $p(t)$ is passed through a recording system which is resonant at $f = f_0$, the "apparent" spectral response (i.e., the spectrum of the forced vibration) is of the form $|g_a(f)| = |g(f)| f/\{2\pi(f^2 - f_o^2)\}$. Therefore, at frequencies much above f_o, the "apparent" energy spectrum for a discontinuous signal has the property

$$S_a(f) \sim |g(f)|^2 \times O(f^{-2}) \sim O(f^{-4}) . \tag{2.4}$$

That is, the spectrum is sustained at a slope of -12 db/octave. The author has remarked in the past (16) that this may be a useful consideration regarding the response to cavitation noise of practical hydrophones, which are always subject to self-resonances at a number of frequencies.

3. SOME POINTS FROM THE THEORY OF CAVITATION IN INCOMPRESSIBLE LIQUIDS

Although the main interest at present is in effects due to the compressibility of water, it will be helpful to fix our ideas by recalling some simple results according to incompressible-flow theory. In awareness of the severe difficulties arising when compressibility is taken into account, problems of radial bubble motion in an incompressible liquid appear on the whole refreshingly straightforward; of course they are not always easy, particularly as the equations of bubble motion are non-linear; but, generally speaking, treatments of these problems are clear-cut and unequivocal. The subject is well-covered by the contributions of Rayleigh (11), Lamb (17,18), Cole (19), Plessett (20), Noltingk and Neppiras (21), Robinson and Buchanan (22) and numerous others.

Consider first an empty spherical cavity of radius $R(t)$ collapsing to zero volume in a large expanse of incompressible fluid. In the later stages of the collapse, the motion becomes insensitive both to the way it was started and to the (hydrostatic) pressure P at a large distance from the centre. The only crucial parameter is the total energy E of the motion, which may be assumed to be approximately constant. Note that if the cavity starts contracting with a radius R_o and if P is a constant P_o, then obviously $E = 4/3\ \pi P_o(R_o^3 - R^3)$ which tends to a constant value as $R \to 0$. If P varies during the initial stages of the motion, E will vary in a rather more complicated fashion, but the final stages occur so rapidly that E will again be practically constant towards the end. Putting the kinetic energy of the fluid equal to E, we have

$$2\pi\rho R^3 \dot{R}^2 = E , \tag{3.1}$$

where $\dot{R}$ denotes dR/dt and ρ the density. This equation has the integral

$$R = \left(\frac{25\ E}{8\pi\rho}\right)^{\frac{1}{5}} (-t)^{\frac{2}{5}} \tag{3.2}$$

on the assumption that the collapse is completed at $t = 0$. This gives the well-known result that $-\dot{R} \to O(-t)^{-3/5}$ as $R \to 0$. The velocity of the cavity wall thus becomes infinite in the limit. Further, the pressure throughout the fluid becomes momentarily infinite, but the velocity at a fixed point is zero at the instant of collapse, being proportional to $R^2\dot{R}$. Thus, in the limit as $R \to 0$, the energy of the motion becomes concentrated wholly at the centre.

It can easily be shown that the pressure p at a radius r from the centre is given by

$$p = P\left[1 - \frac{R}{r}\right] + \frac{1}{2}\rho\dot{R}^2\left[\frac{R}{r} - \frac{R^4}{r^4}\right] . \tag{3.3}$$

This expression demonstrates some features of the pressure distribution which have particular interest in the present context. Note first that $p = 0$ at $r = 0$, which is consistent with the assumption of an empty cavity. Next note that in the later stages of the collapse, P is negligibly small compared with $\rho\dot{R}^2$; and so the term in P on the right-hand side of Eq. (3.3) can be neglected in any region not too far from the cavity. Thus, the pressure is seen to rise from zero at the cavity wall to a maximum value $3 \times 4^{-11/6}\rho\dot{R}^2$ at $r = 4^{1/3}R$, and to decrease steadily for greater r. Figure 1 illustrates changes in the pressure distribution as R decreases. The vertical and horizontal scales are arbitrary, but the figure is easily interpreted. The three curves represent the pressures when, respectively, the cavity radius is 1, 1.5, and 2 in arbitrary units. Since $\dot{R}^2$ is proportional to R^{-3}, the maximum pressure in the last case is eight times that in the first. The figure shows clearly how towards the end of the collapse the pressure increases very rapidly in magnitude and its distribution becomes less dispersed.

Now, if we allow that behaviour at least qualitatively similar to this may occur in a compressible liquid, an interesting possibility is suggested. Positive pressure distributions of the kind illustrated in Fig. 1 suffer a tendency to steepen in the direction away from the centre, and so develop into shock waves. Admittedly, the wave-form is being carried rapidly towards the centre of collapse, and the process of shock formation necessarily occupies a certain time. But it seems at least conceivable that a compressible liquid moving inward to close an empty cavity could develop a shock at some moment before the instant of final collapse, the shock being convected inwards by a supersonic flow. This point will be taken up again in a later part of the discussion.

It scarcely needs to be said that the present simple model of cavitation collapse ceases to be valid well before the cavity disappears. When the velocities and pressures near the cavity become large, the effects of compressibility are no longer

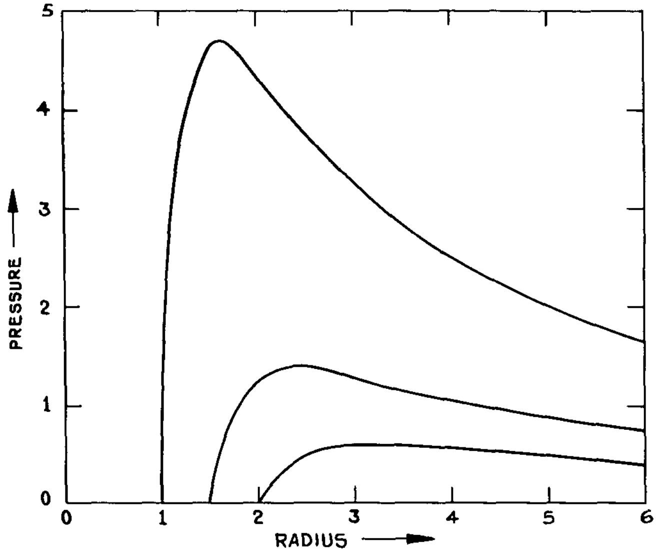

Fig. 1 - Pressure distributions during the collapse of an empty cavity in an incompressible liquid

negligible, and also the assumption of an empty cavity may become unrealistic. However, as Rayleigh (11) was first to point out, the singularity in the incompressible-fluid solution is avoided if the cavity is assumed to enclose a quantity of insoluble gas. The compression of the gas eventually arrests the inward motion, and a "rebound" takes place. It may be shown that as the gas pressure rises, the peak in the pressure distribution illustrated by Fig. 1 is displaced inwards; and ultimately, at the instant when the collapse is halted, the maximum pressure occurs right at the cavity wall.

While serving to remove a mathematical difficulty, the chief merit of this modification is that it represents a definite feature of cavitation in practice (and, incidentally, leads to a very useful account of the pulsations of underwater explosion bubbles (19)). There is plentiful evidence that rebounds often occur (e.g., (13,14)), and presumably they are due to the action of the cavity contents. Of course, the model of a gas-filled cavity surrounded by incompressible liquid still falls far short of reality; for instance, no mechanism for energy dissipation is included, whereas rebounding cavitation bubbles appear always to be considerably smaller than their maximum size before the first collapse. Nevertheless, this model is definitely relevant to cases of fairly "mild" collapse, and gives a useful first approximation to such quantities as the peak noise pressure.

Some numerical results calculated on this basis will be useful for reference later. We consider a cavity initially at rest with $R = R_o$ and containing gas at pressure Q. The hydrostatic pressure P is constant and greater than Q, so that the initial motion is inward. The gas is assumed to be compressed according to the law $pV^\gamma = \text{const}$ $(\gamma > 1)$, i.e., its pressure is given by $p = Q(R_o/R)^{3\gamma}$, and its inertia is neglected. The equation of motion of the cavity is expressible in the form

$$\dot{R}^2 = \frac{2P}{3\rho}\left\{\left(\frac{R_o}{R}\right)^3 - 1 - \frac{Q}{(\gamma-1)P}\left[\left(\frac{R_o}{R}\right)^{3\gamma} - \left(\frac{R_o}{R}\right)^3\right]\right\} . \tag{3.4}$$

From this the minimum radius R_m is easily deduced, since it is a root of $\dot{R} = 0$. Assuming that $(R_o/R_m)^3$ is fairly large, we find that, very approximately,

$$R_m/R_o = (1+K)^{-1/(3\gamma-3)} , \tag{3.5}$$

where $K = (\gamma - 1)P/Q$. The maximum gas pressure is therefore

$$P_m = Q(R_o/R_m)^{3\gamma} = Q(1+K)^{\gamma/(\gamma-1)} . \tag{3.6}$$

Also, the peak pressure rise (i.e., above P) at a radius $r(\gg R_m)$ is $P_m R_m/r$. The maximum velocity of the cavity wall is found by differentiation of Eq. (3.4). This occurs when $R = R'$, this radius being given by

$$R'/R_m = \gamma^{1/(3\gamma-3)} , \tag{3.7}$$

and is very approximately

$$\dot{R}_m = (2P/3\rho)^{1/2}\, \gamma^{-\gamma/(2\gamma-2)} . \tag{3.8}$$

Equations similar to (3.5) and (3.6), although rather less accurate, were given by Noltingk and Neppiras (21). Lamb (17) obtained Eq. (3.6) in a related problem concerning underwater explosions, but gave the formula incorrectly; the mistake was reproduced in his well-known book (18).

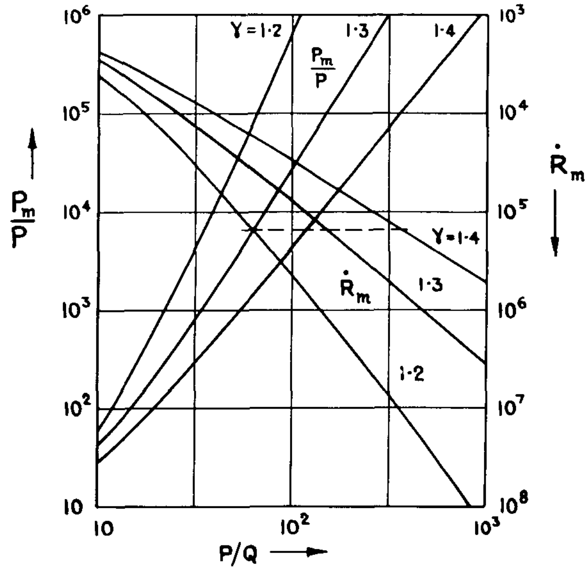

Fig. 2 - Values of the maximum gas pressure and maximum radial velocity occurring during the collapse of a gas-filled cavity in an incompressible liquid

Figure 2 shows graphs of P_m/P vs P/Q for three values of γ. Graphs of the value in cm/sec of $\dot{R}_m$ corresponding to $P = 1$ bar and $\rho = 1$ g/cm^3 are also shown: for other values of P or ρ, $\dot{R}_m$ can be scaled proportionately to $(P/\rho)^{1/2}$. On the graph of $\dot{R}_m$, the dashed line denotes the speed of sound in water at a pressure of 1 bar. Figure 2 illustrates clearly how the maximum pressures generated during a collapse may greatly exceed the hydrostatic pressure. For example, if a cavity contains air at an initial pressure of one one-hundredth the hydrostatic pressure and the air is compressed adiabatically ($\gamma = 1.4$), the air pressure at the end of the collapse is 4420 times the hydrostatic pressure. When we consider that P is of the order of 1 bar in many instances of cavitation, the extent of the magnification of pressure is most impressive. The feature of Fig. 2 which has particular interest later in the discussion is that for $P_m/P = O(10^3)$, the magnitude of $\dot{R}_m$ is still a reasonably small fraction of the speed of sound, so that the error due to neglect of the compressibility of water is probably still fairly small.

4. THE NATURE OF THE PROBLEM FOR COMPRESSIBLE LIQUIDS

To develop a theory on a secure basis, the attempt must be made to solve the exact equations of radial motion in a compressible fluid. We shall now derive a second-order approximation to the effects of compressibility; but, as remarked in Section 1, this is done primarily as an illustration of method and to show how the way is clear to higher approximations.

For spherically symmetric isentropic flow, the velocity potential ϕ satisfies

$$c^2\left(\phi_{rr} + \frac{2}{r}\phi_r\right) - \phi_{tt} = 2\phi_r\,\phi_{rt} + \phi_r^2\phi_{rr}\,, \tag{4.1}$$

where c is the local velocity of sound which in general varies with pressure. (It is worth noting that Lamb (18, § 285) presents this equation in its correct form, but under the unnecessary restriction that c is constant.) As a reasonable first step towards a solution, one may use a linearized (acoustic) theory in which the terms on the right-hand side of Eq. (4.1) are neglected and c is taken to be a constant c_o, the sound velocity in the undisturbed fluid. According to this theory, the characteristic curves in the (r,t)-plane are straight lines, given by $t - r/c_o = \text{const}$ for outgoing waves. But there are two respects in which a linearized theory is inadequate. The first, the obvious one, is that for the very rapid flow near a cavity in the later stages of collapse, the non-linear terms in Eq. (4.1) may have an effect which is by no means negligible. Second, the linearized solution is well known to become progressively less valid at large distances from the origin. Even though the magnitude of φ decreases steadily (like r^{-1}), the effect of the non-linear terms accumulates and eventually predominates over that of the linear terms. For instance, the characteristics at the head of a positive pressure wave diverge, tending to form a shock wave if one is not already present. The linearized theory is therefore not even a valid first approximation at large distances.

A great deal has previously been written on this matter (7,8,25), and there is no need to go into any general detail here. To obtain an approximate solution of Eq. (4.1) which is uniformly valid over the whole flow, the Lighthill-Whitham method is used; the solution is expressed in terms of an implicit exact characteristic η, to which successive approximations are made. This approach seems the only way to proceed without ambiguity to high-order approximations to the equation of the cavity motion. It appears that the method would remain tractable even when a fully developed shock wave is present, but we shall not deal explicitly with this case here, being content to examine the circumstances of the initial formation of a shock.

If p denotes the pressure relative to the constant pressure P far from the centre of the motion (so that $p \to 0$ for large r and $p = -P$ at the wall of an empty cavity), the Bernoulli integral of the equations of motion can be written

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} u^2 = -\int_0^p \frac{dp}{\rho}, \tag{4.2}$$

where $u = \partial\phi/\partial r$ is the velocity. Now, the density ρ is a function of p only; and so, if ρ^{-1} is expanded as a Taylor series and integrated term by term, the right-hand side of Eq. (4.2) becomes

$$-p/\rho_o + \frac{1}{2} p^2/\rho_o^2 c_o^2 + \cdots .$$

Here ρ_o is the density where $p = 0$ and $c_o^2 = (dp/d\rho)_{p=0}$. Only these first two terms of the expansion are required at present; the third term is $O(c_o^{-4})$, and we are going to develop the theory only as far as $O(c_o^{-2})$. Thus, a sufficient approximation to Eq. (4.2) is

$$\phi_t + \frac{1}{2}\phi_r^2 = -\frac{p}{\rho_o}\left(1 - \frac{1}{2}\frac{p}{\rho_o c_o^2}\right). \tag{4.3}$$

The sound velocity is next expanded as

$$c = c_o + \left(\frac{\partial c}{\partial p}\right)_o p + \cdots . \tag{4.4}$$

Again, terms after the second can be neglected, being of fourth order. Hence, using Eq. (4.3), we may write

$$c = c_o \left\{ 1 - \frac{\alpha}{c_o^2} \left(\varphi_t + \frac{1}{2} \varphi_r^2 \right) \right\}. \tag{4.5}$$

Here α is a dimensionless coefficient of the order of unity; for water it is about 2.7. (Experimental data on the properties of water under high pressures (see (6) for a review) show that a linear relation between sound velocity and pressure, as assumed here, holds approximately up to pressures of the order of 2000 bars. To deal adequately with the case of pressures much higher than this, the expansion Eq. (4.4) would have to be taken further than the first two terms; but this is not justified in a second-order theory. In fact, it will appear that the variation of c does not affect the cavity motion to the second order, although this has an important effect on the surrounding pressure field.)

We next suppose the characteristic curves in the (r, t)-plane to have the form $\eta(r, t) = \text{const}$. The equation determining η is then

$$(u + c) \left(\frac{dt}{dr} \right)_{\eta = \text{const}} - 1 = 0 . \tag{4.6}$$

We may also arbitrarily introduce the condition

$$\eta = t \quad \text{on} \quad r = R, \tag{4.7}$$

which lends to η the useful interpretation of a "retarded time" measured with respect to the cavity wall.

To complete the basic details of the mathematical problem, the boundary conditions at the cavity wall will now be written down. First, there is the kinematical condition

$$\left(\frac{\partial \varphi}{\partial r} \right)_{r=R} = \dot{R} = \frac{dR}{d\eta} . \tag{4.8}$$

Second, there is the condition that the (relative) pressure of the cavity contents, say $\rho_o S$, is the same as the pressure just inside the liquid (surface tension being neglected in this account). According to Eq. (4.3), this implies that

$$\left(\frac{\partial \varphi}{\partial t} \right)_{r=R} + \frac{1}{2} \dot{R}^2 = - S \left(1 - \frac{1}{2} S/c_o^2 \right). \tag{4.9}$$

We also have, of course, that $\varphi \rightarrow 0$ for $r \rightarrow \infty$.

5. THE LINEARIZED THEORY

This has been used by several authors (23,24) to derive an equation of cavity motion with a first-order correction for the effects of compressibility. The right-hand side of Eq. (4.1) is neglected, and one puts $c = c_o$. The solution then is

$$\varphi = - \frac{f(\eta)}{r} \tag{5.1}$$

together with

$$\eta = t - \frac{r}{c_o} + \frac{\dot{R}}{c_o}. \tag{5.2}$$

(Note that Eq. (5.2) is the first-order solution of Eq. (4.6) subject to the boundary condition Eq. (4.7)).

From Eq. (5.1) it follows that

$$u = \frac{f(\eta)}{r^2} + \frac{f'(\eta)}{c_o r}. \tag{5.3}$$

When this is substituted in Eq. (4.8), $f(\eta)$ can easily be calculated as far as terms in c_o^{-1}. A first estimate gives $f(\eta) = R^2(\eta)\ \dot{R}(\eta)$; and a second gives

$$f = R^2\dot{R} - \frac{1}{c_o}(R^3\ddot{R} + 2R^2\dot{R}^2). \tag{5.4}$$

From Eqs. (5.1) and (5.4), we get

$$\begin{aligned}\left(\frac{\partial\phi}{\partial t}\right)_{r=R} &= \left[-f(\eta)\left\{\frac{1}{r} + \frac{\dot{R}}{c_o}\right\}\right]_{r=R} \\ &= -R\ddot{R} - 2\dot{R}^2 + \frac{1}{c_o}(R^2\dddot{R} + 6R\dot{R}\ddot{R} + 2\dot{R}^3),\end{aligned} \tag{5.5}$$

which, when substituted in Eq. (4.9), leads immediately to

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 - \frac{1}{c_o}(R^2\dddot{R} + 6R\dot{R}\ddot{R} + 2\dot{R}^3) = S. \tag{5.6}$$

This is exact to $O(c_o^{-1})$. To the same order, the equation may be arranged in several other ways, for instance in the form

$$R\ddot{R}\left(1 - \frac{\dot{R}}{c_o}\right) + \frac{3}{2}\dot{R}^2\left(1 - \frac{\dot{R}}{3c_o}\right) = S + \frac{1}{c_o}\left(R\frac{\partial S}{\partial t} + \dot{R}S\right). \tag{5.7}$$

6. HIGHER-ORDER SOLUTIONS

To proceed to higher approximations, one may write

$$\phi = \phi_1(r,\eta) + \frac{1}{c_o^2}\ \phi_2(r,\eta) + \frac{1}{c_o^3}\phi_3(r,\eta) + \cdots, \tag{6.1}$$

where $\phi_1(r,\eta)$ is the linearized solution given above, and then solve Eq. (4.1) to successively higher powers of c_o^{-1} (i.e., φ_2 is first determined, then φ_3, etc.). At the same time, successive approximations are made to the characteristic determined by Eq. (4.6). The solution of the latter equation is most suitably expressed in the form

$$t = F(r,\eta), \tag{6.2}$$

which, of course, in the linearized theory reduces to

$$t = \eta + \frac{r - R(\eta)}{c_o}. \tag{6.3}$$

Thus, the present method is clearly seen to be an instance of Lighthill's general technique of "coordinate perturbation" (7,25). In applying the method, it is helpful to rewrite the equation for φ in terms of partial derivatives with respect to r (i.e., with η constant) and η (with r constant). Thus, Eq. (4.1) is replaced by

$$\varphi_{rr} + \frac{2}{r}\varphi_r + 2\eta_r\left(\varphi_{r\eta} + \frac{1}{r}\varphi_\eta\right) + \eta_{rr}\varphi_\eta + \eta_r^2\varphi_{\eta\eta} - \frac{1}{c^2}(\eta_t^2\varphi_{\eta\eta} + \eta_{tt}\varphi_\eta) = \text{non-linear terms,} \tag{6.4}$$

the terms on the right-hand side being equivalent to c^{-2} times the non-linear terms on the right-hand side of Eq. (4.1).

Let us now work out the solution as far as terms in c_o^{-2}. We first observe, on substituting Eq. (4.5) in Eqs. (4.1) or (6.4), that the variation of c does not affect the equation for φ to this order. Next, the linearized solution is used to obtain the second-order approximation to the right-hand side of Eq. (6.4): this is

$$\frac{1}{c_o^{\,2}}\left\{\frac{2ff'}{r^4} - \frac{2f^3}{r^7}\right\} \tag{6.5}$$

with $f(\eta) = R^2(\eta)\,\dot{R}(\eta)$.

An approximation to the explicit form of Eq. (6.2) is now found. When Eqs. (5.1) and (4.5) are substituted in Eq. (4.6), the integral of the equation to $O(c_o^{-3})$ is

$$t = \frac{r}{c_o} + \frac{f(\eta)}{rc_o^{\,2}} - \frac{1}{c_o^{\,3}}(1+\alpha)\,f'(\eta)\,\log r + \frac{(2+\alpha)f^2(\eta)}{2r^4c_o^{\,3}} + A(\eta).$$

where $f(\eta)$ is given by Eq. (5.4) and $A(\eta)$ is an arbitrary function arising in the integration. If A is chosen so as to satisfy the boundary condition Eq. (4.7), we have

$$t = \eta + \frac{r - R(\eta)}{c_o} + \frac{f(\eta)}{c_o^{\,2}}\left(\frac{1}{r} - \frac{1}{R}\right) - \frac{(1+\alpha)f'(\eta)}{c_o^{\,3}}\log\frac{r}{R} + \frac{(2+\alpha)f^2(\eta)}{6c_o^{\,3}}\left(\frac{1}{R^3} - \frac{1}{r^3}\right) \tag{6.6}$$

This expansion is far more accurate than we need at present, but it will be useful later.

When Eqs. (6.1), (6.5), and (6.6) are substituted in Eq. (6.4), the first term on the right-hand side (i.e., the first term in Eq. (6.5)) is cancelled, and collection of the second-order terms gives

$$\frac{\partial^2\phi_2}{\partial r^2} + \frac{2}{r}\frac{\partial\phi_2}{\partial r} = -\frac{2R^6\dot{R}^3}{r^7}, \tag{6.7}$$

which has the solution

$$\phi_2 = \frac{F(\eta)}{r} - \frac{R^6\dot{R}^3}{10\,r^5}. \tag{6.8}$$

The arbitrary function $F(\eta)$ is determined from Eq. (4.8). Hence, the complete second-order solution is found to be

$$\phi = -\frac{1}{r}\left\{R^2\dot{R} - \frac{1}{c_o}(R^3\ddot{R} + 2R^2\dot{R}^2) + \frac{1}{c_o^{\,2}}(R^4\dddot{R} + 7R^3\dot{R}\ddot{R} + \frac{7}{2}R^2\dot{R}^3)\right\} - \frac{R^6\dot{R}^3}{10\,c_o^{\,2}\,r^5}. \tag{6.9}$$

This result is exact to the second-order. The last term on the right-hand side is negligible far from the cavity, but near the cavity it has the same importance as the other second-order terms. The appearance of a term in r^{-5} in this solution does not imply that powers between r^{-1} and r^{-5} are absent from higher-order solutions: in fact terms in r^{-2} arise in the next approximation (compare the solution of the blast wave problem given by Whitham (8, see also 25)). Note that the present result discloses an error in the Kirkwood-Bethe hypothesis, which states $r\phi_t$ to depend only on η.

An equation for $R(t)$ corresponding to Eq. (5.6) can now be obtained by putting Eq. (6.9) in Eq. (4.9). However, at this point it is convenient to introduce a simplification which has been suggested by Dr. Proudman (it was in fact used in his work referred to in the Introduction). We have remarked earlier that the motion of a cavity towards the end of a collapse depends very little on the hydrostatic pressure, but is controlled essentially by the total energy E. Therefore, for the concluding stages of the collapse of an empty cavity, it is permissible to put $S = 0$ in Eq. (4.9). The vital parameter E then enters as a constant of integration in a first integral of the differential equation for R. (The case of a gas-filled bubble near the end of its collapse may be simplified in a similar way by putting S equal to the absolute (instead of relative) pressure of the gas contents, which is justified when this pressure becomes very large compared with the pressure P far from the centre; however, to illustrate the theory we shall consider here only the case of an empty cavity.) The step of putting $S = 0$ can also be justified by considering that the motion in the final stages would be the same approximately as for a cavity which collapses from infinite size under zero pressure, but which is initially given a finite kinetic energy. Hence, after a certain amount of straightforward calculation, we arrive at the equation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{\dot{R}^3}{c_o} + \frac{13}{20}\frac{\dot{R}^4}{c_o^2} = 0 . \tag{6.10}$$

Consider now the expression

$$\frac{k}{R^3} = \dot{R}^2\left(1 + A\frac{\dot{R}}{c_o} + B\frac{\dot{R}^2}{c_o^2}\right),$$

where k, A, and B are constants. On differentiation this is found to satisfy Eq. (6.10) to $O(c_o^{-2})$ if $A = -4/3$ and $B = 9/10$. Also, k can be identified with $E/2\pi\rho_o$; for we must have $2\pi\rho_o R^3\dot{R}^2 \sim E$ when the cavity is still large enough for the velocity to be small and the effect of compressibility negligible (see Eq. (3.1)). Thus we have

$$E = 2\pi\rho_o R^3\dot{R}^2\left(1 - \frac{4}{3}\frac{\dot{R}}{c_o} + \frac{9}{10}\frac{\dot{R}^2}{c_o^2}\right). \tag{6.11}$$

This result shows clearly that compressibility of the liquid slows down the cavity motion: i.e., at a given radius R, the magnitude of the (negative) velocity $\dot{R}$ is less than if the terms in $\dot{R}/c_o$ were zero.

Now, there are various ways one might interpret this equation. If it were supposed to remain a valid approximation right up to the instant of total collapse, one would conclude that $-\dot{R} \to O(R^{-3/4})$ as $R \to 0$, so that the velocity ultimately becomes infinite. However, this argument is obviously unsound. Equation (6.11) is only a second-order approximation in terms of $\dot{R}/c_o$; and clearly in successive higher approximations the series in powers of $\dot{R}/c_o$ would be carried on indefinitely. Thus, if $-\dot{R}$ is in fact infinite in the limit, the truncated series would be incapable of determining the exact nature of the singularity. On the other hand, there is an approximate solution of Eq. (6.11), satisfying the equation to the second order in R/c_0, which gives a finite velocity in the limit as $R \to 0$. At first sight it seems possible that this solution is a uniformly valid approximation right to the end of the collapse, so that the finite terminal velocity is a realistic result; for we note that in order to keep the right-hand side of Eq. (6.11) constant as $R \to 0$, it may be sufficient for the series in $\dot{R}/c_0$—of which only the first three terms are given in the present approximation—to diverge in a certain way for some finite value of $\dot{R}$.

To obtain the result in question, we introduce the dimensionless variables $x = R(2\pi\rho_0 c_0^2/E)^{1/3}$ and $\delta = (-5t/2)^{-3/5}(E/2\pi\rho_0 c_0^5)^{1/5}$. In terms of these, a solution accurate to the second-order is found to be

$$x = \delta^{-2/3}\left(1 - \frac{2}{3}\delta - \frac{29}{90}\delta^2\right), \tag{6.12}$$

$$\frac{\dot{R}}{c_0} = -\delta\left(1 + \frac{1}{3}\delta + \frac{29}{45}\delta^2\right). \tag{6.13}$$

According to Eq. (6.12), the collapse ends at $\delta = 1.008$. When this value is put in Eq. (6.13), the velocity at the end is given as $-\dot{R} = 2.01\, c_0$.

It was noted in the Introduction that this speculation about the terminal velocity of collapse is probably wrong. The author became convinced of this only very recently. Previously the available analytical evidence seemed substantially in favour of a finite collapse velocity—for instance, this interpretation was consistent with a fourth-order approximation obtained by Dr. Proudman (which at first sight would appear more reliable than a theory based on the Kirkwood-Bethe hypothesis). However, sufficient evidence for rejecting this interpretation was provided by a numerical solution of the problem which Mr. C. Hunter at Cambridge has found by use of the EDSAC II digital computer. The numerical results indicate that the velocity $-\dot{R}$ increases without bound as $R \to 0$, which is as Gilmore (5) suggested a number of years ago. This temporary mistake might nevertheless be said to fit naturally into the story of recent work on this problem; and even though a finite collapse velocity may not after all be a true theoretical result, it is not inconsistent with the physical picture of events at the centre of a cavity collapse. The chief physical implication of a theoretically finite velocity (in fact the only really significant aspect of such a result) would be that there is no concentration of energy at the centre, as we noted in Section 3 occurs in the case of an incompressible liquid. This apparently is a valid conclusion. Some recent work, to be summarized in the Appendix, has demonstrated that the flow field surrounding a collapsing empty cavity possesses an inner core which holds a vanishingly small share of the total energy of the motion. It therefore appears that the pressure wave radiated from a practically empty cavity would be scarcely affected by events near the centre immediately preceding the final closure of the cavity; the properties of the wave are mainly determined by the flow outside the central core, for which part of the flow the above theory provides a valid second approximation.

It remains to reconsider the possibility that a shock is formed before the end of the collapse, as suggested earlier in the paper. If this occurred, calculations of these present sort would subsequently become invalid. Analytical work on this question has so far been inconclusive, but the machine calculations organized by Mr. Hunter have indicated that no shock is in fact formed up to the instant when $R \to 0$.

7. THE PRESSURE WAVE

An advantage of the preceding method of analysis is that it provides a uniformly valid approximation for the pressure pulse transmitted away from the cavity, even when there is distortion of the wave-form due to non-linear effects. To illustrate the theory, we shall now focus attention on the question whether a rebounding gas-filled cavity may generate a shock wave.

At a large distance from the cavity, the pressure due to the radial motion is

$$p = \rho_0 \frac{f'(\eta)}{r}, \tag{7.1}$$

where $f(\eta)$ is prescribed on the cavity wall and is $R^2\dot{R}$ to a first approximation; also, according to Eq. (6.6), η is given by

$$t = \eta + \frac{r}{c_0} - \frac{(1+\alpha)f'(\eta)}{c_0^{\,3}} \log \frac{r}{R}. \tag{7.2}$$

The last term in this expression has the effect that in a positive pressure wave ($f' > 0$), the characteristics progressively diverge from the approximate characteristic $t - r/c_0$ given by the acoustic theory. Thus, as we have remarked earlier, the acoustic approximation always breaks down when the wave has travelled far enough.

The result of the diverging of characteristics is that a positive wave is made skew in the outward direction and may eventually develop a shock as its slope becomes infinite. Now, since $f'(\eta)$ is continuous, we can only have $\partial p/\partial t \to \infty$ if $\partial\eta/\partial t \to \infty$; i.e., if $\partial t/\partial\eta \to 0$. Therefore, from Eq. (7.2), the condition for the initial formation of a shock is approximately (taking $\log(r/R)$ to be large)

$$0 = 1 - \frac{(1+\alpha)f''(\eta)}{c_0^{\,3}} \log \frac{r}{R}. \tag{7.3}$$

This gives the radius at which a shock first appears. Clearly, the shock develops at the point in the wave where the slope $f''(\eta)$ is positive (i.e., at the head of the wave in time) and a maximum—i.e., at a point of inflexion. It might be supposed that for a positive pressure pulse such as produced by a rebounding cavity, a shock always occurs, because the condition Eq. (7.3) can always be satisfied by taking r to be large enough. However, if $f''(\eta)$ is everywhere small, the required value of r becomes enormous, which means that in practice there will be no shock. The effects of dispersion by viscosity and heat conduction will eventually cancel the very gradual tendency to build up a shock; for, although the magnitude of these effects on spherical waves is extremely small, they depend on r in contrast to $\log r$ for the latter effect (13).

We are now in a position to estimate the practical conditions under which a rebounding cavitation bubble will give rise to a shock. The maximum value of $f''(\eta)$ is required; and as a first approximation it is sufficient to take $R^2\dot{R}$ for f. We assume for simplicity that the pressure inside the bubble is given by $P_m(R_m/R)^4$, where P_m is the maximum pressure and R_m the minimum radius (this assumption implies an adiabatic compression with $\gamma = 4/3$; it is found that other values of γ do not significantly change the end results of the present calculation). By a straightforward method along the same lines as the calculations at the end of Section 3, the maximum of $f''(\eta)$ is estimated to be simply $0.556\,(P_m/\rho_0)^{3/2}$. Next, values of c_0 and α for water are substituted in Eq. (7.3). Hence, we deduce that the condition for a shock to be formed at radius r in water is

$$P_m > 13.6/[\log (r/R)]^{2/3} \text{ kilobars.} \tag{7.4}$$

The value of R in Eq. (7.4) is nearly the same as R_m, but it is really unimportant which value precisely is considered. The following table shows some values of P_m just satisfying Eq. (7.4), given as a function of r/R:

r/R	10^8	10^7	10^6	10^5	10^4
P_m (kilobars)	1.9	2.1	2.3	2.7	3.1

We see that the necessary value of P_m varies very little over orders of magnitude of r/R. This table indicates a good practical criterion for shocks to form within "reasonable distances"; this is simply that the maximum pressure in the bubble should be at least about 2 or 3 kilobars. If P_m is much smaller than this, shock waves are unlikely to be detectable, both because dispersion may overcome the tendency towards shock formation and because anyway the pressure wave will be of extremely small amplitude at the distance where a shock might appear.

As a check on this calculation, we observe from Fig. 2 that for value of P_m around 2 kilobars the maximum velocity of the bubble is still reasonably small compared with the sound velocity. Therefore compressibility of the water will not have any important effect on the bubble motion; and so the estimate of $f''(\eta)$ from the incompressible-fluid theory is probably quite adequate.

After a shock is initiated, it develops rapidly since its velocity is less than for a continuous wave of the same amplitude; thus, the shock is continually "fed" by wavelets arriving from behind. In the present problem, it would be feasible to account for the progress of a shock after incipience by using the methods given by Whitham (10).

8. CONCLUSION

The present theoretical work is intended to throw some further light on the physical mechanism of the collapse of cavitation bubbles and the noise arising therefrom. It has been shown that the theory of cavity collapse in compressible liquids can with advantage be developed by means of the Lighthill-Whitham technique of coordinate perturbation, successive approximations being obtained in powers of the parameter $1/c_o$. For the hypothetical case of an empty cavity, the approximate theory admits the interpretation that the collapse is terminated at a finite radial velocity; and although this is now believed to be an incorrect theoretical result, it is nevertheless true that the "imperfect" theory provides a uniformly valid physical approximation, since the finite terminal velocity is a reasonable representation of the apparently true fact that only a limitingly small part of the overall energy is carried right to the centre of collapse (in other words, most of the energy is stored in a compressive wave before the collapse ends).

In reality the concluding stages of a "severe" collapse (i.e., where the cavity has a negligible gas content) will be greatly affected by compression of the enclosed vapour and by other effects which have not been included in the present theory. But, in the manner suggested above, compressibility of the water may have enfeebled the ending of the collapse before the latter effects become important, so that they play an insignificant role in the overall mechanism. By similar considerations, the possibility of a complete collapse without rebound from a certain minimum cavity size now appears fairly reasonable. Of course, as soon as the collapse is completed, a strong shock wave will be formed (the shock will probably originate from the centre; but as it expands it rapidly becomes independent of the "inner core" left unexplained by the simple theory); and this shock wave will have a rarefaction phase behind it which might re-open a cavity. The question whether or not rebounds occur in usual circumstances remains one of the most debatable topics in cavitation studies. In the author's experiments, no evidence has ever been found of a complete collapse without any rebound. Several other experimenters have, however, reported such evidence.

Another aspect of cavitation which has been considered concerns the case of a comparatively mild collapse. It has been shown that even a gas-filled bubble can generate a shock wave if the pressure of the gas rises beyond a certain limit at the instant of maximum compression when the rebound begins. Experiments by the author have revealed cases of incipient shock formation; and the observations appear to support the theoretical estimate of the critical conditions, although a precise comparison has not so far been achieved.

Finally, I feel this is an appropriate place to acknowledge my indebtedness to the British Admiralty, which has supported my work on cavitation for a number of years. In particular, I should like to express my gratitude to Dr. Jackson, Dr. Vigoureux, and Dr. Byard of the Royal Naval Scientific Service, whose advice and encouragement on many occasions in the past have greatly sustained my interest in cavitation.

APPENDIX

The methods given earlier in this paper were shown to lead to what is very probably an incorrect answer to the theoretical question of how the collapse of an empty cavity is terminated. In view of the results of Mr. Hunter's calculations with EDSAC II, this question has been reconsidered by him along different lines, and as some progress has been made since the date of the meeting in Washington, it seems worthwhile to include the following summary of what has been done so far.

Suppose the collapse ends at $t = 0$, and suppose the velocity $-\dot{R}$ of the cavity does in fact increase without bound as $R \to 0$. The pressure just inside the liquid then also increases indefinitely; and since the sound velocity c increases with pressure, eventually the situation develops where the value c_0 right at the cavity wall is insignificant compared with the values of c inside the liquid. It is therefore reasonable to assume that the motion very near the centre is the same as that of a gas flowing into a vacuous spherical cavity; for the gas the value of c is actually zero at the boundary of the cavity, in contrast to the "insignificant" value c_0 which the liquid still possesses at zero pressure. Thus, the liquid motion contains an inner core which tends to the solution of this corresponding gas-dynamical problem in the limit as $R \to 0$.

For such a gas flow there is a "similarity solution" (i.e., a "progressive wave") of the type considered by Guderley (26) and others in studies of strong spherical and cylindrical shock waves (see Courant and Friedrichs (27), Chapter 6C, for a general account of the subject). We take the motion of the contracting inner boundary to follow the simple power law $R = A(-t)^{\alpha}$, where A and α are positive constants, and assume a solution of the form

$$u = \frac{\alpha r}{t} U(\xi), \qquad c = -\frac{\alpha r}{t} C(\xi), \tag{A1}$$

where $\xi = r/R = r/A(-t)$. The inclusion of a minus sign in the expression for c makes $C(\xi)$ a positive quantity over the whole flow, since c is necessarily positive. In the

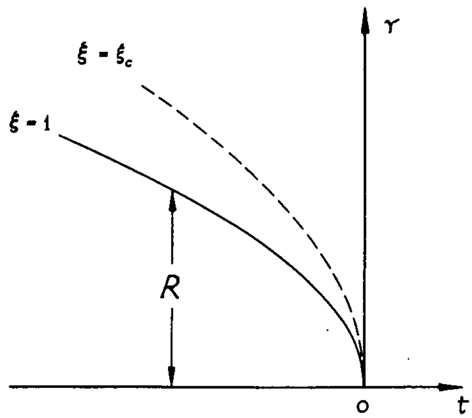

Fig. A1 - The loci of the boundary $\xi = 1$ and the "critical characteristic" $\xi = \xi_c$ in the (r, t)-plane.

(r, t)-plane, the lines ξ = const spread from the origin as illustrated in Fig. A1, the inner boundary being $\xi = 1$. The boundary conditions are $U(1) = 1$ and $C(1) = 0$ at the inner boundary $r = R$, and also $U(\infty) = C(\infty) = 0$.

As shock waves are absent, the flow can be taken to be isentropic; hence, p/ρ^γ is constant everywhere and c is proportional to $p^{(\gamma-1)/2\gamma}$. To apply the solution of the gas-dynamical problem to the cavitation problem, we shall take $\gamma = 7$, recalling the approximate homentropic equation of state for water $p + B = \text{const} \times \rho^7$, where B = 3000 bars (in effect, the present argument assumes that inside the central core of the liquid motion, pressures become much larger than B).

Now, there is a particular line $\xi = \xi_c$ in the (r, t)-plane which, if it occurs inside the flow, has a crucial role in the determination of the solution Eq. (A1)—as it does in shock-wave theory (26). This is the ξ-line whose slope dr/dt is $u - c$, and which is therefore a characteristic curve passing through the origin (see Fig. A1). Hence, according to Eq. (A1), we have

$$U_c + C_c = 1 , \tag{A2}$$

where U_c and C_c denote the values of U and C at $\xi = \xi_c$. Also, the relevant characteristic equation (see p. 46 in (27)) for isentropic flow can be put in the form

$$k \frac{dc}{dt} - \frac{du}{dt} + \frac{2cu}{r} = 0 , \tag{A3}$$

in which $k = 2/(\gamma - 1)$ and the differentiations are performed along the characteristic, i.e., along $\xi = \xi_c$. The substitution of Eq. (A1) into Eq. (A3) now leads, with the use of Eq. (A2), to

$$2 U_c^2 + [k - 3 + (1 - k)\alpha^{-1}] U_c + \alpha^{-1} - 1 = 0 . \tag{A4}$$

Some important properties of the solution are implied directly by this quadratic equation for U_c.

We wish particularly to know the possible values of α in this problem, since from them the asymptotic behaviour of a collapsing cavity in a liquid might be inferred. The first possibility which need be considered is that $\alpha = 1$. This value clearly is admissible according to Eq. (A4) whose roots become $U_c = 0, 1$, indicating that there is no "critical characteristic" inside the flow; or rather it could be said that the characteristic has become $\xi_c = 1$ and thus coincides with the boundary $r = R$. For $\alpha = 1$, the velocity dR/dt is constant, as are all velocities along rays r/t = const in the (r, t)-plane. This case would therefore be relevant to the cavitation problem if the terminal velocity of collapse were finite yet very large compared with c_o. Courant and Friedrichs ((27), p. 431) have noted that the present gas-dynamical problem does have a solution with $\alpha = 1$, at least for the value $\gamma = 1.4$ appropriate to air. On further investigation, however, it appears there is no solution applicable to the cavitation problem. This fact can most easily be demonstrated as follows.

When the expressions Eq. (A1) are inserted into the dynamical and continuity equations for spherical isentropic flow (see (27), p. 37) and use is made of the fact that $d\rho/dc = k\rho/c$, we obtain the equations

$$\xi(U-1)U' + k\xi UC' = -U^2 + \alpha^{-1}U - kC^2 , \tag{A5}$$

$$\xi CU' + k\xi(U-1)C' = -C\,\{(3+k)U - k\alpha^{-1}\} . \tag{A6}$$

We recall that $U(1) = 1$ and $C(1) = 0$; hence, for $\alpha = 1$, we find from Eq. (A6) that

$$U'(1) = -\frac{3}{1+k}; \tag{A7}$$

and from Eqs. (A5) and (A7) we find that

$$k[C'(1)]^2 = -U'(1)[1+U'(1)] = \frac{3(k-2)}{(k+1)^2} . \tag{A8}$$

This result shows that for a real solution it is necessary that $k \geq 2$, which means that $\gamma \leq 2$. With $\gamma = 7$ as at present, there is therefore no solution with $\alpha = 1$.

Further investigation shows that there is only one admissible value of $\alpha\,(< 1)$; but there appears to be no simple way of calculating this value. However, a lower limit to α can at once be specified on consideration that Eq. (A4) must have real roots. With $k = 1/3$, corresponding to $\gamma = 7$, the conditions for real roots become

$$\alpha^{-1} \leq 7 - 3\sqrt{3} ,$$

$$\text{i.e.,} \quad \alpha \geq 0.55437 . \tag{A9}$$

By means of some fairly simple analysis, details of which we shall omit here, one may conclude that the actual value of α is only slightly greater than the value in Eq. (A9). But, to obtain α accurately, the following course is necessary.

From Eqs. (A5) and (A6), a differential equation can be obtained in which U and C occur, respectively, as independent and dependent variables and in which ξ is absent. This has the form

$$\frac{dC}{dU} = f(C, U) , \tag{A10}$$

where k and α are involved as parameters in the function f of C and U. It turns out $\xi = \xi_c$ (which implies $C = 1 - U$; see Eq. (A2)) corresponds in general to a singularity of this differential equation; and Eq. (A4) accordingly assumes the role of a condition on the parameters k, α which is necessary for a physically realistic solution. The correct value of α can apparently only be found by substituting trial values in Eq. (A10) and observing whether corresponding solution meets all the requirements of the physical problem (this procedure recalls the work of Guderley (26)). As the solutions to Eq. (A10) can be found only by numerical integration, the task of finding α entails a great deal of computation. But this work has been done by Mr. Hunter at Cambridge; and for $\alpha = 7$ he has determined the value of α at about 0.5552.

Thus it appears fairly well established what the asymptotic behaviour should be in the cavitation problem <u>if</u> the terminal velocity of collapse is in fact infinite. Note that the above value of α implies that $-dR/dt \sim R^{0.801}$ as $R \rightarrow 0$, this exponent being $\alpha^{-1} - 1$. However, it has not been proved that the terminal velocity is infinite; this has been inferred only because the results of the machine computation seem to indicate it to be so.

The total energy of the gas flow considered above is found to be infinite; but this fact is not necessarily inconsistent with our application to the cavitation problem, since only the motion very near the centre is supposed to be the same in the two cases. Actually, a very important conclusion with respect to the cavitation problem follows on considering how the energy of the gas flow is distributed. With $\gamma = 7$ and $\alpha = 0.5552$, it can be shown that in the limit as $\underline{R} \rightarrow 0$ the total kinetic and compressional energy within a sphere of radius $\underline{r}$ is proportional to $\underline{r}^{1.23}$. Thus, evidently the energy of the liquid motion also is spread away from the centre of collapse, there being no concentration of energy at the centre (as happens in the case of incompressible flow) despite the infinite velocity which occurs there. The physical significance of this fact was noted earlier in this paper.

REFERENCES

1. Fitzpatrick, H.M., and Strasberg, M., "Hydrodynamic Sources of Sound," p. 241 in "Symposium on Naval Hydrodynamics," Sherman, F.S., editor, Nat. Acad. Sci.—Nat. Res. Council, Publ. 515, Washington, 1957

2. Benjamin, T. Brooke, "Noise from Cavitation Generated by Vibrating a Mass of Water," Abs. in "Programme of the 55th Meeting of the Acoust. Soc. Am. (1958)

3. Mellen, R.H., "Ultrasonic Spectrum of Cavitation Noise in Water," J. Acoust. Soc. Am. 26:356 (1954)

4. Mellen, R.H., "An Experimental Study of the Collapse of a Spherical Cavity in Water," J. Acoust. Soc. Am. 28:447 (1956)

5. Gilmore, F.R., "The Growth and Collapse of a Spherical Bubble in a Viscous Compressible Liquid," Calif. Inst. Tech., Hydromechanics Lab., Report No. 26-4 (1952)

6. Flynn, H.F., "Collapse of a Transient Cavity in a Compressible Liquid," Harvard University, Acoustics Res. Lab., Tech. Mem. No. 38 (1957)

7. Lighthill, M.J., "A Technique for Rendering Approximate Solutions to Physical Problems Uniformly Valid," Phil. Mag. (7)40:1179, (1949). See also ref. (25).

8. Whitham, G.B., "The Propagation of Spherical Blast," Proc. Roy. Soc. A, 203:571 (1950)

9. Whitham, G.B., "The Propagation of Weak Spherical Shocks in Stars," Comm. Pure Appl. Math. 6:397 (1953)

10. Whitham, G.B., "On the Propagation of Weak Shock Waves," J. Fluid Mech. 1:290 (1956)

11. Lord Rayleigh, "On the Pressure Developed in a Liquid During the Collapse of a Spherical Cavity," Phil. Mag. 34:94 (1917)

12. Harrison, M., "An Experimental Study of Single-Bubble Cavitation Noise," Taylor Model Basin Report No. 815 (1952). See also J. Acoust. Soc. Am. 24:776 (1952)

13. Benjamin, T. Brooke, "Cavitation in Liquids," Ph.D. Dissertation, University of Cambridge (1954)

14. Knapp, R.T., and Hollander, A., "Laboratory Investigations of the Mechanism of Cavitation," Trans. A.S.M.E. 70:419 (1948)

15. Bromwich, T.J.I'a., "An Introduction to the Theory of Infinite Series," 2nd Ed., § 174 (see particularly Example 5). London:Macmillan (1926)

16. Benjamin, T. Brooke, Contribution to the discussion following Ref. (1).

17. Lamb, H., "The Early Stages of a Submarine Explosion," Phil. Mag. 45:257 (1923). See also Ref. (18), 91.

18. Lamb, H., "Hydrodynamics," Cambridge University Press, 6th Ed. (1932)

19. Cole, R.H., "Underwater Explosions," Princeton University Press (1948)

20. Plesset, M.S., "Dynamics of Cavitation Bubbles," J. Appl. Mech. 16:277 (1949)

21. Noltingk, B.E., and Neppiras, E.A., "Cavitation Produced by Ultrasonics," Proc. Phys. Soc. B, 63:674 (1950); 64:1032 (1951)

22. Robinson, R.B., and Buchanan, R.H., "Undamped Free Pulsations of an Ideal Bubble," Proc. Phys. Soc. B, 45:893 (1956)

23. Herring, C., "Theory of Pulsations of a Gas Bubble Produced by an Underwater Explosion," OSRD Report No. 236 (1941); also in "Underwater Explosion Research," Vol. II, Washington, D.C., Office of Naval Research (1950)

24. Trilling, L., "The Collapse and Rebound of a Gas Bubble," J. Appl. Phys. 23:14 (1952)

25. Tsien, H.S., "The Poincare-Lighthill-Kuo Method," Advances in Appl. Mech. 4:281 (1956)

26. Guderley, G., "Starke kugelige und zylindrische Verdichtungsstosse inder Nahe des Kugelmittelpunktes bzw. der Zylinderaschse," Luftfahrtforschung 19:302 (1942)

27. Courant, R., and Friedrichs, K.O., "Supersonic Flow and Shock Waves," New York:Interscience (1948)

NOTE ADDED IN PROOF. A complete study of the asymptotic theory referred to in the Appendix is presented in a paper by C. Hunter, "On the Collapse of an Empty Cavity in Water," J. Fluid Mech. 8:241 (1960).

* * * * *

DISCUSSION

C. A. Gongwer (Aerojet General Corporation)

At Aerojet we have been able to generate a transient bubble of four-feet maximum diameter which is a true cavitation bubble, i.e., one with negligible permanent gas content. Because it's so large, the time scale is expanded and we can take a lot of time studying it. I would estimate from movies, that the velocity of collapse is on the order of several hundred feet per second. What gas is present is possibly air which is drawn out of the water. It becomes incandescent on recompression and there is a flash. This is possibly the source of the luminescence occasionally observed in cavitation experiments. When the bubble becomes small it looks like a "porcupine" in that sharp spikes extend radially from it. There is debris in and on the surface of the bubble which is overtaken by this large, collapse acceleration and is left behind projecting like radial spikes all over the bubble.

The bubble center is 5.5 feet deep and 10 feet away. The charge is about an ounce of this non-gassing explosive.

Figures D1 and D2 are representative pictures of the bubble at maximum size and near the final collapse.

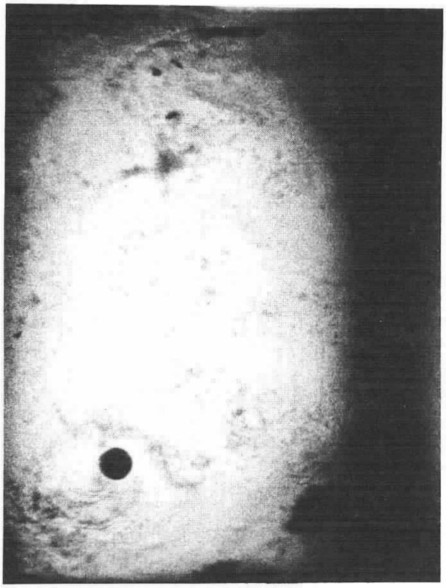

Fig. D1 - Maximum expansion; 1 oz Alclo charge, reel 16631, time: 50 ms, bubble radius: 18.7 inches

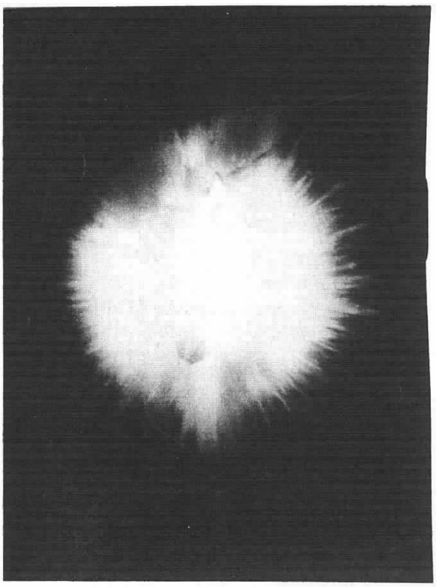

Fig. D2 - Near final collapse; 1 oz Alclo charge, reel 16631, time: 95 ms, bubble radius: approx. 4 inches

This bubble lasts about one-tenth of a second, and one sees a little jet extending upward with a few bubbles in it. This is because the integrated displacement (vs time) of the bubble provides an upward impulse to the water comprising the virtual mass.

The implosion shock is of the same order as that produced by the explosion. The collapsing bubble (see Fig. 2) has sharp projections on it, sticking out radially like quills on a porcupine, as it comes together. One sees a little residual burning, but when the luminescence disappears there is virtually no gas. For our purposes the bubble is empty.

There is enough dissolved air on the surface of the 4-foot bubble in a layer of water 5/1000 of an inch thick to supply what appears to be the residual gas. This bubble is about 4 feet in diameter. It represents a potential energy at this stage of approximately 150,000 foot-pounds.

Further analysis of the film gives the radius-time curves, shown in Fig. D3, plus a derived curve of pressure vs. time in Fig. D4. This data was analyzed by Mr. J. Levy of Aerojet. Further work is being done on the explosive to reduce even further the amount of residual gas and thus approximate more closely the conditions presented in Dr. Benjamin's paper.

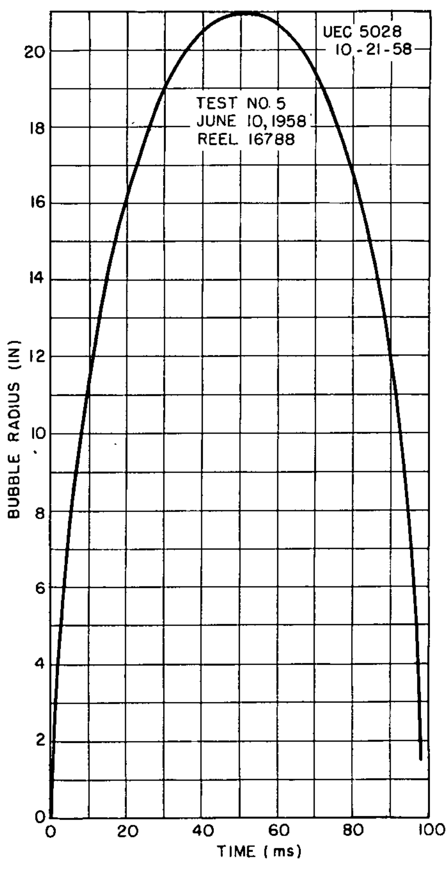

Fig. D3 - Bubble radius vs time. 1 oz Alclo pinger; depth: 5.5 feet.

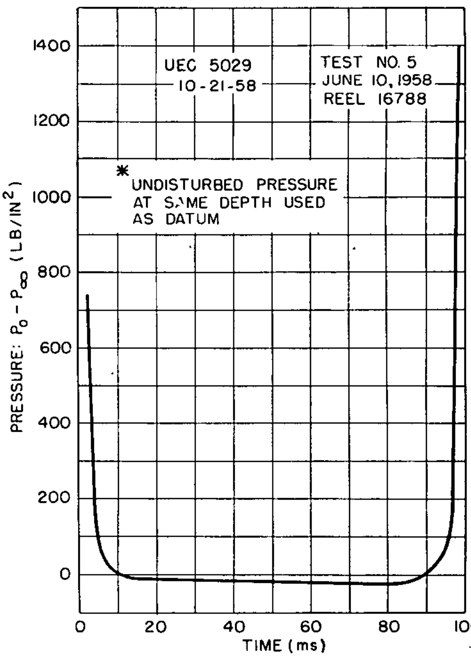

Fig. D4 - Bubble pressure vs time. 1 oz Alclo pinger; depth: 5.5 feet.

F. S. Burt (Admiralty Research Laboratory)

With respect to the question as to whether or not rebound follows the collapse of a vapor cavity, it may be worth while referring to some experimental work which was carried out at the Admiralty Research Laboratory in 1952 by Mr. A. L. Kendrick. In these experiments, single vapor cavities were initiated by a nucleus of permanent gas produced electrolytically - the instability being induced thermally.

Photographs taken with a high-speed camera at the rate of 16,000 frames per second showed that any rebound that existed was extremely small. However, several well-defined rebounds could be observed when appreciable quantities of permanent gas were introduced into the cavity.

M. S. Plesset (California Institute of Technology)

Dr. Benjamin's paper contains several remarks which are somewhat obscure and puzzling. I refer to the discussion on the collapse velocity of a bubble and to the remarks about the development of a shock before the end of collapse. In connection with the latter, I was interested to see the remark that "the machine calculations organized by Mr. Hunter have indicated that no shock is in fact formed up to the instant when $R \to 0$." The author has also made a very questionable point to the effect that his analysis reveals an error in the Kirkwood-Bethe approximation which has been applied to this problem.

Dr. Benjamin has apparently not given much consideration to the numerical integrations of the compressible equations of bubble collapse which were carried out some years ago. One of these was made by A. J. R. Schneider* up to a bubble collapse velocity corresponding to a Mach number of 2.2. A summary of these calculations has been given in the paper by F. R. Gilmore† which is referred to by Dr. Benjamin. A numerical integration of the compressible equations was also performed subsequent to Schneider's calculation by Dr. Gilmore. (Gilmore's results have been reported in the Proceedings of the First Symposium on Naval Hydrodynamics.)‡ Both Schneider's calculations and those made by Gilmore show rather good agreement between the numerical integrations and the results obtained from the Kirkwood-Bethe hypothesis. As expressed by Gilmore, "the calculated bubble-wall velocity agrees within 6 percent with that given by the analytical theory (based on the Kirkwood-Bethe hypothesis) over the calculated range of 0.2 to 4.9 times the velocity of sound in water. The inward velocity thus appears to increase with the radius to the minus one-half power as the radius becomes small, instead of with the minus first power given by the acoustic theory or the minus three-halves power given by the incompressible theory."

T. Brooke Benjamin

Apart from expressing admiration for the resources of Aerojet, I wish to make only one minor comment upon the exciting experiments described by Dr. Gongwer. This is that the buoyancy effects are probably not duplicated by typical small cavitation bubbles owing to the great differences in respective values of the gravitational scale factor $P_0/\rho g R_0$, where P_0 is the environmental pressure and R_0 is a linear measure of bubble size. Otherwise Dr. Gongwer's bubble appears to provide a most effective model of a cavitation bubble.

*A. J. R. Schneider, "Some Compressible Effects in Cavitation Bubble Dynamics," Thesis, California Institute of Technology, 1949.

†F. R. Gilmore, "The Growth or Collapse of a Spherical Bubble in a Viscous Compressible Liquid," Hydrodynamics Laboratory Report No. 26-4, California Institute of Technology, 1952.

‡Symposium on Naval Hydrodynamics, Sept. 24-28, 1956, Publication 515, National Academy of Sciences - National Research Council, Washington, D. C. (See in particular p. 320 and p. 280.)

Mr. Burt has done a valuable service in drawing attention to A. L. Kendrick's experiments, which I have long regarded as an important contribution to cavitation studies and which deserves wide recognition. It is hoped that a full account of these experiments may yet be published, even after so many years have passed since the work was done.

I hope that the several changes incorporated in the final draft of my paper will help to allay some of the misgivings felt by Prof. Plesset. But I still fail to see what is questionable about my remark quoted in Prof. Plesset's third sentence: I still do not find it in any way directly obvious that the shock could not form shortly before—rather than immediately at—the instant of final collapse of an empty cavity, even though I fully accept the evidence of the machine calculations that it does not. Prof. Plesset's comment seems to imply that this fact is self-evident.

Again, I see nothing questionable about my demonstration of the second-order error in the Kirkwood-Bethe approximation. This error certainly exists, though in the paper I have not considered exactly how important it may be. I also purposely refrained from making a critical examination of the theories which, as Prof. Plesset mentions, his two colleagues have worked out on the basis of the Kirkwood-Bethe approximation. The principal aim of my paper was merely to show how better approximations than this one can be made, so that a theory of cavity collapse can be developed on a more secure basis. As one definite criticism, however, I may point out that since the Kirkwood-Bethe approximation becomes wildly inaccurate for highly supersonic flows with velocities approaching infinity, particularly near the centre where the effects of spherical divergence are largest, the asymptotic law of velocity variation referred to by Prof. Plesset is obviously a spurious result. If the collapse velocity does become infinite, the correct law can only be established by some self-consistent asymptotic theory such as the one outlined in the Appendix to my paper.

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