ON THE PROBLEM OF MINIMUM WAVE RESISTANCE FOR STRUTS AND STRUT-LIKE DIPOLE DISTRIBUTIONS

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In this paper we consider the problem of minimizing the wave resistance of a strut of fixed length and volume-per-unit depth at a given Froude number. The work of others has shown that a satisfactory solution of the problem within the framework of the classical linearized theory is not possible. In this paper we show that by regarding the dipole distribution rather than the form as the unknown function, and by correcting the classical thin-ship relation between the dipole distribution and the form, a satisfactory solution can be found. The universal minimum curve of C_w vs f is shown, as well as C_w vs f for a number of optimum and non-optimum forms. Among the problems left unanswered in this paper are the influence of three-dimensional effects and other corrections to the linearized theory.

1. INTRODUCTION

The linearized theory of ship wave resistance has been developed by a series of students beginning with Michell [1]. Accounts of this theory together with extensive bibliographies will be found in Refs. 2-4. Based on the assumption that the fluid is inviscid and that the ship is thin enough to generate only waves of small amplitude, the theory imposes the linearized free surface condition on the velocity potential that characterizes the flow. By means of these assumptions, the problem is made mathematically tractable and expressions for the wave resistance are derived. The expressions have the form of integrals involving either the functions that define the shape of the hull or the functions that define the distribution of sources and sinks by which the hull is generated. Consequently, if we wish to find the hull of minimum wave resistance within the linear approximation, we have to solve a problem in the calculus of variations. Specifically, a quadratic functional (the integral representing wave resistance) is to be minimized subject to a suitable side condition. The need for imposing some constraint on the minimization process becomes apparent upon observing, for example, that a hull of zero beam has zero wave resistance.

Previous studies have been of two types. One type depends on the fact that Michell's integral for the wave resistance can be evaluated in terms of tabulated functions for ship

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forms defined by a certain class of polynomials. The wave resistance is then a function of the polynomial coefficients, and minima can be found by simple computations (see Ref. 5). The other type of investigation has been concerned with minimization within the class of vertical struts of infinite depth having fixed length, fixed volume-per-unit depth, and an otherwise arbitrary form. The latter problem, with which this paper is concerned, has been considered by a number of authors. Their results have been unsatisfactory for one or more of the following reasons:

1. The problem was found to have no solution at all.

2. The shapes obtained were infinitely wide at bow and stern.

3. Shapes (which in some cases have negative width) were obtained by solving an integral equation numerically without regard for the fact that solutions of the equation are known to be (in general) singular.

The cause of these difficulties is a nonuniformity in the accuracy of the perturbation procedure used to linearize the problem. As in the case of thin air foils (see Ref. 6) for fixed position along the air foil the results are arbitrarily accurate for sufficiently small thickness. However, the convergence is not always uniform with respect to position, the difficulty occurring at the ends of the foil.

In this paper the problem is formulated with dipole density instead of strut shape as the unknown function. The dipole density satisfies one of the integral equations studied by previous workers, and is generally singular at the ends of the interval in which it is defined. The associated shape, defined by the closed streamlines in the flow generated by the dipoles, is approximately proportional to the dipole density (as in the usual theory) except at the ends. The shape has the required length and volume-per-unit depth, to first order in the perturbation parameter, and finite positive width. Hence, although refinements can still be made in the method of calculating the shape the basic problem is regarded as solved.

In Section 2 we formulate the problem and derive the integral equation for the dipole density. In Section 3 we discuss the optimum shape for large Froude number. In Section 4 we describe the numerical solution of the integral equation. In Section 5 we determine the shape of the optimum form. In Section 6 we compare the shape and wave drag of various forms.

2. FORMULATION AND THE INTEGRAL EQUATIONS

We reproduce here the usual theory of the thin strut of minimum wave resistance in order to motivate the formulation given later in this section. Throughout the discussion, the fluid is assumed to be incompressible and inviscid and the flow to be irrotational. Referring to Fig. 1, we introduce a right-handed rectangular coordinate system ($\hat{x}$, $\hat{y}$, $\hat{z}$). The $\hat{x}$-$\hat{z}$ plane ($\hat{y} = 0$) represents the upper surface of the fluid, the acceleration of gravity is downwards and the flow velocity is in the $+\hat{x}$-direction.

It is convenient also to introduce the dimensionless coordinates x, y, z defined by

$$\hat{x} = Lx, \qquad \hat{y} = Ly, \qquad \hat{z} = Lz \tag{1}$$

where L is the length of the strut's horizontal section in the $\hat{x}$-direction.

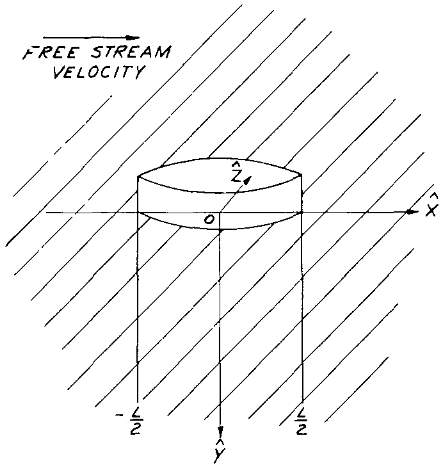

Fig. 1. Coordinate system fixed in a thin strut. The $\hat{x}$ - $\hat{z}$ plane ($\hat{y}$ = 0) represents the upper surface of the fluid, the acceleration of gravity is downward and the flow velocity is in the $+\hat{x}$ - direction.

Now let the strut be represented by the equation

$$\hat{z} = \pm \hat{\zeta}(\hat{x}).^* \tag{2}$$

The extremities of the strut are given by $\hat{x} = \pm L/2$, so that $\hat{\zeta}(\pm L/2) = 0$. Since we wish to perform a perturbation about the strut of zero thickness, we introduce $V/2L$ as a perturbation parameter, where V is the volume-per-unit depth, and write

$$\hat{\zeta}(\hat{x}) = \frac{V}{2L}\, \hat{\zeta}_1(\hat{x}). \tag{3}$$

Since V is the cross-sectional area, $V/2L$ is the average half-beam. The wave resistance of the strut is given by

$$R_w = \frac{1}{2}\,\rho c^2 \left(\frac{V}{2L}\right)^2 C_w \tag{4}$$

where C_w is the wave resistance coefficient. Michell's theory gives [3, p. 115]

$$C_w = 4 \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \frac{d\hat{\zeta}_1(\hat{x})}{d\hat{x}}\, \frac{d\hat{\zeta}_1(\hat{x}')}{d\hat{x}'}\, K\left[\nu(\hat{x} - \hat{x}')\right]\, d\hat{x}d\hat{x}' \tag{5}$$

where $\nu = g/c^2$ and the function $K(t)$ is defined by

$$K(t) = \frac{2}{\pi} \int_1^{\infty} \frac{\cos \lambda t}{\lambda^2 \sqrt{\lambda^2 - 1}}\, d\lambda\,. \tag{6}$$

*The strut is supposed symmetrical about the $\hat{x}$ - $\hat{y}$ plane.

If we introduce the dimensionless variables x and x' into (5), we get

$$C_w = 4 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{d\zeta(x)}{dx} \frac{d\zeta(x')}{dx'} K\left[F(x-x')\right] dx'dx \tag{7}$$

where $\zeta(x)$ is defined by

$$\zeta(x) = \hat{\zeta}_1(Lx) \tag{8}$$

and

$$F = \nu L = \frac{1}{f^2} . \tag{9}$$

The quantity f is called the Froude number.

The problem is to find the function $\zeta(x)$ that minimizes Eq. (7) subject to a suitable constraint. In the case of a strut, a natural constraint is the requirement that a strut of given length L have a given volume V per unit draft. The side condition is thus

$$\int_{-L/2}^{L/2} 2\hat{\zeta}(\hat{x})\, d\hat{x} = V . \tag{10}$$

In terms of $\zeta(x)$, this becomes

$$\int_{-1/2}^{1/2} \zeta(x)\, dx = 1 . \tag{11}$$

The problem therefore is to minimize Eq. (7) among all functions $\zeta(x)$ that satisfy the following conditions:

1. $d\zeta/dx$ must be integrable in $-1/2 \leq x \leq 1/2$ (otherwise Eq. (7) would be meaningless).

2. $\zeta(x)$ must satisfy Eq. (11).

3. $\zeta(\pm 1/2) = 0$.

It is at this point that the central difficulty of the problem appears, in that the minimizing $\zeta(x)$ will now be shown to obey an integral equation that has no solution. Using standard procedures in the calculus of variations, we have at once that if $\zeta(x)$ satisfies the three above conditions, it must also satisfy the equation

$$\int_{-1/2}^{1/2} \frac{d\zeta(x')}{dx'} K(F|x-x'|)\, dx' = kx \tag{12}$$

where the constant k is determined by Eq. (11). Differentiation of Eq. (12) twice with respect to x yields

$$\int_{-1/2}^{1/2} \frac{d\zeta(x')}{dx'} Y_0(F|x-x'|)\, dx' = 0 \tag{13}$$

where $Y_0(t)$ is the Bessel function defined by

$$Y_0(t) = -\frac{2}{\pi}\int_1^\infty \frac{\cos \lambda t \, d\lambda}{\sqrt{\lambda^2 - 1}} .$$

From Eq. (6)

$$\frac{d^2K(t)}{dt^2} = Y_0(t) .$$

Dorr [7] has solved Eq. (13) with an arbitrary given right-hand side; i.e., he has solved

$$\int_{-1/2}^{1/2} h(x') \, Y_0(F|x-x'|)\, dx' = p(x). \tag{14}$$

The change of variables $x = -1/2 \cos \beta$, $x' = -1/2 \cos \beta'$ converts Eq. (14) into

$$\int_0^\pi H(\beta') \, Y_0\left(\frac{F}{2}|\cos \beta - \cos \beta'|\right) d\beta' = P(\beta) \tag{15}$$

where $H(\beta') = (1/2 \sin \beta')\; h(-1/2 \cos \beta')$ and $P(\beta) = p(-1/2 \cos \beta)$.

Dorr then proves that the eigenfunctions of Eq. (15) are the even Mathieu functions of integral order, $ce_n(\beta, F^2/4)$. Since over the integral $(0,\pi)$ these are closed [8], and the kernel $Y_0(F/2|\cos \beta - \cos \beta'|)$ is quadratically integrable over the square $0 \leq \beta' \leq \pi$, $0 \leq \beta \leq \pi$, it follows [9] that there is at most one solution to Eq. (15) in L_2. Moreover, there are no solutions to Eq. (15) not in L_2, as may be deduced from the work of MacCamy [10], who has shown that the only singularities of $h(-1/2 \cos \beta')$ in the interval $0 \leq \beta' \leq \pi$ occur at $\beta' = 0$ or π, these singularities being of the form $1/\sin \beta'$. Thus

$$H(\beta') = \left(\frac{1}{2} \sin \beta'\right) h\left(-\frac{1}{2} \cos \beta'\right)$$

is bounded in the interval $0 \leq \beta' \leq \pi$ and is therefore in L_2.

Summarizing, we have that Eqs. (15), (14), and (13) all have at most one solution. It follows that the unique solution of Eq. (13) is zero. Since any solution of Eq. (12) is a solution of Eq. (13), we conclude finally that Eq. (12) has no solution at all.* Thus, the seemingly natural way in which the problem of the strut of minimum wave resistance has been formulated turns out to be inadequate. We will show that the functions $\zeta(x)$ satisfying the conditions 1, 2, and 3 comprise a class which is too severely restricted for the minimizing $\zeta(x)$ to be found within it.

The problem can also be approached as follows. Since we require $\zeta(\pm 1/2) = 0$, two integrations by parts in Eq. (7) lead to

$$C_w = -4F^2 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \zeta(x)\, \zeta(x')\, Y_0(F|x-x'|)\, dx'dx\,, \tag{16}$$

and the corresponding integral equation is

$$\int_{-1/2}^{1/2} \zeta(x')\, Y_0(F|x-x'|)\, dx' = \alpha \quad \text{on} \quad \left[-\frac{1}{2}, \frac{1}{2}\right] \tag{17}$$

where α is determined so that Eq. (11) holds. As mentioned above it is shown in Ref. 10 that the solutions of Eq. (17) are generally infinite at $x = \pm 1/2$, so that the derivation of Eq. (16) is suspect and the solution physically unacceptable. Nevertheless Pavlenko [12] has solved Eq. (17) numerically by replacing it by a set of algebraic equations and imposing $\zeta(\pm 1/2) = 0$. According to Wehausen, for Froude numbers $f < 0.325$ he finds negative ordinates near the ends. For higher Froude numbers his solutions are fairly good except at the ends, as we shall see. Shen and Kern [13] solved Eq. (17) numerically by assuming that $\zeta(x)$ was a polynomial with three undetermined coefficients. Their results are discussed in Section 6.

We now show that Eqs. (16) and (17), suitably interpreted, are still the correct equations. Let $\hat{\psi}(\hat{x},\hat{y},\hat{z}; V/2L)$ be the potential for the flow generated by a strut advancing into still water. Assuming the strut to be fixed and the free stream directed along the positive $\hat{x}$ axis with velocity c, we have $c\hat{x}$ as the potential of the free stream.† Then

$$\hat{\psi}\left(\hat{x},\hat{y},\hat{z};\, \frac{V}{2L}\right) = \frac{V}{2L}\, \hat{\phi}(\hat{x},\hat{y},\hat{z}) + c\hat{x} \tag{18}$$

where $V\hat{\phi}/2L$ is the "perturbation potential" expressing the amount by which the strut disturbs the free stream. The well-known [3, p. 113] linearized boundary conditions on $\hat{\phi}$ for a thin strut are

$$\frac{\partial^2 \hat{\phi}(\hat{x},0,\hat{z})}{\partial \hat{x}^2} = \frac{g}{c^2}\, \frac{\partial \hat{\phi}(\hat{x},0,\hat{z})}{\partial \hat{y}} \tag{19}$$

at the mean free surface $\hat{y} = 0$ and

*The exceptional case $k = 0$ in Eq. (12) leads to the nontrivial solution ζ = constant, but this fails to satisfy the condition $\zeta(\pm 1/2) = 0$ unless $\zeta = 0$. Sretenskii [11] has concluded that there are no square-integrable solutions, but his reasoning has been criticized by Wehausen [3].

†The flow velocity, $\vec{v}$, is given by $\vec{v} = \operatorname{grad} \psi$.

$$\frac{\partial\hat{\phi}(\hat{x},\hat{y},0\pm)}{\partial\hat{z}} = \pm c\,\frac{d\hat{\zeta}_1(\hat{x})}{d\hat{x}} \tag{19a}$$

along the longitudinal center section of the strut, defined by the half-strip $-L/2 \le \hat{x} \le L/2$, $\hat{y} \ge 0$, $\hat{z} = 0$.

In terms of the dimensionless variables x,y,z, and the function $\phi(x,y,z) = \hat{\phi}(Lx,Ly,Lz)$, Eqs. (19) and (19a) become

$$\frac{\partial^2\phi(x,0,z)}{\partial x^2} = F\,\frac{\partial\phi(x,0,z)}{\partial y}\ ; \qquad y = 0 \tag{20}$$

$$\frac{\partial\phi(x,y,0\pm)}{\partial z} = \pm c\,\frac{d\zeta(x)}{dx}\ ; \quad -\frac{1}{2} \le x \le \frac{1}{2}, \quad y \ge 0, \quad z = 0\,. \tag{20a}$$

ϕ is thus determined by the potential Eqs. (20) and (20a), and the usual conditions at infinity.

We introduce next a Green's function $G(x,y,z;x',y',z')^*$ representing the potential at (x,y,z) of a point sink at (x',y',z') in a uniform flow under a free surface. Then G satisfies Eq. (20) in the variables x,y,z and x',y',z', and the potential $\phi(x,y,z)$ can be expressed in terms of G as follows:

$$\frac{V}{2L}\,\phi(x,y,z) = \frac{c}{4\pi}\,\frac{V}{2L}\int_0^\infty dy'\int_{-1/2}^{1/2} -2\zeta'(x')\;G(x,y,z;\,x',y',0)\,dx'. \tag{21}$$

ϕ as given by Eq. (21) satisfies Eq. (20) because G does. It can also be directly verified to satisfy Eq. (20a). Thus the flow around a thin strut in Michell's theory is characterized by the potential

$$\psi\left(x,y,z;\frac{V}{2L}\right) = cLx + \frac{V}{2L}\,\phi(x,y,z) \tag{22}$$

where ϕ is defined by Eq. (21).

Note that the ship form associated with the potential ϕ is not really $V\zeta(x)/2L$, which is only the approximate form to first order in $V/2L$; the actual ship form is defined by the surface of closed streamlines of the full flow whose potential is the ψ of Eq. (22). Thus if we denote the actual form by

*The coefficient of the singularity in G is unity, i.e., $G = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{-1/2}$ plus a regular function. This coefficient is positive for a sink and negative for a source in contrast to the usual situation, because we take for the flow velocity the gradient of the potential, rather than the negative of the gradient.

$$\hat{z} = \pm s\left(x, y; \frac{V}{2L}\right), \tag{23}$$

its expansion in powers of $V/2L$ is assumed to be (see, for instance, Ref. 14)

$$s\left(x, y; \frac{V}{2L}\right) = \frac{V}{2L}\,\zeta(x) + \text{smaller terms}. \tag{24}$$

Equation (24) reflects the fact that the so-called strut is a strut only to the first order in $V/2L$. The actual shape as determined by the closed streamlines of Eq. (22) has a depth dependence; however, we expect that the shape approaches a strut as a limiting form for great depths.*

At this point it is important to observe that the leading term in the approximation to C_w for the shape $s(x,y;\, V/2L)$, as $V/2L \to 0$, is given by Eqs. (7) and (16).

The basic idea, on which all our results depend, is to consider a class of potentials more extensive than that defined by Eq. (21) but including the latter as a subclass. Specifically, let

$$\frac{V}{2L}\,\mu(x,y,z) = \frac{c}{2\pi}\,\frac{V}{2L}\int_0^{\infty} dy' \int_{-1/2}^{1/2} g(x')\,\frac{\partial G(x,y,z;x',y',0)}{\partial x'}\,dx' \tag{25}$$

where $g(x)$ is integrable and satisfies Eq. (11), i.e.,

$$\int_{-1/2}^{1/2} g(x)\,dx = 1. \tag{26}$$

Otherwise $g(x)$ is to be arbitrary. Equation (25) resembles Eq. (21) integrated by parts. Evidently, μ satisfies Eqs. (20) and (20a) with $d\zeta/dx$ replaced by dg/dx. The derivative dg/dx need not be integrable, and $g(\pm 1/2)$ need not vanish; hence the potentials μ form a larger class than the potentials ϕ, since the functions $\zeta(x)$ appearing in Eq. (21) were assumed to have integrable first derivatives and to vanish at $x = \pm 1/2$. (Obviously, the class of function $\zeta(x)$ is included in the class $g(x)$.) Our expectation is that among the ship forms defined by this larger class of potentials there will be found one with a minimum wave resistance. That this is so will be established shortly.

As in the case of the forms $s(x,y;\, V/2L)$ associated with ϕ, the extended class of forms, $\sigma(x,y;\, V/2L)$ say, associated with μ, are made up of the closed streamlines† of the flow

*For instance a depth equal to a wavelength of a surface wave having velocity c.

†Although we have not proved it rigorously, we believe that there will always be a closed body (i.e., a surface of closed streamlines) formed in the flow defined by Eqs. (25) and (27) whenever the total moment of the dipole distribution on the x axis is negative. Since, as discussed above, the density of the distribution is $-(c/2\pi)(V/2L)\,g(x)$, the assertion is that there will be a closed body if

$$\int_{-1/2}^{1/2} g(x)\,dx > 0.$$

We now observe that all the distributions considered satisfy this latter condition because they all satisfy Eq. (26), the right side of which is always positive. Thus, all the flows defined by Eqs. (25) and (27), subject to Eq. (26), lead to closed bodies.

whose potential is $\theta(x,y,z; V/2L)$ where

$$\theta\left(x,y,z;\frac{V}{2L}\right) = cLx + \frac{V}{2L}\mu(x,y,z). \tag{27}$$

If we could expand $\sigma(x,y; V/2L)$ in powers of $V/2L$, we would expect to find

$$\hat{z} = \pm\sigma\left(x,y;\frac{V}{2L}\right) = \frac{V}{2L}g(x) + \text{smaller terms} = \sigma_1\left(x,y;\frac{V}{2L}\right) + \ldots \tag{28}$$

analogous to Eq. (24).

Just as $V\zeta(x)/2L$ in Eq. (21) is the first order approximation to $s(x,y; V/2L)$, as shown in Eq. (24), so $Vg(x)/2L = \sigma_1$ in Eq. (25) is the first-order approximation to $\sigma(x,y; V/2L)$, as shown in Eq. (28). However, in contrast to $\zeta(x)$, $g(x)$ need merely be integrable and can therefore have integrable singularities, so that the first-order approximation $\sigma = Vg/2L$ is not always uniform in x. On the other hand, in spite of the nonuniformity, we expect that the integral in Eq. (26) when multiplied by $V/2L$ still gives, to first order, the volume-per-unit draft.

Before we can find the function $g(x)$ in Eq. (25) that minimizes the first-order drag coefficient of the shape σ we need an expression for this coefficient. (Equations (7) and (16) apply to the restricted class of shapes, $s(x,y; V/2L)$ but not necessarily to the extended class, $\sigma(x,y; V/2L)$.) To obtain the drag coefficient for σ, we note that $-\partial G(x,y,z;x',y',0)/\partial x'$ in Eq. (25) gives the potential of an x-directed dipole located at $(x',y',0)$.* Therefore, we may interpret μ in Eq. (25) as arising from a distribution of such dipoles over the half-strip $-1/2 \leq x' \leq 1/2$, $y' \geq 0$, $z' = 0$, the density of the distribution being $-(c/2\pi)(V/2L)\,g(x')$.

It is possible to calculate the force exerted on any one of the above dipoles by the combination consisting of the remaining dipoles and the uniform flow. The force on the entire distribution, i.e., the wave resistance, is then the product of the density function, $-(c/2\pi)(V/2L)\,g(x')$, and the force on a single dipole, integrated over the distribution. When the integration is carried out and the integral divided by $(\rho c^2/2)(V/2L)^2$, the drag coefficient of the distribution and therefore of the shape $\sigma(x,y; V/2L)$ is

$$C_w = -4F^2 \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} g(x)\, g(x')\, Y_0(F|x-x'|)\, dx'\,dx. \tag{29}$$

This result is proved in Appendix A.

Comparison of Eq. (29) with Eq. (16) shows that the expression for the wave drag of the shape $\sigma(x,y; V/2L)$ has the same form as that for the drag of the shape $s(x,y; V/2L)$ with the restricted functions $\zeta(x)$ replaced by $g(x)$. Note, however, that Eqs. (7) and (16), which were equivalent for the restricted class $\zeta(x)$, are no longer so when ζ is replaced by g. Indeed, if dg/dx is not integrable, such replacement makes Eq. (7) meaningless. Thus, only Eqs.

*Recall that G is the potential of a sink.

(16) or (29) represent valid formulas for the drag of the extended class of shapes $\sigma(x,y;\, V/2L)$.*

To minimize this drag among all shapes $\sigma(x,y;\, V/2L)$, we must determine a function $g(x)$ that minimizes C_w in Eq. (29), subject to the side condition Eq. (26). This is a straighforward problem in the calculus of variations and leads to the following integral equation for the minimizing function $g(x)$:

$$\int_{-1/2}^{1/2} g(x')\; Y_0(F|x-x'|)\; dx' = \lambda \tag{30}$$

where λ is a (constant) Lagrange multiplier whose value is determined by the use of Eq. (26). It is easily shown that Eq. (30) is the necessary and sufficient condition for $g(x)$ to minimize Eq. (29).

Once Eq. (30) has been solved, subject to Eq. (26), the resultant $g(x)$ is substituted into Eq. (25) and the potential $\mu(x,y,z)$ obtained. The minimizing shape, $\sigma(x,y;\, V/2L)$, is then given by the set of closed streamlines associated with the potential $cLx + (V/2L)\,\mu(x,y,z)$. As already noted, such a shape will not be a true strut; it is a strut only to the first order in $V/2L$.

It has been proved [10] that any solution to the integral Eq. (30) must become singular at the endpoints $x = \pm 1/2$, the singularity being of the form

$$\frac{1}{\sqrt{\frac{1}{4} - x^2}}\,.$$

Thus the solution cannot have a first derivative which is integrable over the closed interval $-1/2 \le x \le 1/2$, and therefore cannot belong to the set of functions $\zeta(x)$. It is for this reason that we introduced the extended class of functions $g(x)$.

Since the minimizing $g(x)$, i.e., the solution to Eq. (30), becomes infinite at $x = \pm 1/2$, it may seem that such a function can have little to do with the actual minimizing shape, $\sigma(x,y;\, V/2L)$. However, Eq. (28) indicates that $(V/2L)\,g(x) = \sigma_1$ is the first-order (in $V/2L$) approximation to σ. The explanation is that the approximation of σ by σ_1 is not uniform in x, and gets worse and worse toward the endpoints $x = \pm 1/2$. Thus, even for very small $V/2L$, the shape will be well approximated by σ_1 only if x is not too near the ends. The shape is determined by a set of streamlines, as stated above, and its complete determination is discussed in Sections 3 and 5.

We have interpreted $g(x)$ as a measure of the density, $-(c/2\pi)(V/2L)\,g(x)$, of the dipole distribution that gives rise to the shape $\sigma(x,y;\, V/2L)$. In this view, the minimization process consists in the following: among all distributions of x-directed dipoles in the half-strip already defined, whose density is a function of x only and satisfies Eq. (26), we choose that

*Note that if $g(x) =$ const. the body in an infinite fluid would be a Rankine oval, and delta function singularities would have been necessary in Eq. (7).

distribution which has (and hence produces a shape $\sigma(x,y;V/2L)$ which has) the least wave resistance.*

Since the minimizing density is infinite at the endpoints $x = \pm 1/2$, the stagnation points of the flow defined by Eq. (27) lie outside the interval $-1/2 \leq x \leq 1/2$. Hence the minimizing shape $\sigma(x,y;V/2L)$ actually extends beyond the limits of this interval. However, as $V/2L \to 0$ the stagnation points approach $\pm 1/2$, as shown in Sections 3 and 5.

Finally, we remark that any solution of Eq. (30) is an even function of x, so that if free surface effects are disregarded the minimizing struts may be said to be symmetrical fore and aft. The above discussion summarizes our formulation of the problem of the strut of minimum wave resistance.

3. THE MINIMIZING SHAPE FOR LARGE FROUDE NUMBER

When the Froude number is large, it is possible to derive an algebraic expression for the limiting form assumed by the minimizing shape at great depths. In this section we indicate briefly how this can be done.

For large Froude number the parameter F is small, so that the kernel in the integral Eq. (30) can be approximated in the usual way by the formula

$$Y_0(F|x-x'|) \cong \frac{2}{\pi} \log \left(\frac{1}{2}\gamma' F|x-x'|\right) \tag{31}$$

where $\log \gamma' = \gamma =$ Euler's constant $= 0.577\ldots$.

Thus, the integral equation becomes

$$\frac{2}{\pi} \int_{-1/2}^{1/2} g(x') \log \left(\frac{1}{2}\gamma' F|x-x'|\right) dx' = \lambda \,. \tag{32}$$

The solution to this equation is an elementary function. Specifically, we have [15]

$$g(x) = \frac{A}{\sqrt{\frac{1}{4} - x^2}} \tag{33}$$

where A is a constant that depends on λ and F, i.e., it depends on the side condition Eq. (26) and on the Froude number, f. Naturally, when $\lambda = \lambda(F)$ is chosen so that $g(x)$ satisfies Eq. (26) then $A =$ constant independent of F. Wehausen [3] regards the singularity at $|x| = 1/2$ as depriving the solution of physical reality. This criticism is valid in the context

*We recall, that, to first order in $V/2L$, Eq. (26) fixes the volume-per-unit draft. Thus within the limitations of first-order theory, the minimization is still being performed among all struts having a given carrying capacity per unit draft.

of Pavlenko's theory in which $g(x)$ represents the form of the strut. In the present theory $g(x)$ represents a dipole density and the criticism does not apply.

Now, according to the procedure developed in the previous sections, we can obtain the minimizing shape σ by substituting $g(x)$ from Eq. (33) into Eq. (25) and determining the closed streamlines of the potential defined by Eq. (27). However, this procedure can be bypassed if we desire only the limiting form assumed by σ at large depths. At such depths, the flow is very nearly two-dimensional (horizontal) and can therefore be characterized by a complex velocity potential

$$\Omega(\tau) = cL\tau + \frac{V}{2L} w(\tau) \tag{34}$$

where $\tau = x + iz$ and w is the complex perturbation potential arising from the presence of the strut.

In terms of its real and imaginary parts,

$$w(\tau) = \mu(x,z) + i\xi(x,z) \tag{35}$$

where μ is the real perturbation potential of Eqs. (25) and (27) and ξ is the associated real stream function.

We now verify that w has the specific form

$$w(\tau) = \frac{iK}{\sqrt{\frac{1}{4} - \tau^2}} . \tag{36}$$

In Eq. (36), K is a real constant to be determined. The definition of w is completed by stipulating that the τ plane be slit along the x axis from $-1/2$ to $+1/2$, with the square root given the positive determination on the side of the slit for which $z = 0+$.

Differentiating Eq. (36) with respect to z, we get

$$\frac{\partial w}{\partial z} = \frac{dw}{d\tau}\frac{\partial \tau}{\partial z} = \frac{iK\tau}{\left(\frac{1}{4} - \tau^2\right)^{3/2}} \frac{\partial \tau}{\partial z} . \tag{37}$$

Letting $z \to 0+$ in Eq. (37) and introducing Eq. (35), we have

$$\frac{\partial \mu(x,0+)}{\partial z} + i \frac{\partial \xi(x,0+)}{\partial z} = -\frac{Kx}{\left(\frac{1}{4} - x^2\right)^{3/2}} . \tag{38}$$

It follows that

$$\frac{\partial\mu(x,0+)}{\partial z} = -\frac{Kx}{\left(\frac{1}{4} - x^2\right)^{3/2}}, \qquad -\frac{1}{2} < x < \frac{1}{2}. \tag{39}$$

On the other hand, differentiation of $g(x)$ with respect to x in Eq. (33) gives

$$\frac{dg}{dx} = \frac{Ax}{\left(\frac{1}{4} - x^2\right)^{3/2}}. \tag{40}$$

Now we set $K = -Ac$. Then, upon comparing Eqs. (39) and (40), we get

$$\frac{\partial\mu(x,0+)}{\partial z} = c\,\frac{dg}{dx} \tag{41}$$

which is what Eq. (20) becomes when ϕ is replaced by μ and ζ by g. (The process $z \to 0-$ reproduces Eq. (41) with a negative sign on the right, in agreement with Eq. (20).)

We conclude that for large Froude numbers and at large depths, the perturbation potential, μ, of Eqs. (25) and (27) is given by

$$\mu(x,z) = \mathrm{Re}\,\frac{-iAc}{\sqrt{\frac{1}{4} - \tau^2}} = \mathrm{Im}\,\frac{Ac}{\sqrt{\frac{1}{4} - \tau^2}}. \tag{42}$$

The associated perturbation stream function, ξ, is therefore

$$\xi(x,z) = \mathrm{Re}\,\frac{-Ac}{\sqrt{\frac{1}{4} - \tau^2}}. \tag{43}$$

We recall that the constant A appearing in Eqs. (42) and (43) is determined by the Froude number and the side condition Eq. (26). But

$$\int_{-1/2}^{1/2} \frac{A}{\sqrt{1/4 - x^2}}\,dx = 1 \text{ implies } A = \pi^{-1}.$$

From Eqs. (34) and (43) it follows that the stream function, $\Gamma(x,z)$, for the total flow at large depths and large Froude number has the form

$$\Gamma(x,z) = cLz + \frac{V}{2L}\,\zeta(x,z) = cLz - \frac{V}{2L}\,\mathrm{Re}\,\frac{c\pi^{-1}}{\sqrt{\frac{1}{4} - \tau^2}}. \tag{44}$$

The closed streamline defining the minimizing shape, σ, is given by

$$\Gamma(x, z) = 0 \,. \tag{45}$$

To see this, note that the flow is symmetrical about the x-y plane, so that the part of the x axis outside the shape σ belongs to the streamline in question. Now as $x \to \infty$ (which implies $r \to \infty$), $\Gamma(x,z) \to cLz$, as is evident from Eq. (44). But $cLz = 0$ on the x axis, proving that the desired streamline is defined by Eq. (45).

Thus the limiting form assumed by the minimizing shape, σ, at large depths and for large Froude number, is represented by

$$cLz - \frac{V}{2L} \operatorname{Re} \frac{c\pi^{-1}}{\sqrt{\frac{1}{4} - \tau^2}} = 0 \qquad (\tau = x + iz) \,. \tag{46}$$

If we define a thickness parameter $\epsilon = V/2L^2 = \bar{B}/2L$,* where $\bar{B}$ is the average (full) width, we can rewrite Eq. (46) as

$$z = \frac{\epsilon}{\pi} \operatorname{Re} \left(\sqrt{\frac{\frac{1}{2} + \tau}{\frac{1}{2} - \tau}} + \sqrt{\frac{\frac{1}{2} - \tau}{\frac{1}{2} + \tau}} \right) . \tag{47}$$

In bipolar coordinates† ξ and θ we have

$$z = \frac{\frac{1}{2} \sin \theta}{\cosh \xi + \cos \theta} \qquad \text{and} \qquad x = \frac{\frac{1}{2} \sinh \xi}{\cosh \xi + \cos \theta}$$

letting the line $\theta = 0$ have length 1. Also

$$\frac{\frac{1}{2} + \tau}{\frac{1}{2} - \tau} = e^{\xi}(\cos \theta + i \sin \theta)$$

and Eq. (47) becomes

$$\frac{\frac{1}{2} \sin \theta}{\cosh \xi + \cos \theta} = 2 \frac{\epsilon}{\pi} \cosh \frac{\xi}{2} \cos \frac{\theta}{2} \,.$$

*ϵ is also equal to the horizontal-plane area coefficient multiplied by $B/2L$, where B is the usual maximum beam.

†See, for example, Morse and Feshbach, "Methods of Theoretical Physics," New York: McGraw-Hill, 1953, p. 1210.

Finally,

$$\sin\frac{\theta}{2} = \frac{\pi}{8\epsilon\cosh\frac{\xi}{2}}\left(-1 + \sqrt{1 + \frac{64\epsilon^2}{\pi^2}\cosh^4\xi}\right) \tag{48}$$

which is a convenient form for computation. It is easily seen that $\xi = 0$ corresponds to $x = 0$. In order to get an idea of the range of ξ we observe that at the stagnation points we have $y = 0$, $|x| > 1/2$, and hence $\theta = \pi$. Solving Eq. (48) with $\theta = \pi$ approximately for ξ_{max}, the value of ξ at the stagnation point, we find

$$\xi_{max} \cong 2\cosh^{-1}\left[\frac{\pi}{4\epsilon} + \left(\frac{\pi}{4\epsilon}\right)^{1/3}\right]^{1/3} \tag{49}$$

and

$$|x_{max}| \approx \frac{1}{2} + \left(\frac{\epsilon}{2\pi}\right)^{2/3} .$$

The midship beam is given to first order by

$$z(0) \approx \frac{2\epsilon}{\pi} .$$

Since $2\epsilon/\pi$ is the beam given by the Michell theory it is interesting to compare $z(0)$ with $2\epsilon/\pi$. Table 1 shows that the relative error is less than 0.008 for $\epsilon < 0.10$.*

Table 1
Comparison of $z(0)$ and $2\epsilon/\pi$

ϵ	$\frac{2\epsilon}{\pi}$	$z(0)$	$\frac{z(0) - \frac{2\epsilon}{\pi}}{z(0)}$
0.005	0.00318	0.00318	0.0000
0.01	0.00637	0.00637	0.0000
0.02	0.1273	0.1273	0.0000
0.05	0.03183	0.03177	−0.0019
0.10	0.06366	0.06316	−0.0079

The values of x_{max} indicate the extent to which the form fails to have the preassigned length. Figure 2 shows the optimum forms for five values of ϵ ranging from 0.005 to 0.100. They resemble dogbones.

*Typical values of ϵ are 0.04 for a destroyer and 0.048 for a tanker, based on the waterplane section in each case.

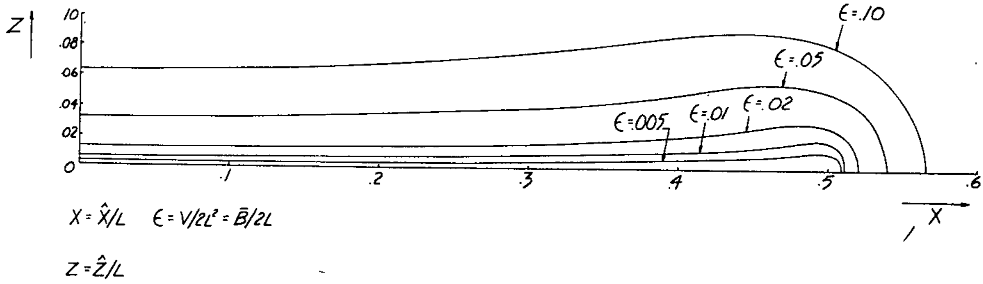

Fig. 2. The optimum shape (at large depth) for large Froude number for various values of the thickness parameter ϵ. This form is obtained by considering the streamfunction of Eqs. (47) and (48). We have Eq. (48), where $z = (1/2)\sin\theta/(\cosh\xi + \cos\theta)$ and $x = (1/2)\sinh\xi/(\cosh\xi + \cos\theta)$.

Equation (28) implies that

$$\int_{-1/2}^{1/2} \hat{z}(x)\, dx \cong \int_{-1/2}^{1/2} \sigma_1(x)\, dx = \int_{-1/2}^{1/2} \frac{V}{2L}\, g(x)\, dx = \frac{V}{2L} \tag{50}$$

or

$$\int_{-1/2}^{1/2} z(x)\, dx \cong \epsilon .$$

To see how well the approximate side condition specifies the volume we have compared

$$\int_{-x_{max}}^{x_{max}} z(x)\, dx,$$

computed by numerical integration, with ϵ, for various values of ϵ. The results are as follows:

ϵ	0.0050	0.0100	0.0200	0.050	0.100
$\int_{-x_{max}}^{x_{max}} z(x)\, dx$	0.0047	0.0093	0.0177	0.044	0.082

Expansion of the exact volume for small ϵ yields

$$\int_{-1/2}^{1/2} z(x)\, dx \cong \epsilon\left[1 - 0.44\,\epsilon^{1/3}\right] \tag{50a}$$

in which the coefficient 0.44 is approximate. The general form (specifically the minus sign) of this relation is in agreement with the theorem of G. I. Taylor relating the dipole moment, the volume, and the added mass in the x-direction, for a line dipole distribution in a flow without free surface.

4. NUMERICAL SOLUTION OF THE INTEGRAL EQUATION

Equation (30), with $\lambda = -1$, was solved by numerical methods as described in Appendix B. The solution depends on Froude number and is denoted by $g_o(x)$ or $g_o(x;f)$. Solutions for other values of λ are given simply by $\lambda g_o(x)/-1$. The desired solution $g(x)$ which satisfies Eq. (26) is given by

$$g(x;f) = \frac{g_o(x;f)}{I_o(f)} \tag{51}$$

where

$$I_o(f) = \int_{-1/2}^{1/2} g_o(x;f)\ dx. \tag{52}$$

$I_o(f)$ and $g(x;f)$ have been computed from $g_o(x;f)$. As indicated in Appendix B

$$g(x) = \frac{h(x)}{\sqrt{\frac{1}{4} - x^2}} \tag{53}$$

where $h(x)$ is a bounded function. The behavior of $h(\pm 1/2)$ as a function of f (see Fig. 5) is of interest since when $h(\pm 1/2)$ is large (small) the bulge in the shape at bow and stern is also large (small), as shown in Section 5. Figure 4 shows $g(x;f)$ for various values of f. Figure 5 shows $h(x;f)$ for various values of f. Figures 3, 4, and 5 show that bluntness is a predominant feature at high speed, but less so at low speeds. However, the forms appear to have bulbs at all values of f yet examined ($f > 0.3$). Note that (in Fig. 3) $h(\pm 1/2) \to \pi^{-1} \cong 0.3183$, as $f \to \infty$, in agreement with Eq. (43) et seq.

The function $h(x)$ (Fig. 5) is of some interest. We have $h(x) \to \pi^{-1}$ as $f \to \infty$, as explained in Section 3. Hence one can qualitatively define a high-speed range as that in which $h(x) \sim \pi^{-1}$. Figures 4 and 5 suggest that $f > 0.6$ is a reasonable definition of the high-speed range. Below $f = 0.6$ there is a rapid transition from the dogbone shape characteristic of high speed to shapes whose maximum beam is amidship.

Let $C_w^{f'}(f)$ be the wave resistance coefficient at Froude number f of the form which is optimum at Froude number f'. $C_w^f(f)$, shown in Fig. 6, is then a universal optimum for struts. Note that $C_w^f(f)$ is a maximum at $f = 0.65$. In Fig. 6 we show also $C_w^{1.0}(f)$ and $C_w^{0.5}(f)$. $C_w^f(f)$ and $C_w^{1.0}(f)$ are very close for $f > 0.6$, which substantiates our identification of $f > 0.6$ as the high-speed range. The divergence of $C_w^f(f)$ and $C_w^{1.0}(f)$ below $f = 0.6$ is substantial. $C_w^f(f)$ and $C_w^{0.5}(f)$ are close in the range $0.46 < f < 0.6$.

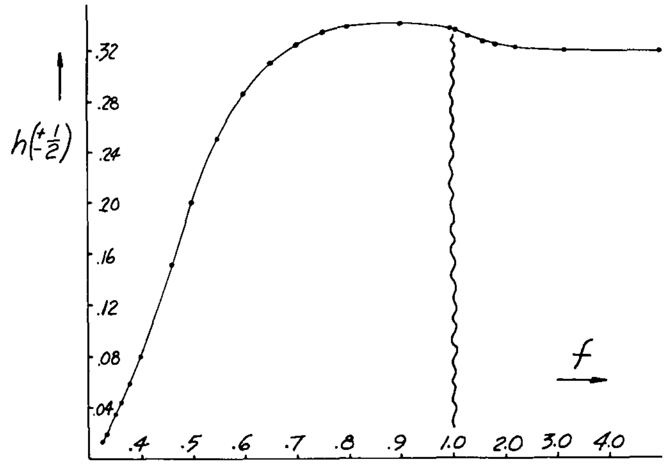

Fig. 3. Graph of $h(\pm 1/2)$ vs f. The optimizing dipole distribution has the form $-(c/2\pi)\, h(x)/(1/4 - x^2)^{1/2} = -cg(x)/2\pi$; see Eq. (53). $h(\pm 1/2)$ is the coefficient of the edge singularity, and is important in determining the shape at the ends.

As indicated in Appendix B the interval of integration (half a boat-length) was divided into six subintervals (seven points). One of the factors affecting the accuracy of the numerical scheme with seven points is the value of F, since as F increases the number of oscillations of $Y_0(F|x|)$ increases. We believe that all our numerical results for $f < 0.4$ should be regarded as questionable. For some lower values of f the numerical solution of the integral equation and the wave resistance coefficient had the wrong sign. Figure 3 shows $h(\pm 1/2)$ decreasing rapidly for $0.3 < f < 0.4$, with intentions of becoming negative. Also, a separate computer program was written for calculating C_w for any strutlike dipole distribution. The mesh in this program was variable, and it was found that seven points was not sufficient to calculate C_w for $f = 0.4$ with an accuracy of the order of 5 percent whereas seven points gave an accuracy of better than 1 percent for $f = 0.5$. A more powerful computer program is in preparation for work on the three-dimensional problem, and we expect that it will allow us to extend the range of accurate calculation to lower values of f.

5. DETERMINATION OF THE SHAPE NEAR THE ENDS

Having found the optimum dipole distribution we are left with the problem of finding the form. While digital methods were available and will play a role in future work we preferred to determine the shape approximately by analytic means. We expect that $z = Vg(x)/2L$ should be a good approximation to the true shape away from the ends, at least for sufficiently small $V/2L^2 = \epsilon$ and sufficiently far from the free surface. Figure 7 shows the exact shape and the linearized shape $g(x)$ in the case of $f = \infty$, for $\epsilon = 0.05$. The agreement even away from the ends is not as good as one would like* but no attempt will be made to improve this aspect of the theory.

*See also Fig. 10. For additional discussion see Section 6.

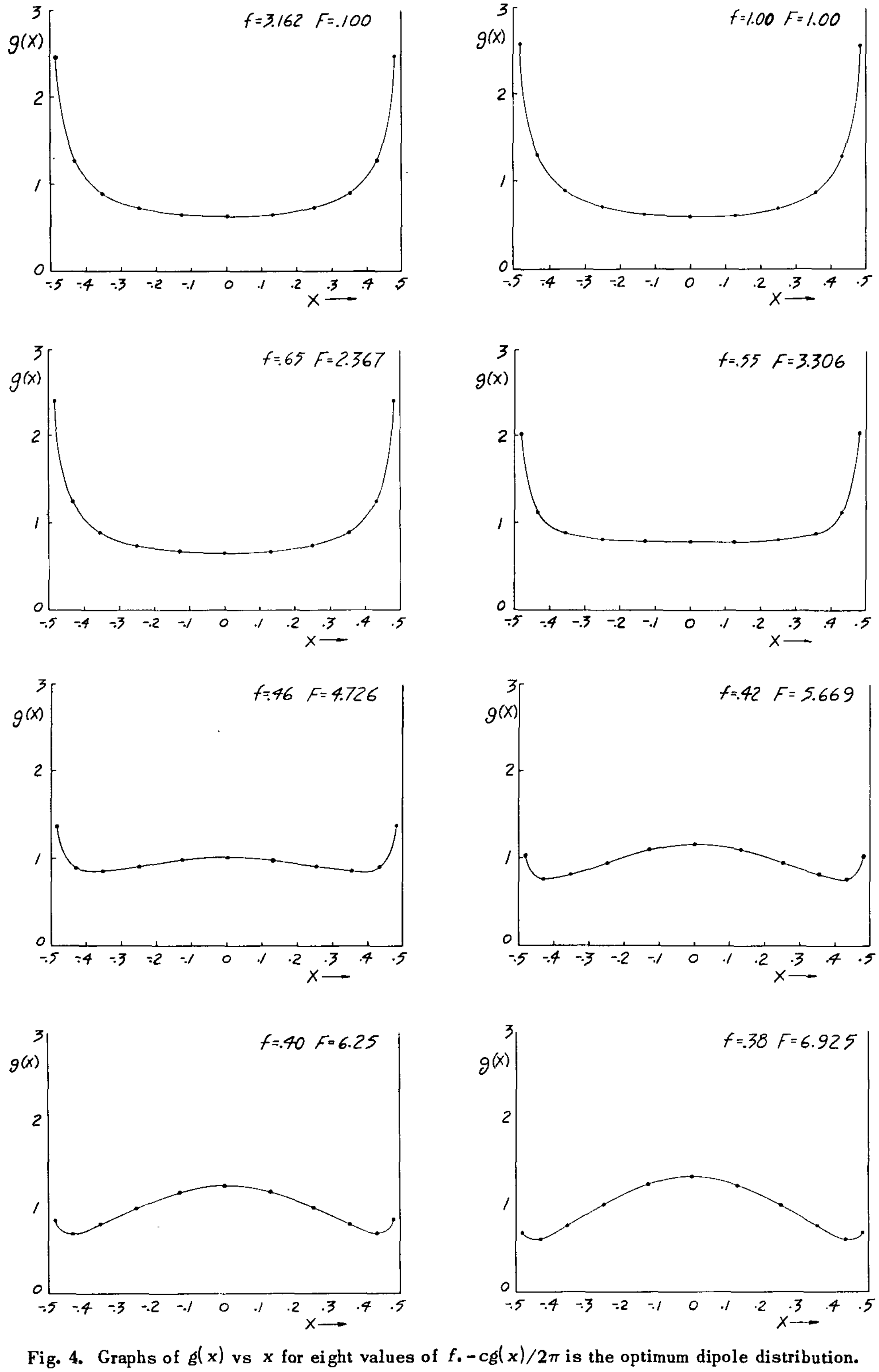

Fig. 4. Graphs of $g(x)$ vs x for eight values of f. $-cg(x)/2\pi$ is the optimum dipole distribution.

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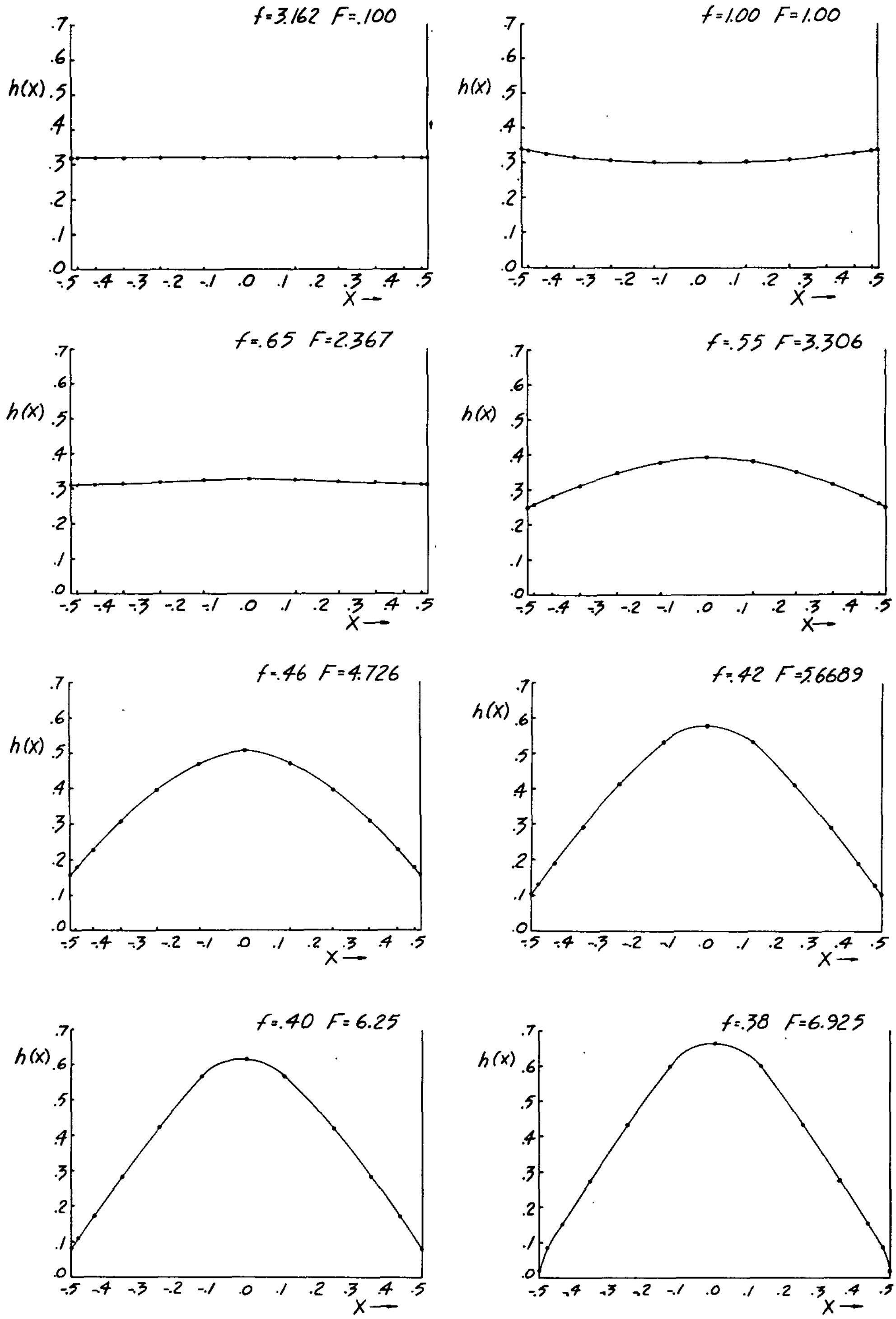

Fig. 5. Graphs of $h(x)$ vs x for eight values of f. $h(x) = g(x)(1/4 - x^2)^{1/2}$. Note that $h(x)$ for $f>0.65$ is almost independent of f.

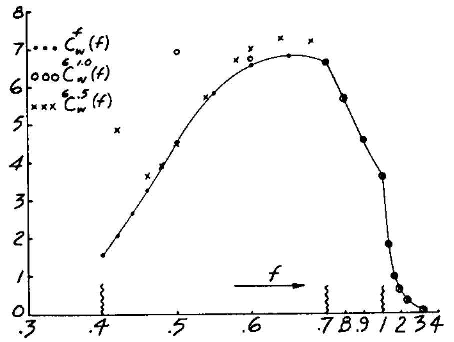

Fig. 6. $C_w^f(f)$, $6C_w^{1.0}(f)$ and $6C_w^{0.5}(f)$ vs f. Notice the changes of f-scale at $f = 0.7$ and 1.0. $C_w^f(f)$ is the universal minimum curve. The figure shows that a dipole distribution which is optimal for $f = 1$ performs well for $0.6 < f < \infty$, but ceases to do so as f decreases. Likewise, a strut which is optimal for $f = 0.5$ performs well in the range $0.46 < f < 0.6$.

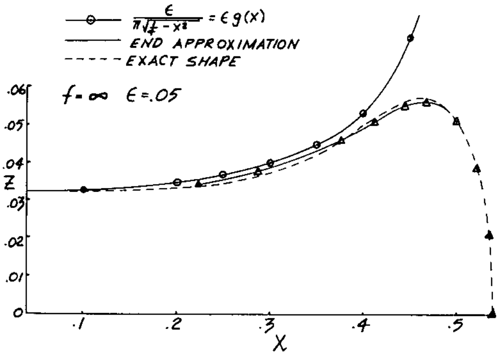

Fig. 7. Graphs of the exact optimum shape, $\epsilon g(x)$, for $f = \infty$ and an approximation to the shape. The figure shows the extent to which the approximate shape at the ends matched the linearized shape, and the way in which we joined the two.

We now prove that the shape at each end is determined by the singularity of $g(x)$ at that end in the following sense. Suppose we have a distribution of x-directed dipoles whose strength is 0 for $x' > 0$ and $-q(x')/\pi(x')^{1/2}$ for $x' > 0$, where $x' = x + L/2$ and

$$q(x') = \frac{V}{2L}\,(A_0 + A_1 x' + A_2 (x')^2 + \ldots)\,. \qquad (54a)$$

If we let $\hat{x}' = \hat{r}\cos\theta$ and $\hat{z}' = \hat{r}\sin\theta$, the stream function of the two-dimensional flow is given by

$$\hat{\xi} = c\hat{r}\sin\theta - \frac{V}{2L}\left(\frac{A_0}{\hat{r}^{1/2}}\cos\frac{\theta}{2} + A_1\hat{r}^{1/2}\cos\frac{\theta}{2} + A_2\hat{r}^{3/2}\cos\frac{3\theta}{2} + \ldots\right). \qquad (55)$$

Let us introduce dimensionless variables $r = \hat{r}/L$, $\xi = \hat{\xi}/L$, so that the dipole density is

$$-\frac{q(x')}{\pi\sqrt{x'}} = -\frac{L\epsilon c}{\pi\sqrt{x'}}\,(B_0 + B_1 x' + \ldots) \qquad (54b)$$

where

$$B_0 = \frac{A_0}{cL^{1/2}}$$

$$B_1 = \frac{A_1 L^{1/2}}{c}$$

and the dimensionless stream-function is given by

$$\xi = c\left[r \sin\theta - \epsilon\left(\frac{B_0 \cos\theta/2}{r^{1/2}} + B_1 r^{1/2} \cos\frac{\theta}{2} + \ldots\right)\right]. \tag{56}$$

If we solve Eq. (54b) when $\xi = 0$ (the streamline passing through the stagnation point) for r as a function of θ we obtain the following expansion of r in terms of θ and ϵ:

$$r = B_0^{2/3}\left(\frac{\epsilon}{2\sin\theta/2}\right)^{2/3} + \frac{2}{3} B_0^{1/3} B_1\left(\frac{\epsilon}{2\sin\theta/2}\right)^{4/3} + \ldots \tag{57}$$

where the successive terms involve ascending powers of ϵ.* We conclude that for small ϵ the shape at one end is determined mainly by the singularity of the dipole distribution at that end. The series $A_0 + A_1\hat{x}' + \ldots$ will have a radius of convergence equal to L, the distance to the other singularity. In order to use Eq. (57) to find the approximate shape at the ends we relate the coefficients B_0 and B_1 to the solution $g(x)$ of the integral equation. Since $\epsilon g(x)$ generates the desired unnormalized shape shrunk by a factor of L in the x-direction, and since the dipole density corresponding to $\epsilon g(x)$ is $-Lc\epsilon g(x)/\pi$, we have from Eq. (54b)

$$-\frac{Lc}{\pi}\epsilon g(x) = -\frac{Lc}{\pi}\epsilon\frac{h(x)}{\sqrt{\frac{1}{4} - x^2}} = -\frac{Lc}{\pi}\epsilon\frac{h(x')}{\sqrt{x'(1-x')}}$$

$$= -\frac{L\epsilon c}{\pi\sqrt{x'}}(B_0 + B_1 x' + \ldots) \tag{58}$$

from which

$$h(x') = (B_0 + B_1 x' + \ldots)\sqrt{1 - x'} \tag{59}$$

and therefore

$$h(x' = 0) = h\left(x = -\frac{1}{2}\right) = B_0 \tag{60a}$$

and

$$\left.\frac{dh}{dx'}\right|_{x'=0} = \left.\frac{dh}{dx}\right|_{x=-1/2} = -\frac{1}{2}B_0 + B_1. \tag{60b}$$

These equations determine B_0 and B_1 in terms of $h(x)$. Since we have only a digital approximation to $h(x)$ we were only able to determine B_0 and B_1 approximately. Figure 7 shows the extent to which the approximate shape at the ends matched the linearized shape, and the way in which we joined the two. This example is for $f = \infty$ but it is fairly typical. Figure 8 shows the resulting shapes for a number of values of f.

*The next term involves all the B_i.

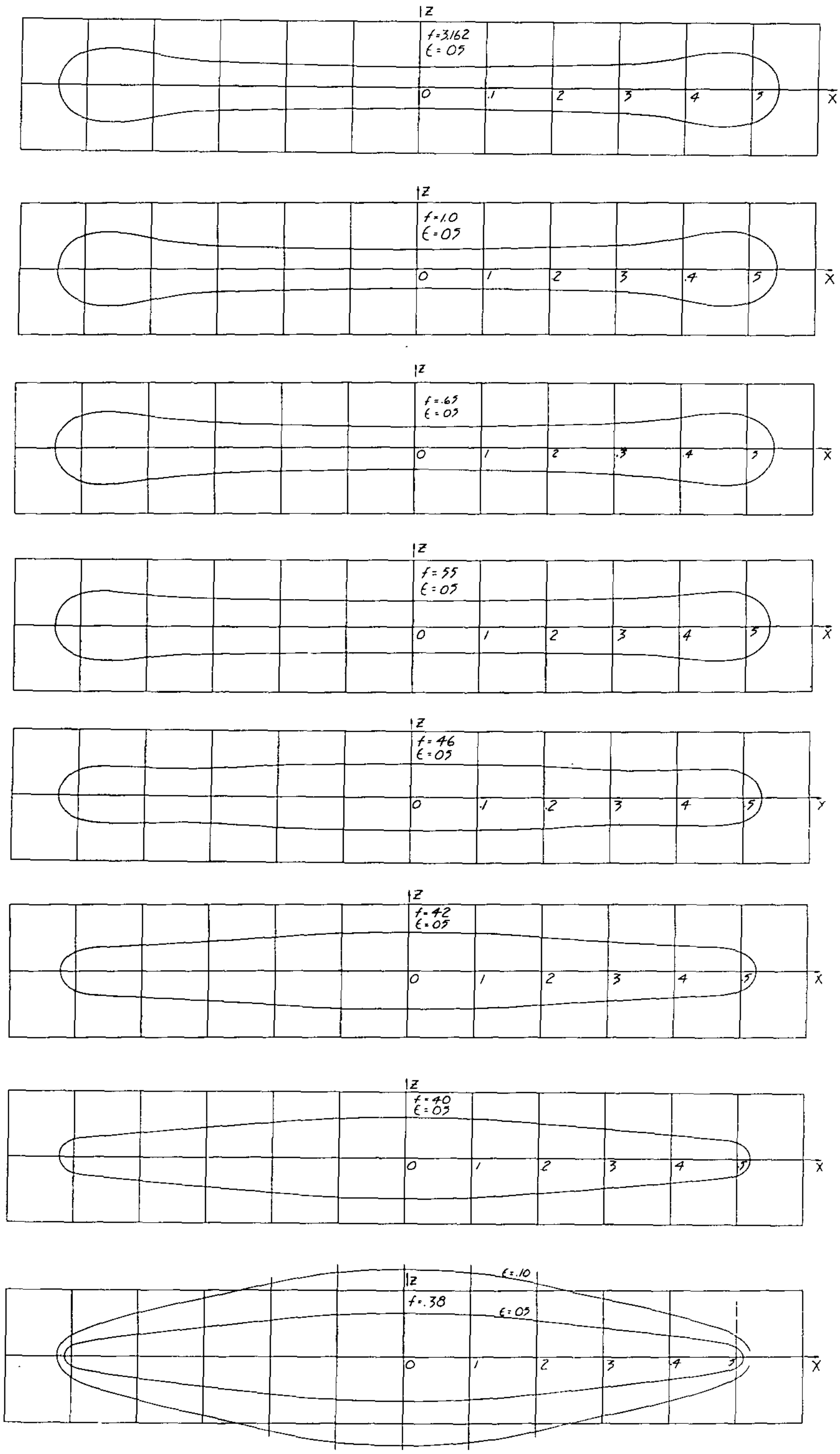

Fig. 8. Approximate optimum shapes at sufficiently large depth for eight values of *f*.

6. COMPARISON OF VARIOUS FORMS

In Section 4 we examined the resistance of some optimum dipole distributions. We now compare these with some others. Figure 9 shows the universal minimum $C_w^f(f)$, $C_w^{0.5}(f)$ and $C_w(f)$ for a parabolic dipole distribution vanishing at the ends (labeled $C_w^P(f)$). It is known that a dipole distribution $-cg(x)/2\pi$, where

$$g(x) = \frac{-\frac{2}{p}\sin\frac{2\pi}{p}\left(1-4x^2\right)^{2/p}}{(1+2x)^{4/p} + (1-2x)^{4/p} - 2\left(1-4x^2\right)^{2/p}\cos\frac{2\pi}{p}} \tag{61}$$

generates (in two-dimensional flow) a lens given by

$$z(x) = \frac{1}{2}\left(\sqrt{\csc^2\frac{p\pi}{2} - 4x^2} + \cot\frac{p\pi}{2}\right). \tag{62}$$

For $p = 1.812$ the full interior lens angle is $(2 - p)\pi = 0.188\pi$, while

$$\int_{-1/2}^{1/2} z(x)\,dx = 0.05 \quad \text{and} \quad \int_{-1/2}^{1/2} g(x)\,dx = 0.057.$$

Figure 9 shows $C_w(f)$ for a dipole distribution $-cg(x)/2\pi$ (labeled $C_w^{LD}(f)$) and also $C_w(f)$ for a dipole distribution $-cz(x)/2\pi$ (labeled $C_w^{LS}(f)$). This last quantity is Michell's wave resistance for the shape $z(x)$, since $-cz(x)/2\pi$ is the linearized dipole distribution defined by the shape $z(x)$. Figure 10 shows $g(x)$ and $z(x)$, and we see that even for these thin forms the approximation $g(x) \sim z(x)$ is not very accurate. Comparing $C_w^{LD}(f)$ with $C_w^{LS}(f)$ in Fig. 9 we find a discrepancy of up to 25 percent. We notice that $C_w^{0.5}(f)$ is about half of $C_w^P(f)$, $C_w^{LD}(f)$, or $C_w^{LS}(f)$ over a considerable range of f.

Some information on Pavlenko's work was obtained from Refs. 4 (p. 115) and 12. The forms shown by Pavlenko are reminiscent of ours, and show the same trend as a function of f (for $f \leq 0.43$, the maximum f considered in these references). Shen and Kern [13] solved the integral equation approximately, by determining three coefficients in a polynomial approximation to the solution. They carried out the computations for $F = 0.1, 0.2, 0.4, 0.6$

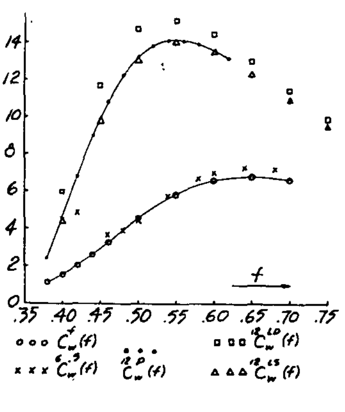

Fig. 9. Graphs of

$$C_w^f(f),\ {}^6C_w^{0.5}(f),\ {}^{12}C_w^P(f),\ {}^{12}C_w^{LD}(f),\ \text{and}\ {}^{12}C_w^{LS}(f)$$

vs f. The upper curve and the points nearby show $C_w(f)$ for a parabolic dipole distribution

$$\left(-\frac{c}{2\pi}\left[\frac{1}{6}\left(\frac{1}{4}-x^2\right)\right]\right),$$

for the dipole distribution generating a lens (in an infinite fluid), and for the linearized dipole distribution defined by the lens. The lower curve and nearby points show the universal minimum $C_w^f(f)$ and also $C_w(f)$ for the dipole distribution which is optimum at $f = 0.5$. A reduction in $C_w(f)$ by 50 percent has been achieved.

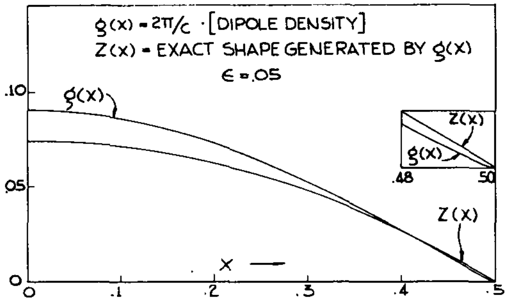

Fig. 10. Graphs of the shape function for a lens and $-2\pi/c$ times the exact dipole distribution which generates it in an infinite fluid.

(f = 3.162, 2.236, 1.581, 1.292). The forms shown by them are somewhat reminiscent of ours. The forms of Pavlenko and of Shen and Kern differ from ours in having finite entrance angles.

In a series of pioneering papers Weinblum has considered the problem of minimizing the wave resistance of more realistic, three-dimensional forms. He has defined various families of polynomials, imposed constraints on the coefficients by specifying some geometrical properties of the form, and used the remaining degrees of freedom to minimize the resistance. Again, in spite of these differences in approach, our forms show the same general variation with Froude number as his waterplane sections.

(It should be noted that all wave resistance coefficients of dipole distributions are normalized with respect to dipole moment (multiplied by $-2\pi/c$) rather than volume. Hence to the extent that volume differs from dipole moment (multiplied by $-2\pi/c$) a comparison of wave resistance coefficients does not provide an exact comparison of resistance values based on equal volume. This discrepancy, like the length discrepancy for optimum forms, does not affect our theoretical results, which are asymptotically valid for $\epsilon \to 0$.)

7. SUMMARY

A universal curve of minimum wave resistance for infinite strutlike dipole distributions has been found (see Fig. 6). The optimizing distributions have been found, and turn out to be infinite at the ends. Despite this last mentioned feature the corresponding shapes, at large depths, are determined. The shapes are blunt at the ends, which extend beyond the interval on which the dipoles are distributed (see Fig. 8). Attention is called to the distinction between shape and dipole distribution, a distinction which is of fundamental importance in the present context. Attention is also called to the discrepancy between the wave resistances calculated using Michell's approximate dipole distribution proportional to shape and the exact dipole distribution yielding that shape, especially for ships with nonzero entrance angle (see Figs. 9 and 10). Some of the objectionable features of the analysis given by previous investigations have been removed.

ACKNOWLEDGMENT

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Appendix A

PROOF OF THE WAVE RESISTANCE FORMULA

We wish to show that Eq. (29) gives the correct formula for the drag coefficient of a distribution of horizontal dipoles in the half-strip $-1/2 \leq x \leq 1/2$, $0 \leq y < \infty$, $z = 0$, the density of the distribution being $-(c/2\pi)(V/2L)\, f(x)$ with $f(x)$ integrable but not necessarily bounded.

In what follows, we revert temporarily to the unnormalized variables $\hat{x}$ and $\hat{y}$. Consider first a finite number, n, of sources located in the above half-strip at the points $(\hat{x}_s', \hat{y}_s', 0)$ $(s = 1, 2, \ldots, n)$ and having strengths $q_s (s = 1, 2, \ldots, n)$. Then the wave resistance, R, of such an assemblage in a steady stream flowing in the $\hat{x}$ direction with velocity c, is shown by Lunde [2] to be

$$R = 16\pi\rho\nu^2 \int_0^\infty \left(I_n^2 + J_n^2\right) \cosh^2 u \, du \tag{A1}$$

where ρ is the fluid density, $\nu = g/c^2$, and

$$I_n = \sum_{s=1}^{n} q_s \cos(\nu \hat{x}_s' \cosh u)\, e^{-\nu \hat{y}_s' \cosh^2 u} \tag{A2}$$

$$J_n = \sum_{s=1}^{n} q_s \sin(\nu \hat{x}_s' \cosh u)\, e^{-\nu \hat{y}_s' \cosh^2 u} \quad . \tag{A3}$$

We now evaluate I_n and J_n when the source collection is arranged to form a finite set of $\hat{x}$-directed dipoles. For the purpose, we must assume that n is even, and we locate the dipoles at the points $(\hat{x}_p, \hat{y}_p, 0)$ $(p = 1, 2, \ldots, n/2)$. The coordinates $\hat{x}_p, \hat{y}_p$ are related to $\hat{x}_s', \hat{y}_s'$ by the equations

$$\left.\begin{array}{rcll} \hat{x}_p & = & \hat{x}_s' & (s = 2p-1) \\ \hat{x}_p + \hat{h} & = & \hat{x}_s' & (s = 2p) \\ \hat{y}_p & = & \hat{y}_s' & (s = 2p-1 \text{ or } 2p) \end{array}\right\} p = 1, 2, \ldots, \frac{n}{2} . \tag{A4}$$

The quantity $\hat{h}$ is a small distance which will eventually approach 0.

The dipole moments, m_p, are related to the source strengths, q_s, by

$$\frac{m_p}{\hat{h}} = q_s \qquad (s = 2p - 1)$$
$$-\frac{m_p}{\hat{h}} = q_s \qquad (s = 2p) . \tag{A5}$$

Introducing Eqs. (A4) and (A5) into (A2) and (A3), we obtain

$$I_n = \frac{1}{\hat{h}} \sum_{p=1}^{n/2} m_p \left\{ \cos(\nu \hat{x}_p \cosh u) - \cos\left[\nu(\hat{x}_p + \hat{h}) \cosh u\right] \right\} e^{-\nu \hat{y}_p \cosh^2 u} \tag{A6}$$

$$J_n = \frac{1}{\hat{h}} \sum_{p=1}^{n/2} m_p \left\{ \sin(\nu \hat{x}_p \cosh u) - \sin\left[\nu(\hat{x}_p + \hat{h}) \cosh u\right] \right\} e^{-\nu \hat{y}_p \cosh^2 u} . \tag{A7}$$

For small $\hat{h}$,

$$\cos\left[\nu(\hat{x}_p + \hat{h}) \cosh u\right] \approx \cos(\nu \hat{x}_p \cosh u) - \hat{h}\nu \cosh u \sin(\nu \hat{x}_p \cosh u) \tag{A8}$$

$$\sin\left[\nu(\hat{x}_p + \hat{h}) \cosh u\right] \approx \sin(\nu \hat{x}_p \cosh u) + \hat{h}\nu \cosh u \cos(\nu \hat{x}_p \cosh u). \tag{A9}$$

Therefore in the limit, as $\hat{h} \to 0$,

$$I_n = \nu \cosh u \sum_{p=1}^{n/2} m_p \sin(\nu \hat{x}_p \cosh u)\, e^{-\nu \hat{y}_p \cosh^2 u} \tag{A10}$$

$$J_n = -\nu \cosh u \sum_{p=1}^{n/2} m_p \cos(\nu \hat{x}_p \cosh u)\, e^{-\nu \hat{y}_p \cosh^2 u} . \tag{A11}$$

These equations represent the forms assumed by I_n and J_n when the original sources (and sinks) are combined so as to form a collection of $n/2$ $\hat{x}$-directed dipoles in the $\hat{x}$-$\hat{y}$ plane, the dipole moments being m_p ($p = 1, 2, \ldots, n/2$). The wave resistance of such a collection is obtained by substituting Eqs. (A10) and (A11) into (A1). Before substituting, we go to the limit of an infinite number of dipoles of infinitesimal strength, continuously distributed with a density $m(\hat{x})$ in the half strip $-L/2 \leq \hat{x} \leq L/2$, $0 \leq y < \infty$, $\hat{z} = 0$. To this end, it is convenient first to imagine the $n/2$ dipoles arranged in a rectangular array of r rows each containing q dipoles. Then (A10) and (A11) can be replaced by the double sums

$$I_n = \nu \cosh u \sum_{i=1}^{q} \sum_{j=1}^{r} m_i \sin (\nu \hat{x}_i \cosh u) \; e^{-\nu \hat{y}_j \cosh^2 u} \tag{A12}$$

$$J_n = -\nu \cosh u \sum_{i=1}^{q} \sum_{j=1}^{r} m_i \cos (\nu \hat{x}_i \cosh u) \; e^{-\nu \hat{y}_j \cosh^2 u} . \tag{A13}$$

Passing to the continuous case with the $\hat{y}$ integration extended to infinity, we get, instead of (A12) and (A13),

$$I = \nu \cosh u \int_{-L/2}^{L/2} \int_0^{\infty} m(\hat{x}) \sin (\nu \hat{x} \cosh u) \; e^{-\nu \hat{y} \cosh^2 u} \; d\hat{y} d\hat{x} \tag{A14}$$

$$J = -\nu \cosh u \int_{-L/2}^{L/2} \int_0^{\infty} m(\hat{x}) \cos (\nu \hat{x} \cosh u) \; e^{-\nu \hat{y} \cosh^2 u} \; d\hat{y} d\hat{x}. \tag{A15}$$

Note that in Equations (A14) and (A15), we have assumed only the integrability of $m(\hat{x})$ and not its boundedness.

The $\hat{y}$ integration can be performed separately, so that (A14) and (A15) become

$$I = \frac{1}{\cosh u} \int_{-L/2}^{L/2} m(\hat{x}) \sin (\nu \hat{x} \cosh u) \; d\hat{x} \tag{A16}$$

$$J = -\frac{1}{\cosh u} \int_{-L/2}^{L/2} m(\hat{x}) \cos (\nu \hat{x} \cosh u) \; d\hat{x}. \tag{A17}$$

Finally, we reintroduce the normalized variable $x = \hat{x}/L$. Then

$$I = \frac{L}{\cosh u} \int_{-1/2}^{1/2} M(x) \sin (Fx \cosh u) \; dx \tag{A18}$$

$$J = -\frac{L}{\cosh u} \int_{-1/2}^{1/2} M(x) \cos (Fx \cosh u) \; dx. \tag{A19}$$

In these equations, $M(x)$ is defined to be $m(Lx)$, while $F = L = gL/c^2$ as before.

According to (A1) and the above discussion, the wave resistance of a continuous distribution of x-directed dipoles in the aforementioned half-strip is

$$R = 16\pi\rho\nu^2 \int_0^\infty (I^2 + J^2) \cosh^2 u \, du \tag{A20}$$

where I and J are given by (A18) and (A19).

Squaring (A18) and (A19), we get

$$I^2 = \frac{L^2}{\cosh^2 u} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} M(x)\, M(x') \sin(Fx \cosh u) \cdot \sin(Fx' \cosh u)\, dxdx' \tag{A21}$$

$$J^2 = \frac{L^2}{\cosh^2 u} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} M(x)\, M(x') \cos(Fx \cosh u) \cdot \cos(Fx' \cosh u)\, dxdx'. \tag{A22}$$

Thus

$$I^2 + J^2 = \frac{L^2}{\cosh^2 u} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} M(x)\, M(x') \cdot \cos\left[F(x-x') \cosh u\right] dxdx'. \tag{A23}$$

Substituting from (A23) into (A20), we have

$$R = 16\pi\rho F^2 \int_0^\infty \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} M(x)\, M(x') \cdot \cos\left[F(x-x') \cosh u\right] dxdx'du. \tag{A24}$$

We perform the u integration first, using the identity

$$\int_0^\infty \cos(t \cosh u)\, du = -\frac{\pi}{2} Y_o(|t|). \tag{A25}$$

Then (A24) becomes

$$R = -8\pi^2 \rho F^2 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} M(x)\ M(x')\ Y_o(F|x-x'|)\ dxdx'. \tag{A26}$$

Now let the density $M(x) = -(c/2\pi)(V/2L)f(x)$, as in Eq. (25) of the text. Then

$$R = -2\rho c^2 \left(\frac{V}{2L}\right)^2 F^2 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} f(x)\ f(x')\ Y_o(F|x-x'|)\ dxdx'. \tag{A27}$$

The drag coefficient, C_w, is obtained by dividing R by

$$\frac{1}{2}\rho c^2 \left(\frac{V}{2L}\right)^2 .$$

Thus, the final result is

$$C_w = -4F^2 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} f(x)\ f(x')\ Y_o(F|x-x'|)\ dxdx' \tag{A28}$$

in agreement with Eq. (29).

Appendix B
NUMERICAL SOLUTION OF THE INTEGRAL EQUATION

The equation to be solved is Eq. (30) of the text, viz.,

$$\int_{-1/2}^{1/2} g_o(x')\ Y_o(F|x-x'|)\ dx' = -1. \tag{B1}$$

It is easily shown that $g_o(x) = g_o(-x)$, so that (B1) can be written

$$\int_0^{1/2} g_o(x') \left\{Y_o(F|x-x'|) + Y_o[F(x+x')]\right\} dx' = -1 . \tag{B2}$$

As already noted, MacCamy has proved [10] that $g_o(x)$ has the form

$$g_o(x) = \frac{1}{\sqrt{1/4 - x^2}} h(x) \tag{B3}$$

where $h(x)$ is a regular function in $-1/2 \leq x \leq 1/2$. The singularity in $g_o(x)$ at $x = 1/2$ introduces a loss of accuracy into the numerical solution of (B2); we therefore convert (B2) into an equation for the regular function, h, by means of the following change of variables:

$$x = 1/2 \sin \beta, \qquad x' = 1/2 \sin \beta'. \tag{B4}$$

Equation (B2) then becomes

$$\int_0^{\pi/2} H(\beta') \left\{ Y_o\left(\frac{F}{2} |\sin \beta - \sin \beta'|\right) + Y_o\left[\frac{F}{2}(\sin \beta + \sin \beta')\right]\right\} d\beta' = -1 \tag{B5}$$

where $H(\beta) = h(1/2 \sin \beta)$. Thus the unknown in (B5) is a regular function.

To perform the integration in Eq. (B5) numerically, we first divide the interval of integration, $0 \leq \beta' \leq \pi/2$, into an even number, n, of equal subintervals of length $\pi/2n$. We denote the values assumed by β' at the endpoints of these subintervals by $\beta_j (j = 1, 2, \ldots, n+1)$. Then $\beta_1 = 0$ and $\beta_{n+1} = \pi/2$. We further denote by H_j the values of the unknown function $H(\beta')$ at β_j, i.e., $H_j = H(\beta_j)$. Finally, we allow the variable β to assume the values $\beta_i (i = 1, 2, \ldots, n+1)$. We thereby obtain $n + 1$ linear algebraic equations for the $n + 1$ unknowns H_j, each equation corresponding to a different value of the index, i. These equations are constructed as follows:

The part of the integration in (B5) that involves $Y_o[F/2(\sin \beta + \sin \beta')]$ can be performed at once by Simpson's rule:

$$\int_0^{\pi/2} H(\beta') Y_o\left[\frac{F}{2}(\sin \beta_i + \sin \beta')\right] d\beta' \approx \frac{\pi}{6n}\left\{H_1 Y_0\left[\frac{F}{2}(\sin \beta_i + \sin \beta_1)\right] + 4H_2 Y_0\left[\frac{F}{2}(\sin \beta_i + \sin \beta_2)\right] + 2H_3 Y_0\left[\frac{F}{2}(\sin \beta_i + \sin \beta_3)\right] + \ldots H_{n+1} Y_o\left[\frac{F}{2}(\sin \beta_i + \sin \beta_{n+1})\right]\right\} \quad (i = 2, 3, \ldots n+1). \tag{B6}$$

(We have excluded the case $i = 1$ in (B6) because it is discussed separately below.)

On the other hand, the logarithmic singularity in Y_o prevents the remaining integral in (B5) from being represented by a formula like that of (B6); when $i = j$, the expression $Y_o(F/2|\sin \beta_i - \sin \beta_j|)$ is meaningless. We therefore break up the integral and write (when $i = 1$ or $n + 1$)

$$\int_0^{\pi/2} H(\beta')\; Y_o\left(\frac{F}{2}\left|\sin\beta_i - \sin\beta'\right|\right) d\beta'$$

$$= \int_{\beta_1}^{\beta_{i=1}} H(\beta')\; Y_o\left(\frac{F}{2}\left|\sin\beta_i - \sin\beta'\right|\right) d\beta'$$

$$+ \int_{\beta_{i-1}}^{\beta_{i+1}} H(\beta')\; Y_o\left(\frac{F}{2}\left|\sin\beta_i - \sin\beta'\right|\right) d\beta'$$

$$+ \int_{\beta_{i+1}}^{\beta_{n+1}} H(\beta')\; Y_o\left(\frac{F}{2}\left|\sin\beta_i - \sin\beta'\right|\right) d\beta' \qquad (i = 2,3,\ldots,n). \tag{B7}$$

Since the total number of subintervals, n, is even, the first and third integrals on the right of (B7) (these contain no singularity) either both extend over an even or both extend over an odd number of subintervals. In the even case, the integrals are evaluated by Simpson's rule. In the odd case, they are evaluated by the trapezoidal rule over the first subinterval and by Simpson's rule over the rest. (Note that when $i = 2$, the first integral on the right of (B7) vanishes, while when $i = n$, the third integral vanishes.)

The second integral on the right of (B7) is approximated by first writing

$$\int_{\beta_{i-1}}^{\beta_{i+1}} H(\beta')\; Y_o\left(\frac{F}{2}\left|\sin\beta_i - \sin\beta'\right|\right) d\beta'$$

$$\approx H_i \int_{\beta_{i-1}}^{\beta_{i+1}} Y_o\left(\frac{F}{2}\left|\sin\beta_i - \sin\beta'\right|\right) d\beta'. \tag{B8}$$

This approximation is sufficiently accurate for our purpose when $n \geq 6$. The integral on the right of (B8) is evaluated by expanding the Bessel function about the point $\beta' = \beta_i$. The result is

$$\int_{\beta_{i-1}}^{\beta_{i+1}} Y_o\left(\frac{F}{2}\left|\sin\beta_i - \sin\beta'\right|\right) d\beta' \approx \frac{2}{n}\left[\log\left(\frac{\pi\gamma' F}{8n}\cos\beta_{\mathrm{i}}\right)\right]\left(1 - \frac{\pi^2F^2\cos^2\beta_{\mathrm{i}}}{192n^2}\right)$$

$$- \frac{2}{n}\left[1 + \frac{\pi^2}{n^2}\left(\frac{1}{72} + \frac{1}{96}\tan^2\beta_i\right) - \frac{\pi^2F^2\cos^2\beta_i}{144n^2}\right] \qquad (i = 2,3,\ldots,n) \tag{B9}$$

where $\log\gamma' = \gamma =$ Euler's constant $= 0.577\ldots$.

By these procedures, we can perform numerically the integration on the left side of (B7) for $i = 2, 3, \ldots, n$. Adding the result to the right side of (B6) and (in accordance with the integral equation (B5)) equating the entire sum to -1, we obtain $n - 1$ linear algebraic equations for the $n + 1$ unknowns $H_j\,(j = 1, 2, \ldots, n + 1)$, each equation corresponding to one of the i values, $i = 2, 3, \ldots, n$.

The remaining two equations, associated with $i = 1$ and $i = n + 1$, are arrived at separately. When $i = 1$,

$$Y_o\left(\frac{F}{2}\,|\sin\beta_1 - \sin\beta'|\right) = Y_o\left(\frac{F}{2}\,|\sin\beta_1 + \sin\beta'|\right)$$

since $\beta_1 = 0$; thus (B5) becomes

$$\int_0^{\pi/2} 2H(\beta')\,Y_o\left(\frac{F}{2}\sin\beta'\right)d\beta' = -1 \qquad (i = 1). \tag{B10}$$

This equation can be rewritten

$$\int_{\beta_1}^{\beta_2} 2H(\beta')\,Y_o\left(\frac{F}{2}\sin\beta'\right)d\beta' + \int_{\beta_2}^{\beta_{n+1}} 2H(\beta')\,Y_o\left(\frac{F}{2}\sin\beta'\right)d\beta' = -1. \tag{B11}$$

The first integral on the left is approximated by setting $H(\beta') = H_1$ and expanding Y_o about $\beta' = 0$. The result is

$$\int_{\beta_1}^{\beta_2} 2H(\beta')\,Y_o\left(\frac{F}{2}\sin\beta'\right)d\beta' \approx \frac{2H_1}{n}\left[\log\left(\frac{\pi\gamma' F}{8n}\right)\right]\left(1 - \frac{\pi^2F^2}{192n^2}\right)$$
$$- \frac{2H_1}{n}\left[1 + \frac{\pi^2}{n^2}\left(\frac{1}{72} - \frac{F^2}{144}\right)\right]. \tag{B12}$$

The second integral in (B11), which is free of singularities, extends over an odd number of subintervals and is therefore evaluated by the combination of the trapezoidal and Simpson's rules already described. Upon adding the result to the right side of (B12) and substituting into (B11), we obtain the algebraic equation corresponding to $i = 1$. Finally, when $i = n + 1$, (B5) becomes

$$\int_{\beta_1}^{\beta_{n+1}} H(\beta')\,Y_o\left(\frac{F}{2}\,|\sin\beta_{n+1} - \sin\beta'|\right)d\beta'$$
$$+ \int_{\beta_1}^{\beta_{n+1}} H(\beta')\,Y_o\left[\frac{F}{2}\left(\sin\beta_{n+1} + \sin\beta'\right)\right]d\beta' = -1 \qquad (i = n+1). \tag{B13}$$

The second integral on the left of (B13) is given by (B6) with $i = n + 1$. The first integral in (B13) can be written (since $\sin \beta_{n+1} = 1$)

$$\int_{\beta_n}^{\beta_{n+1}} H(\beta') Y_o\left(\frac{F}{2}|1 - \sin \beta'|\right) d\beta' = \int_{\beta_1}^{\beta_n} H(\beta') Y_o\left(\frac{F}{2}|1 - \sin \beta'|\right) d\beta'$$

$$+ \int_{\beta_n}^{\beta_{n+1}} H(\beta') Y_o\left(\frac{F}{2}|1 - \sin \beta'|\right) d\beta'. \qquad \text{(B14)}$$

The first integral on the right of (B14) contains no singularities and extends over an odd number of subintervals; we therefore evaluate it by the combination of the trapezoidal and Simpson's rules. The second integral must be specially dealt with because the first derivative of $1 - \sin \beta'$ with respect to β' vanishes at $\beta' = \beta_{n+1} = \pi/2$. If we are to maintain consistency in order of accuracy to which the integral is evaluated, we cannot merely replace $H(\beta')$ by H_{n+1} in the interval $\beta_n \leq \beta' \leq \beta_{n+1}$. Rather, a linear approximation to $H(\beta')$ must be employed, viz.,

$$H(\beta') \approx H_n + \frac{H_{n+1} - H_n}{\beta_{n+1} - \beta_n} (\beta' - \beta_n)$$

$$= H_n + \frac{2n}{\pi} (H_{n+1} - H_n) \left(\beta' + \frac{\pi}{2n} - \frac{\pi}{2}\right) \qquad (\beta_n \leq \beta' \leq \beta_{n+1}). \qquad \text{(B15)}$$

Upon substituting from (B15) into the integral and expanding Y_o about the point $\beta' = \pi/2$, we obtain the following approximation:

$$\int_{\beta_n}^{\beta_{n+1}} H(\beta') Y_o\left(\frac{F}{2}|1 - \sin \beta'|\right) d\beta' \approx \frac{H_n}{2n} \left[\log\left(\frac{\gamma' F \pi^2}{32n^2}\right) - 1 - \frac{\pi^2}{96n^2}\right]$$

$$+ \frac{H_{n+1}}{2n} \left[\log\left(\frac{\gamma' F \pi^2}{32n^2}\right) - 3 - \frac{\pi^2}{288n^2}\right]. \qquad \text{(B16)}$$

Addition of this result to the first integral on the right of (B14), followed by substitution from (B14) into (B13), gives the final algebraic equation for the quantities H_j. This equation corresponds to $i = n + 1$. All our calculations were made with $n = 6$.

Having obtained $g_o(x)$ the computer finds

$$I_o(f) = \int_{1/2}^{1/2} g_o(x; f)\, dx$$

and $g(x) = I_o^{-1}(f)\, g_o(x;f)$. The wave resistance is found easily from

$$C_w^f(f) = 4F^2 I_o^{-1}(f)\ , \tag{B17}$$

which follows from (29) and (B1). Since one can prove that Eq. (29) is a positive definite expression it follows that $I_o(f) > 0$, which is not obvious from the integral equation (B1).

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DISCUSSION

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Unfortunately advance copies of these papers were only available yesterday. This prevents any criticism of the mathematical work, but there are some points of importance which should be mentioned regarding the practical use of these calculations of forms of minimum resistance.

Firstly the wave resistance as calculated by the Michell or equivalent formulas is that wave resistance which would exist in a perfect fluid.

Secondly it is assumed that for a ship the wave resistance is to be simply added to another resistance due to viscosity which does not change greatly with a change of form.

Regarding the first point raised, in fact, except at very high speeds where the Froude number is above 0.4, the effects of viscosity on the wave formation are serious, causing a decrease in the efficiency of the afterbody as a wavemaker.

Regarding the second point, the nonwave resistance may be considered as consisting of the sum of two terms, the first term depending only on the wetted surface and speed and being some 90 to 95 percent of the total, and the second term, sometimes called the form resistance, depending as well on the shape of the form. Very little is known as to the variation of this form resistance with speed, although its value at very low speed is shown by the difference between the calculated frictional resistance and that actually measured, since the wave resistance can be neglected at such speeds. It is suggested by M. Guilloton (see Trans. I.N.A., London, vol. 1952, pp. 352 and 353) that the form resistance is increased at the higher speeds owing to the effect of the wave motion on the flow round the form, a symptom of this influence being the change of attitude of the form during motion.

The effects of these considerations on the actual minima of resistance are well shown by the simple question of the optimum position for the center of buoyancy. From the mathematical theory, as given by these papers today, the optimum position of the center of buoyancy would be amidships at all speeds. In practice, it is found that, at the lower speeds up to a Froude number of about 0.25 where the wave resistance is small the optimum position of the center of buoyancy lies forward of amidships, since the form resistance is thus diminished. As the speed increases the optimum position moves aft and lies aft of amidships for a range of Froude number up to about 0.4, owing to the lower efficiency in wavemaking of the afterbody, which evidently causes an advantage when displacement is moved from the forebody to the afterbody. At still higher speeds the best position tends to agree with the theoretical position at amidships.

It may be of some interest to compare the results for struts of infinite draft found by Messrs. Kotik, Karp, and Lurye with some experimental and calculated results for bulbous bows I published in the Proceedings of the North-East Coast Institution of Engineers & Shipbuilders in 1936. These calculations were made for a spheroid added to a calculable ship form, and the experiments were made with a fairing between the spheroid and the form which would tend to diminish any additional form resistance. Also the change of form was hoped to be insufficient to change appreciably the frictional correction to the wave resistance. Under these precautions the calculated and experimental curves showed, for the best position and size of the bulb, a definite decrease in resistance over a range of Froude number from 0.25 to 0.5.

The conclusion of these comments is that, owing to the uncertainty of the application of such results as those in the papers under discussion, it is advisable that they should be checked by actual measurements before any practical use be made of them.

G. Weinblum (Institut für Schiffbau, Hamburg)

May I express my sincerest thanks to the authors of this paper and of the preceding paper for the aesthetic pleasure presented to me. Some general remarks may be to the point:

1. It is a well-known proposition that ship theory can become too difficult for naval architects.

2. It is therefore advisable for naval architects to love mathematicians.

3. This love, however, should not be unreciprocated.

4. The present speaker feels much obliged that his mathematic colleagues have given such a kind credit to his earlier work. Obviously these attempts were formally rather poor; nonetheless, they embrace some physical and technical ideas which have proved to be fruitful in the further development. Besides the authors mentioned, Dr. Guilloton has contributed to the discussion of the problem which is thrilling both from the point of view of ship hydrodynamics as well as mathematics.

Twenty-five years ago a contribution was made by Prof. von Karman at the 4th International Congress in Cambridge, England, in which he pointed out difficulties encountered when dealing with the exact solution of the minimum problem. Among other things, he found curves with infinite horns shown in the paper by Prof. Karp and others. This finding caused a lot of confusion in the professional world. Unfortunately, Prof. von Karman has forgotten to publish these interesting investigations.

A decisive progress has been made in the meanwhile by distinguishing between the shape of the singularity distribution and the actual body form. Although this difference is well known and has been clearly illustrated by Havelock, e.g., in the case of the general ellipsoid, no use was made of this fact in many earlier publications on our subject. I suggested in vain a thesis on this topic some 12 years ago but it was not completed. So we are essentially indebted to Prof. Inui, because he has dealt with the problem at stake in a consequent and efficient if approximate manner. His severe remark that Wigley and I have hampered the progress in wave resistance research by neglecting the distinction between hull form and distribution is perhaps too hard since our conclusions are primarily based on the sectional area curve and this, fortunately, is less affected by the difficulties mentioned than that of the actual ship form.

I am dwelling at some length on this subject as a warning example: one should not proceed too long in a well-established groove of thought and therefore lose the connection with facts.

Prof. Timman and Mr. Vossers have, as far as I understand, rigorously remained within the concept of the Michell ship. We are looking forward for explicit results. Prof. Karp and colleagues have followed rather the way indicated by Havelock, distinguishing between body form and distribution. Although the results apply to a rather restricted case of a deeply submerged strut, they are in principle extremely interesting. The optimum cross sections plotted correspond to some extent to those which have been derived by approximate methods, except, however, that they all are more exaggerated and show a rounded nose which could not be obtained by the low degree polynomials used before. It would be interesting to have similar cross sections for Froude numbers below 0.38.

This lower range appears to have important practical consequences. Using my earlier work my collaborator Kracht has investigated the influence of the bulbous bow (and stern). Contrary to experimental results and earlier findings by Mr. Wigley and myself he has established that moderate bulbs may have a beneficial influence on wave resistance even at low Froude numbers. We are continuing with these investigations. To get an independent check, wholly submerged bodies of revolution moving in the vicinity of the free surface have been considered. I have already treated this problem several times earlier. However, while in the case of the Michell surface ship it is plausible from physical reasoning to assume zero end ordinates for the distributions, there is no need to introduce this restriction for bodies of revolution. Therefore, together with Dr. Eggers and Mr. Sharma, I have investigated singularities systems which include continuous line doublet distributions, concentrated sources, and, following Wigley, doublets located on the axis of the body.

The plots in Fig. D1 show the optimum longitudinal distribution calculated for a constant area coefficient $\varphi = 0.60$ (except for plot (g)), three depths of immersion ratios $2f/L$, and two Froude numbers.

The symbol $<2, 4, 6, 8\ D>$ indicates the powers of the polynomial terms; D stands for concentrated dipoles at the ends of the axis with a doublet moment a_D.

Computations have been made under three assumptions:

1. The end ordinates of the dipole distribution, $\eta(\pm 1) = 0$.

2. End ordinates $-\eta(\pm 1)$ are not prescribed but $a_D = 0$; a result $\eta(\pm 1) > 0$ means a concentrated source at the ends.

3. The dipole moment $a_D \neq 0$. In this case the area of the circle at the ends of the axis represents the dipole moment to the scale of the distribution curve.

Figures D1(a)-(c) show the optimum doublet distributions for $\gamma_o = 1/2F^2 = 6$, $F = 0.289$. These distributions coincide sometimes nicely for different $2f/L$ (Fig. D1(a)), sometimes

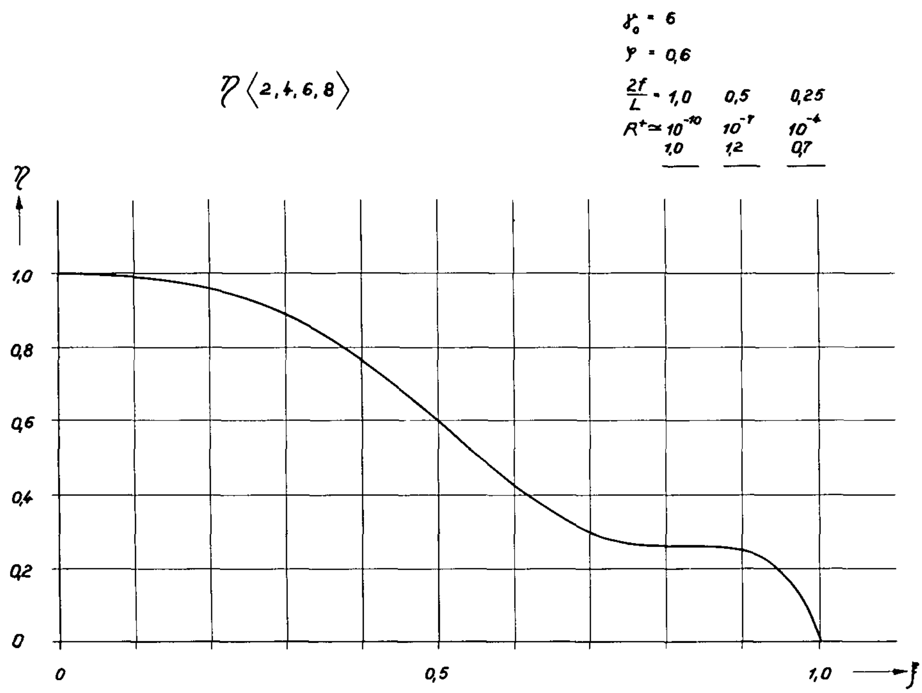

Fig. D1(a). Optimum longitudinal distribution

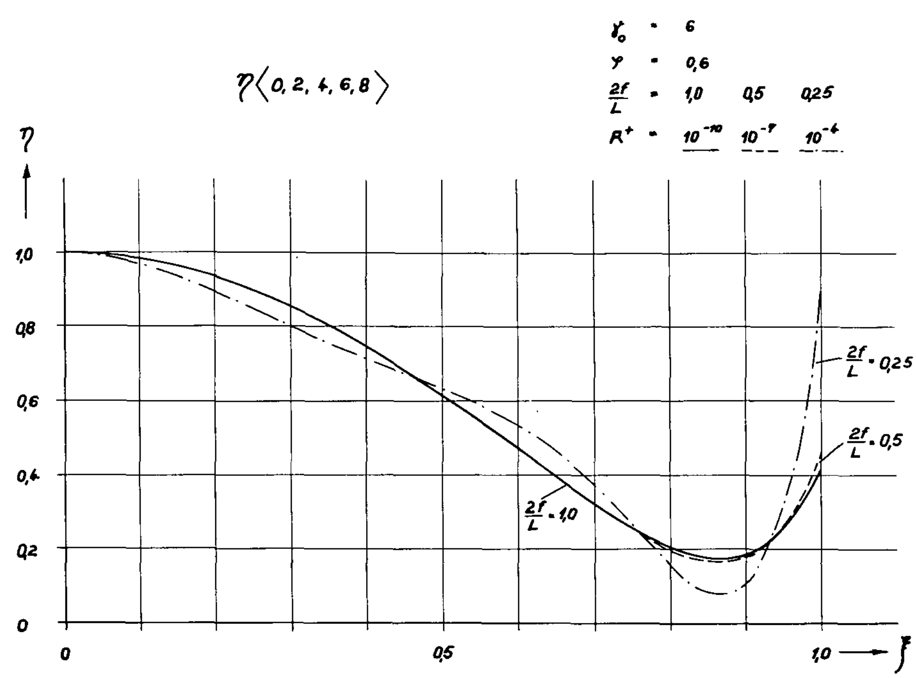

Fig. D1(b). Optimum longitudinal distribution

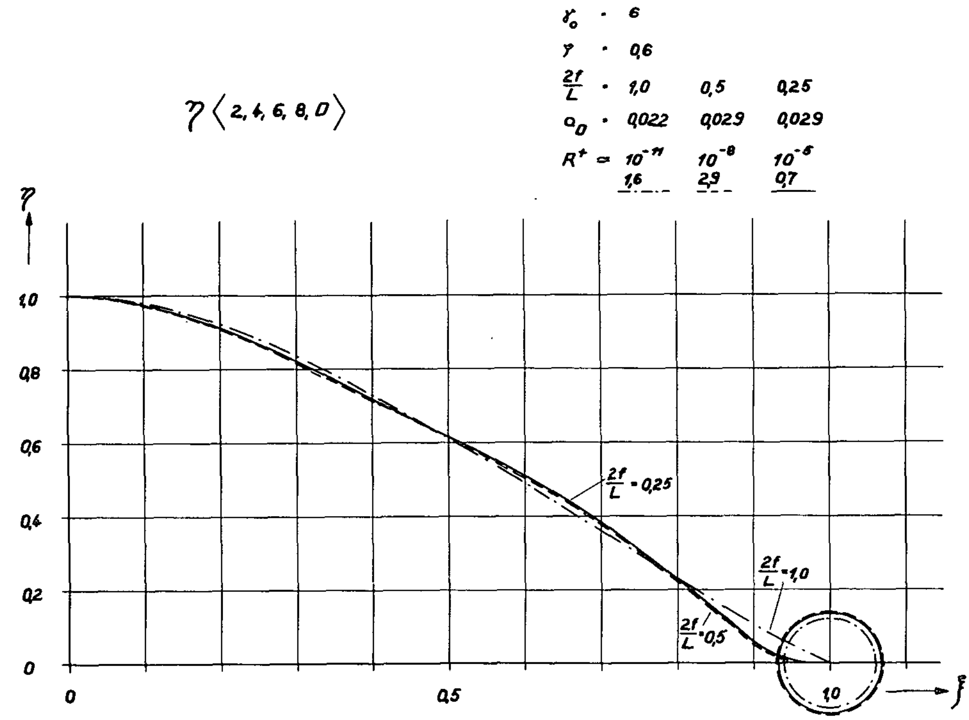

Fig. D1(c). Optimum longitudinal distribution

they differ. Occasional erratic behavior has been disregarded for present purposes. We are primarily interested in the variation of the shape with the three assumptions listed. Figure D1(a) ($\eta(\pm 1) = 0$, $a_D = 0$) shows the well-known swan-neck form; the curve for $2f/L = 0.25$ in Fig. D1(b) may be doubtful.

In the range of low Froude numbers $\gamma_o = 15$, $F = 0.183$ we find in Fig. D1(d) the orthodox hollow distribution ($\eta(\pm 1) = 0$, $a_D = 0$); in Figs. D1(e) and D1(f) we see the relatively small end ordinates and doublets respectively.

Figure D1(g) finally shows the optimum forms for a high prismatic $\varphi = 0.80$ and $\gamma_o = 1/2F^2 = 15$. In general the results do not present great surprises except for the appearance of a concentrated doublet at small F and the large reduction in resistance due to the former as compared with the assumption $a_D = 0$. An explanation can be found by comparing the generated body shapes which display marked differences. The resulting forms and data for other Froude numbers will be the subject of a more elaborate report.

The numbers R^+ stand for dimensionless wave resistance values. Obviously, they indicate that the latter is negligible for $2f/L = 1$ and in some cases for $2f/L = 0.5$ at the Froude numbers considered. Thus the investigations presented by the authors have contributed to clarify a basic problem of ship theory. For purpose of practice we shall be forced to investigate more general forms than the Michell ship. E.g., in the range of higher Froude numbers conclusive experiments have shown that by using a flat stern a definite improvement can be reached. A further step in this direction may be the investigation of the so-called Hogner interpolation formula although some essential difficulties must be overcome.

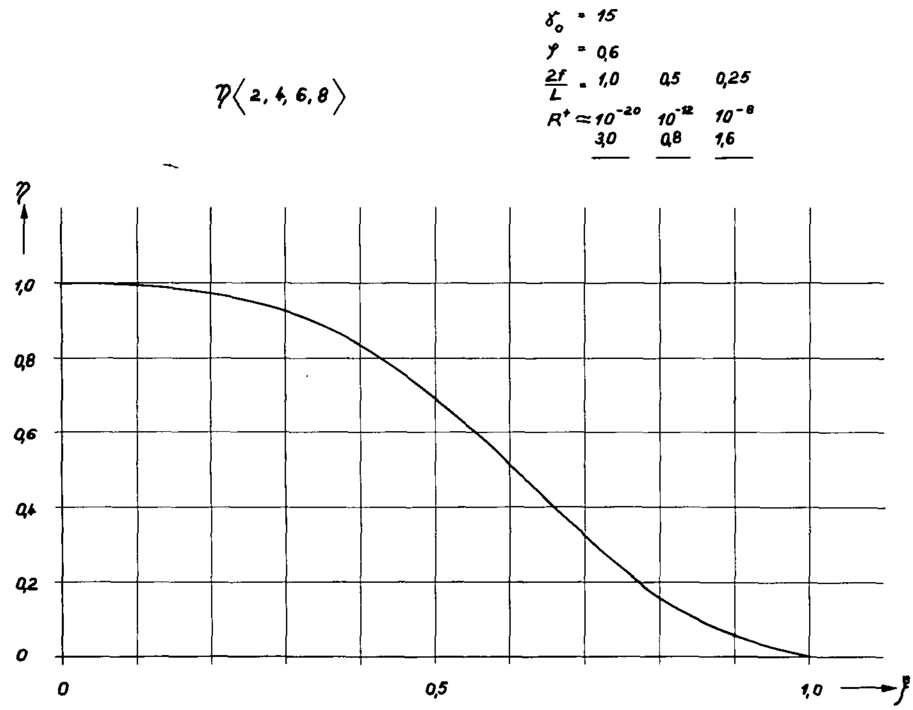

Fig. D1(d). Optimum longitudinal distribution

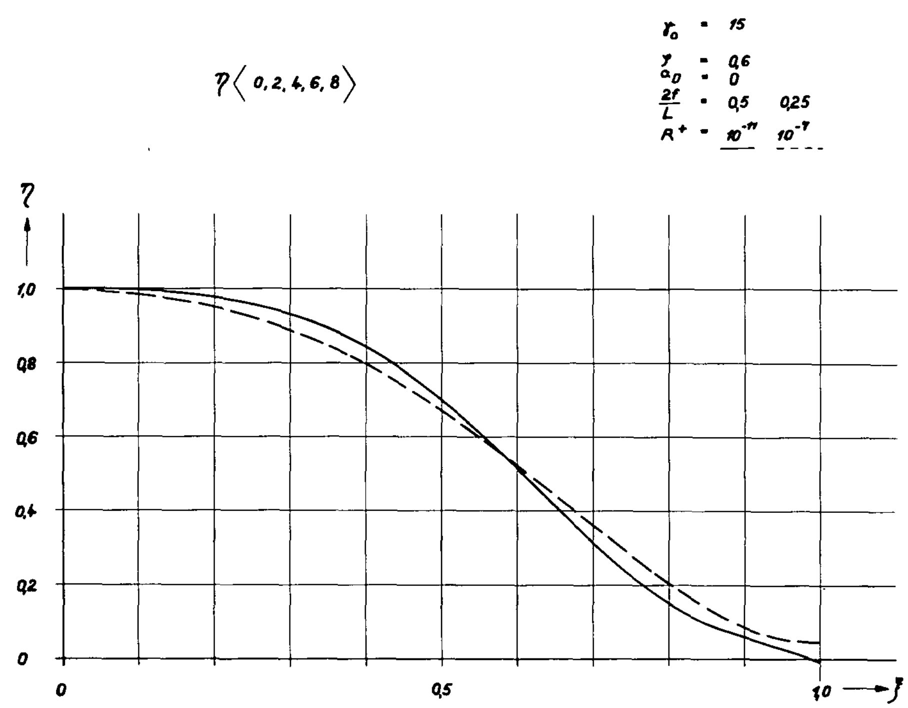

Fig. D1(e). Optimum longitudinal distribution

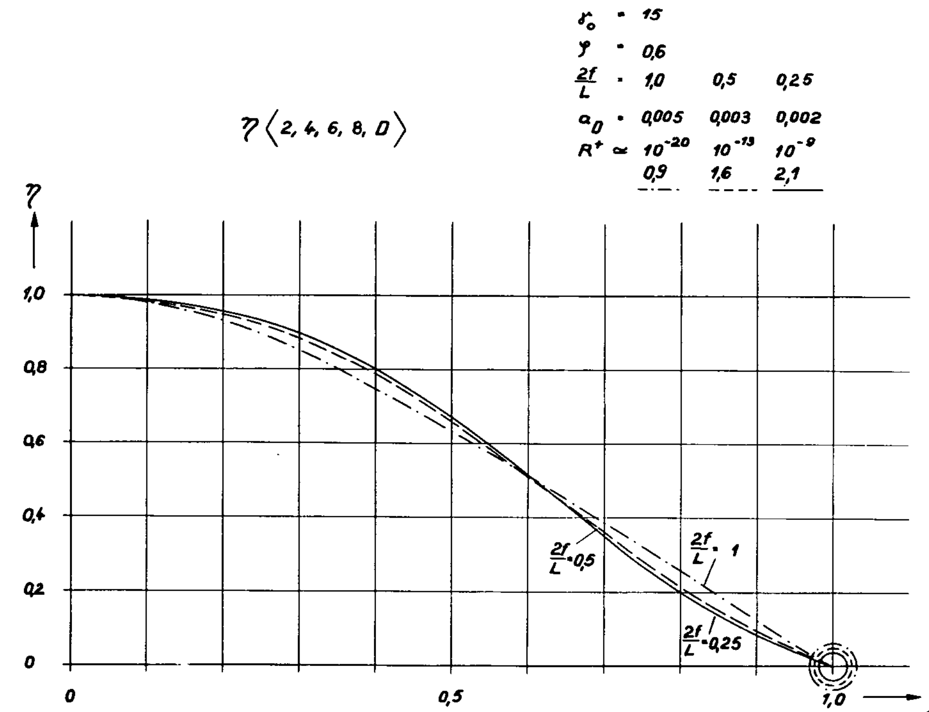

Fig. D1(f). Optimum longitudinal distribution

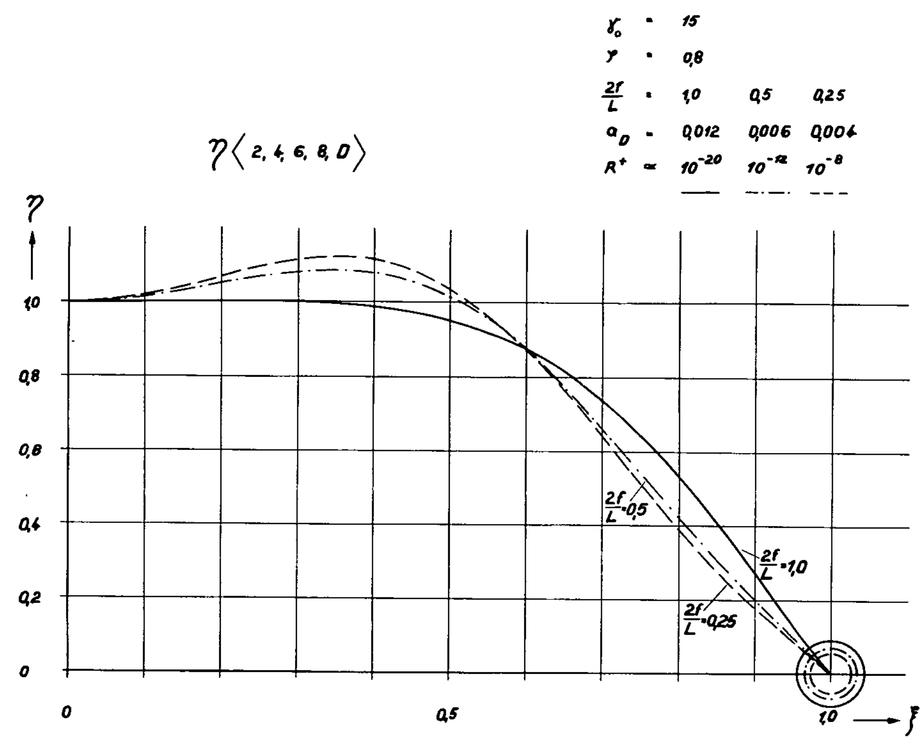

Fig. D1(g). Optimum longitudinal distribution

J. Kotik

Dr. Weinblum has asked a question as regards an arithmetical error in Fig. 10. The parameter ϵ is defined as the average half-beam divided by the length, rather than the average beam. Perhaps this factor of 2 will accommodate Prof. Weinblum's doubts. In reference to Dr. Weinblum's descriptions of his work on the wave resistance of submerged boats, or of dipoles distributed on a submerged horizontal line segment, I have the following comment. It turns out that if one fixes the length and submergence of this horizontal line segment, distributes dipoles on it, and tries to minimize the wave resistance within the family of dipole distributions having a fixed dipole moment (or approximately a fixed volume), one is led to an integral equation which has no solution. We cannot say at present why this apparently well-posed problem has no solution.

S. Karp

I would like to ask Prof. Timman whether the solution of the integral equations were Mathieu functions.

R. Timman

There is still a big gap between the mathematical approach to the problem of minimum resistance and the engineering point of view. With regard to the influence of viscosity, I am quite sure that the purely mathematical approach is not feasible.

T. Inui (University of Tokyo)

I should like to make some general remarks regarding future possible developments in our "practical" study of wavemaking resistance.

Firstly, I wish to point out the necessity of improving our measuring techniques such as so-called stereophotography for recording the actual wavemaking phenomena which we can observe in our daily model basin work. Figure D2 is one example of such stereo slides. It is more than five years since I first tried to take a picture of model waves at the Tokyo University Tank.

Secondly, I should like to point out the special importance of coordination between theory and experiment in our study of wavemaking resistance. If we had no theoretical basis, we could not go any further in our study, even if we have succeeded in taking beautiful pictures of model waves. In this connection, Sir Thomas Havelock's contribution in the field of wavemaking theory is most valuable.

Thirdly and finally, I wish to mention our desire of having an international panel to make use of high speed computers for preparing many kinds of mathematical tables which may be valuable not only to our basic studies but also to our practical design work.

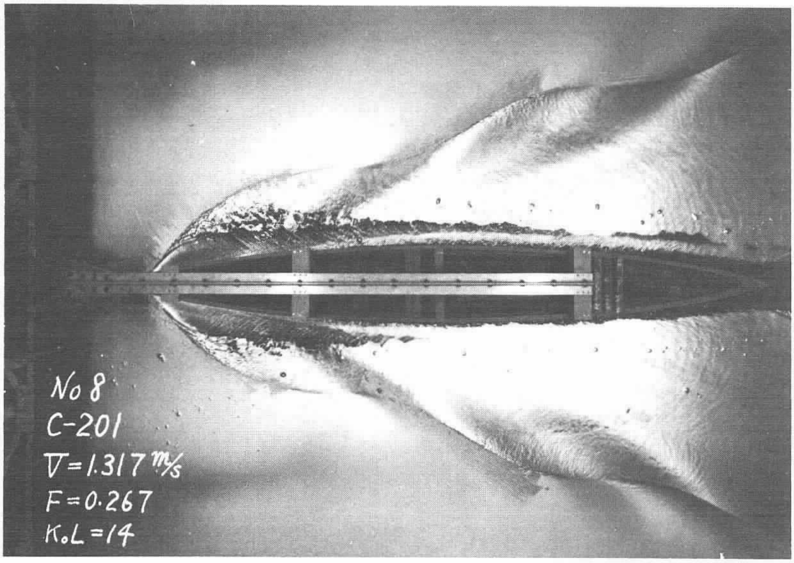

Fig. D2. A black-and-white reproduction of a colored stereo slide

John M. Ferguson (John Brown and Co., Glasgow)

The authors of these two preceding papers must be admired for the ease and sureness of handling the complicated mathematics of this problem of minimum wave resistance. Yet the feeling of admiration is tempered with a little disappointment. In what way do these papers, as they stand, assist the practical designer in his daily work of producing the best form for a given set of conditions.

A typical example is as follows: A form is designed. To this form a model is made and tested. The form may not come up to expectation. In what way can this model be altered to improve its performance? Somewhere in R. E. Froude's published work he states that the quality of performance of a model depends mainly on the shape of the sectional area curve and the load water plane. If the mathematicians could devise some method of analyzing the sectional area curve so that the analysis could indicate the features of performance and thereby suggest the manner in which the area curve could be modified for the better, then their mathematics would be really worthwhile.

The mathematician has all the time he needs. The practical designer often has to produce the answer today, if not sooner.

Some years ago, a young colleague of mine was interested in this problem. He devised a method in which a sectional area curve was analyzed by Runge's scheme into a sine and cosine series. It was found that the coefficients of the sine series could be related to certain values of the speed-length base and that the humps and hollows of the coefficients bore some close relationship to the humps and hollows of the typical resistance curve. If a hump on the resistance curve occurred at, say a trial or contract speed and there was a hump in the Runge coefficient at that point, it was suggested that a reduction of the hump in the resistance curve might result from a reduction in the value of the nearest Runge coefficient. The series would then be rearranged to give the original value of the prismatic coefficient of the sectional area curve. This rearrangement of the series then resulted in an alteration in the sectional area curve from a rebuilding of the series. In two or three cases where this was tried there was some improvement in performance, but there were some failures as well.

In one particular example where a drastic change was made in the sines with the intention of completely removing a large hump in the resistance curve, a reversal of the analysis gave a sectional area curve with a large hollow in it at or about midships. From the practical point of view this was laughable. Yet some work by Prof. Weinblum and others have produced such sectional area curves.

An intriguing thought pertaining to this hollow is provided by the fact that the sectional area curve for the model in motion, taking the wave profile into account, can show such peculiar hollows.

I throw these thoughts to the mathematicians in the hope that they may be able to adapt their powerful mathematical methods to the daily needs of the less expert practitioners. It might be better if these experts could recognize the needs of their humbler brethren – said in all sincerity. On the other hand it would be as helpful if we more practical people paid greater attention to the more general but more fundamental work of the mathematicians.

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