

II

RIVER WAVES

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1. Introduction

There are few pastimes more pleasant than water-watching, and fortunately most people live near enough to some river or stream to be able to enjoy it. Like other aesthetic pleasures, this is one that is greatly enhanced if one makes an effort to learn something of the underlying causes and effects. My aim in this lecture is to describe the results of reading and thinking undertaken in this spirit, and combined

with happy observation of many British rivers as well as respectful study through the printed page of some great American ones.

2. Frequency dispersion and amplitude dispersion

I shall have a lot to say about the effect of friction on river waves, but it is best to begin, I think, with a survey of water-wave motions in the absence of friction, as these are already fairly complicated [19, 15]. Waveforms on water tend to change shape as a result of two kinds of dispersion, which I propose to call frequency dispersion and amplitude dispersion. The word dispersion is normally used to mean frequency dispersion, according to which components of the wave of different frequencies are propagated with different velocities. Long waves, however, suffer also amplitude dispersion, according to which high values of the surface elevation are propagated with greater velocity than lower values. This amplitude dispersion, producing steepening of the front of the wave, is familiar also in sound waves. We shall see that there are important kinds of water waves for which the deforming effects of frequency dispersion and amplitude dispersion are in competition.

3. Gravity waves of small amplitude

To illustrate frequency dispersion one may draw a graph of wave velocity c against wavelength λ , beginning with Cauchy's relation*

$$c = \sqrt{\frac{g\lambda}{2\pi}} \quad (1)$$

which is valid for gravity waves of small amplitude if the depth exceeds half a wavelength. (In fact, in the Cauchy solution, the disturbance half a wavelength below the surface is reduced to only 4% of its surface amplitude.) This wave velocity c is the velocity of individual crests, but the energy in the wave is propagated at the group velocity**

$$U = c - \lambda \frac{dc}{d\lambda}, \quad (2)$$

which is $\frac{1}{2}c$ for these waves. This is of course why the group of waves generated say by throwing a stone into a pond has its energy travelling slower than the individual crests, which therefore have to disappear on reaching the front of the group and passing out of the energy-containing region. Similarly, new crests have to appear at the back of the group.

When the wavelength exceeds $2h$, where h is the depth, the bottom begins to influence the waves, but we still have a simple solution if the depth is uniform, when the wave velocity is

$$c = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}} \quad (3)$$

tending to the constant value \sqrt{gh} as λ/h becomes large (see the righthand part of the lower full-line curve in Fig. 1). The group velocity

$$U = c \left(\frac{1}{2} + \frac{2\pi h/\lambda}{\sinh 4\pi h/\lambda} \right) \quad (4)$$

* Plotted as the broken line in Fig. 1 below.

** The most general proof of this statement is that given in the Appendix to Rayleigh's Theory of Sound, which applies to an arbitrary conservative system capable of transmitting sinusoidal waves of all frequencies.

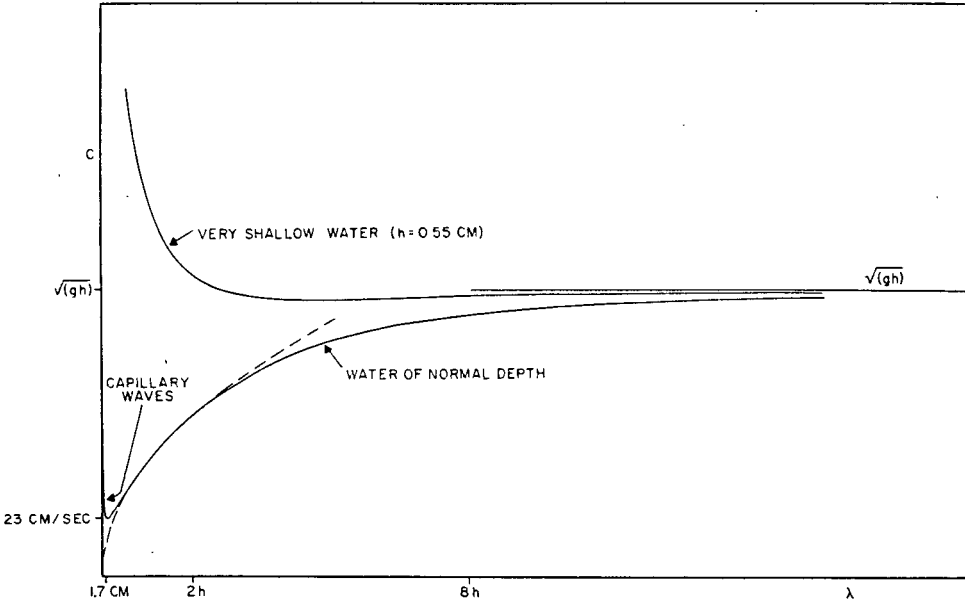


Figure 1. Wave velocity c versus wavelength λ for waves of small amplitude on water. (Note that the relative position of the points 1.7 cm. and $2h$ on the wavelength scale is variable; so is that of the points 23 cm./sec. and \sqrt{gh} on the wave-velocity scale.)

is then closer to c , and indeed one observes that the fading-out of wavecrests at the front of a group of waves on a pond becomes less pronounced when the group moves into the shallower water at the edge.

4. Capillary waves

For very small wavelengths surface tension becomes comparable with gravity as a force tending to restore the horizontal condition of the surface, and then g must be replaced (if $T =$ surface tension, $\rho =$ density) by

$$g + \frac{T}{\rho} \frac{4\pi^2}{\lambda^2} \quad (5)$$

throughout. Then (Fig. 1) c has a minimum of 23 cm/sec. at $\lambda = 1.7$ cm. and tends to infinity as $\lambda \rightarrow 0$. The group velocity U exceeds c for $\lambda < 1.7$ cm. I should remark, however, that for water of very small depth h (about half a centimetre) the rise in c due to surface tension and the fall due to finite λ/h occur for the same value of λ and largely cancel out, and there is then very little frequency dispersion.* This

* Expanding $c = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{T}{\rho} \frac{2\pi}{\lambda} \right) \tanh \frac{2\pi h}{\lambda}}$ in powers of h/λ gives

$c = \sqrt{gh} + O(h^2/\lambda^2)$ provided $h = \sqrt{3T/\rho g} = 0.5$ cm. for water. To illustrate the conflicting tendencies better, the curve is shown for a slightly larger value of the depth (0.55 cm.) in Fig. 1.

makes water of the particular depth 0.5 cm. specially suitable for producing flows closely similar to gas flows.

5. Viscous attenuation

Viscous effects modify the flow near the free surface and near the bottom in boundary layers of thickness about

$$\sqrt{\frac{\nu\lambda}{c}} \quad (6)$$

(where ν = kinematic viscosity) and cause damping by a factor of e^{-1} in amplitude in a time

$$\frac{\lambda^2}{8\pi^2\nu} \approx (\text{wavelength in cm.})^2 \quad (7)$$

Hence the capillary waves with $\lambda < 1.7$ cm. travel only a short distance before being damped out by viscosity. The longer waves are not seriously affected, but of course in a river *turbulent* friction is a much more important source of damping for these, as we will be discussing later.

6. Wave-making resistance to two-dimensional obstacles

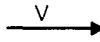
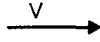
The most striking evidence of frequency dispersion in river waves is to be found by observing the wave pattern due to obstacles in the stream or steps in the bed. We may consider first two-dimensional obstacles or steps (see Fig. 2). In steady flow these can produce only waves whose wave-velocity c equals the velocity V of the stream, so that their crests remain stationary. The energy in the waves is propagated upstream at the group velocity U and swept downstream with the stream velocity V , which equals c . Thus the energy input due to the resistance of the obstacle travels upstream for capillary waves with $U > c$, which therefore are found in front. The energy travels downstream, however, for the gravity waves with $U < c$, which therefore are found behind, and for larger obstacles the bulk of the wave-making resistance of the obstacle leaks back through these at the speed $c - U$. On the other hand, for supercritical streams, with $V > \sqrt{gh}$, no *gravity* wave with $c = V$ exists, and so the wave-making resistance of a two-dimensional obstacle is lowered. Similarly, the drag of a barge on a narrow canal is reduced when its speed rises above \sqrt{gh} , and Scott Russell [28] describes how this was first discovered due to a horse taking fright, and galloping off pulling its barge behind it, when it was observed "to the proprietor's astonishment, that the foaming stern surge which used to devastate the banks had ceased, and the vessel was carried on through water comparatively smooth, with a resistance very greatly diminished." The proprietor went on to achieve "a large increase in revenue" as a result of this scientific observation.

7. Stationary waves on non-uniform streams

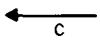
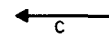
I might remark that Mr. J. C. Burns and I investigated [4] what relation corresponding to $c = V$ a stationary wave must satisfy in a stream whose velocity varies with depth according to a one-seventh power distribution typical of turbulent flow over a smooth bottom. For small λ/h the waves are on the surface only, so that for them to be stationary we must have $c = \text{surface velocity} = 1.143 V_m$ ($V_m = \text{mean velocity}$). As λ/h increases, c drops (Fig. 2) till it equals V_m for $\lambda/h = 10$ and is $0.966V_m$ when $\lambda/h \rightarrow \infty$; this lower value is because the oscillatory motion for very long waves tends to be slightly stronger in the low-speed region near the bottom.

VELOCITY

OF STREAM

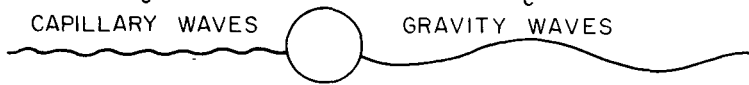


OF CRESTS RELATIVE
TO MEDIUM

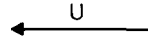


CAPILLARY WAVES

GRAVITY WAVES



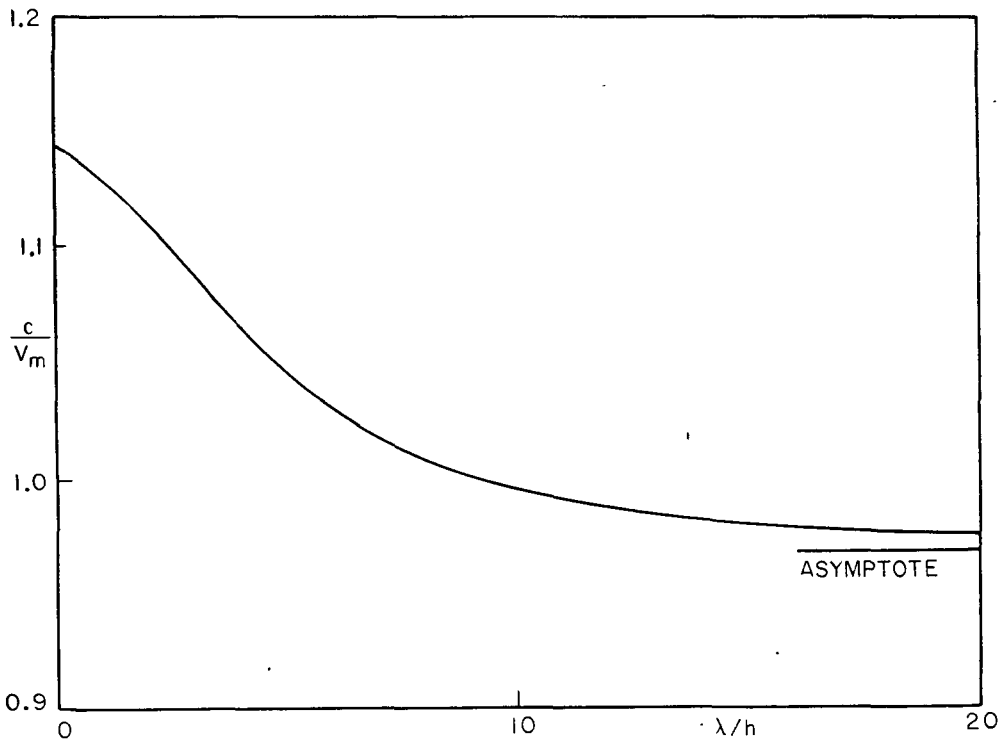
OF WAVE ENERGY
RELATIVE TO MEDIUM



OF WAVE ENERGY
RELATIVE TO OBSTACLE



(a) Pattern of waves generated.



(b) Graph which plots against λ/h , for stationary waves of length λ on a stream of depth h , with one-seventh power distribution of velocity and mean speed V_m , the ratio to V_m of the wave velocity c which waves of that length would have on still water.

Figure 2. Two-dimensional wave making.

8. Three-dimensional wave pattern due to an obstacle

I will now discuss briefly the three-dimensional pattern of stationary waves due to an obstacle at a point in a running stream (Fig. 3). Oblique waves are then present, and for them to be stationary those of velocity c must set themselves at an angle

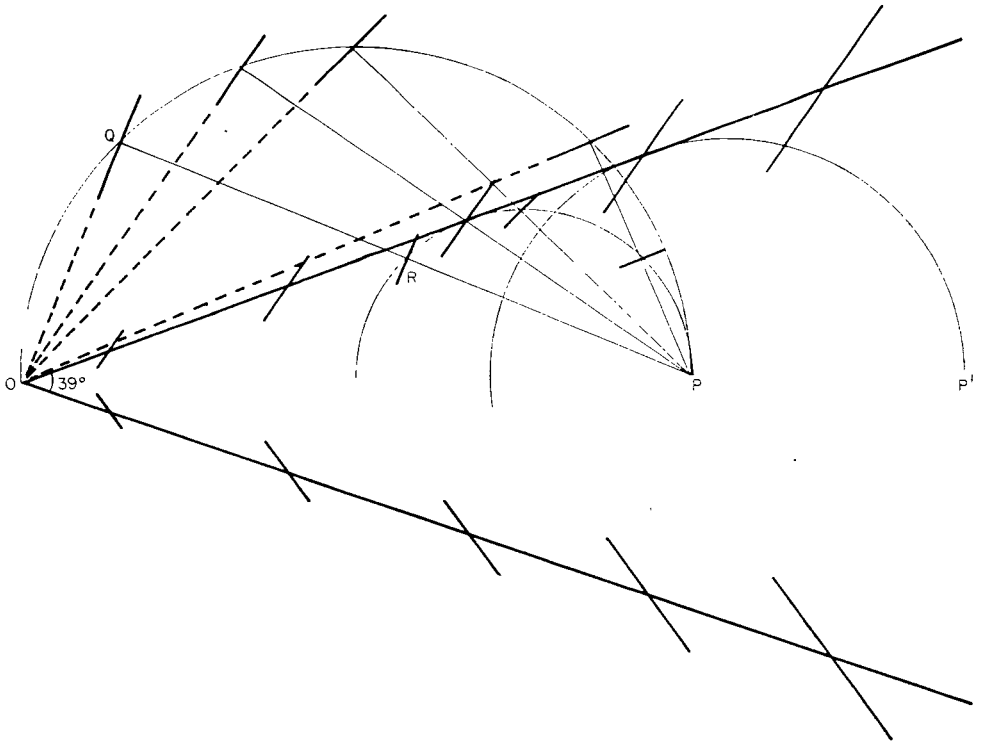


Figure 3. Three-dimensional gravity-wave making due to an obstacle O . Waves generated t seconds ago are shown. $OP = Vt$, $PQ = ct$, since $\angle QOP = \sin^{-1}(c/V)$, $\angle OQP$ must be a right angle; so locus of Q is a semicircle. Locus of R (mid-point of PQ) is therefore also a semicircle. Similarly waves generated t' seconds ago lie on semicircle through P' , and the different semicircles envelop a wedge of total angle 39° , within which all waves are found. Waves on boundary of wedge are set as shown.

$\sin^{-1}(c/V)$ to the stream. At a time t after they have been emitted they might be expected to be at a distance ct from the centre of the disturbance, which however has drifted a distance Vt downstream. In this case their crests would lie along a line through the disturbance, like a "Mach line" in supersonic aerodynamics. All the waves emitted at the same instant would be on their Mach lines, and the whole lot would lie on the semicircle through the obstacle O and the point P to which the water passing the obstacle at the instant of emission has drifted. However, this neglects the fact that each wave group proceeds at the group velocity, and has travelled a distance Ut , not ct , since it was emitted. As a consequence the waves have not got as far as the semicircle. For stream speeds which are well sub-critical all the waves produced are essentially "deep-water" waves with $U = \frac{1}{2}c$, and then the waves lie on an inner semicircle of half the diameter. Every wave, produced at any time, lies on one such semicircle, and all the semicircles lie in a wedge of total angle $2 \sin^{-1}(\frac{1}{3}) = 39^\circ$; a characteristic pattern is formed by the specially strong waves at the boundary of the wedge, which are set at an angle of 55° to the stream. However, as the stream speed V approaches \sqrt{gh} this semi-circle gets elongated at the front, and the wedge expands. At supercritical speeds the wedge contracts again because only waves with $c < \sqrt{gh}$ are present, and it is then a genuine Mach wedge of semi-angle $\sin^{-1}(\sqrt{gh}/V)$.

I have no time to mention the details of the charming pattern of capillary waves which forms ahead of the obstacle, which is especially prominent if the obstacle is small, like Rayleigh's famous "fish-line" [19], but this can be understood by directly similar arguments.

9. Waves of greatest height

I pass now from the frequency spectrum to what I may call the "amplitude spectrum" of water waves. This is best displayed on a diagram (Fig. 4) in which the abscissa is λ/h and the ordinate a/λ , where a is the vertical height between crest and trough. The wave velocity c for small a/λ is exhibited under the diagram as a reminder.

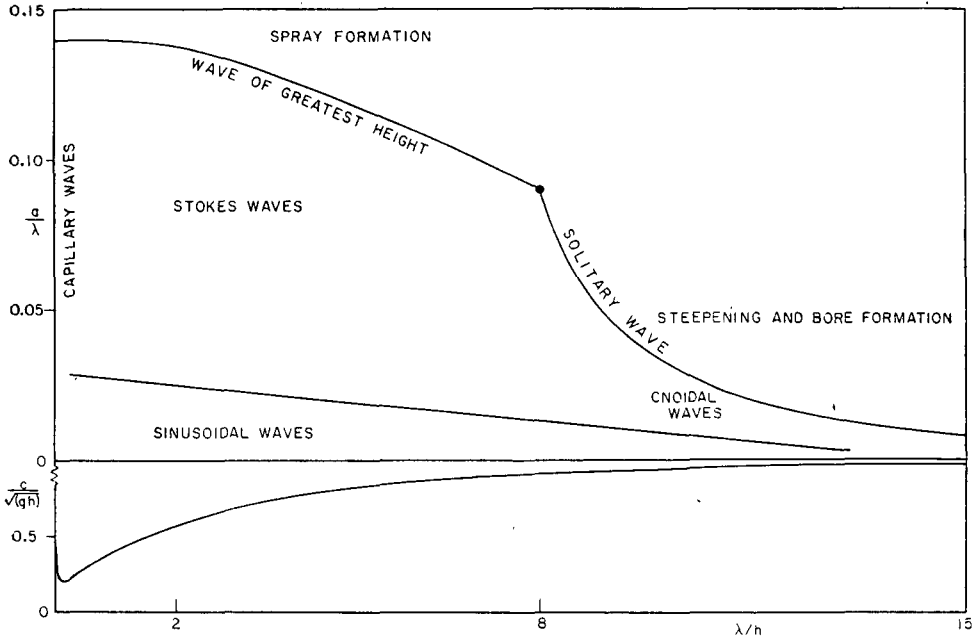


Figure 4. Amplitude-wavelength diagram for water waves. Under it is shown the wave-velocity versus wavelength diagram for waves of small amplitude (Fig. 1) as a reminder.

Now, there are two kinds of *breaking* which can limit the possible existence of periodic waves (Fig. 5). At the deep-water end of the scale there is the formation of the "wave of greatest height" which when it occurs as a stationary wave on a stream brings the water locally to rest at its wedge-shaped crest.* Further increase in the energy of these waves goes into spray formation. Stokes [30] predicted this wave and also observed it, writing: "In watching many years ago a grand surf which came rolling in on a sandy beach near the Giant's Causeway, without any storm at the place itself, I recollect being struck with the blunt wedge-like form of the waves where they first lost their flowing outline, and began to show a little broken water at the very summit. It is only I imagine on an oceanic coast, and even there on somewhat rare occasions, that the form of waves of this kind, of nearly the maximum height, can be studied to

* Values of a/λ for waves of greatest height in Fig. 4 were obtained as follows: for deep-water waves from Michell [25], for the solitary wave from McCowan [24] and for intermediate cases from Kishi [16].

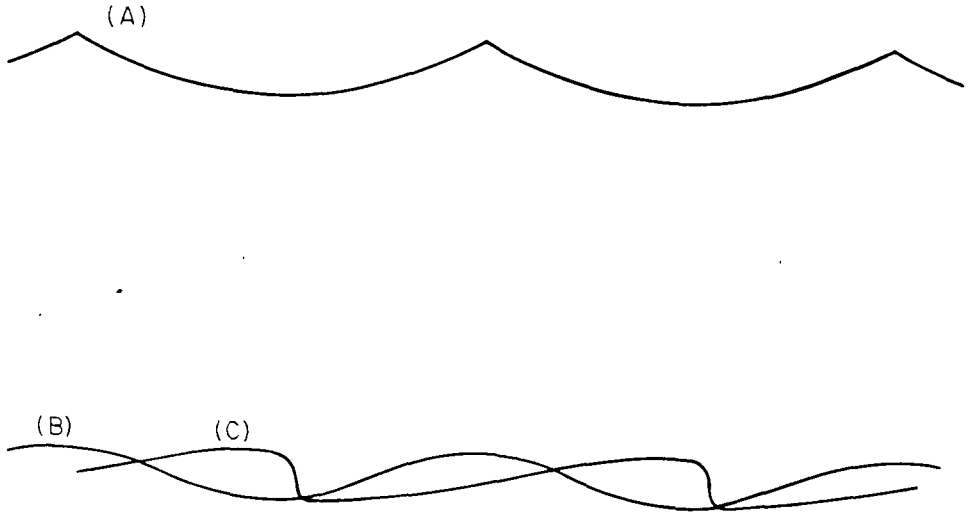


Figure 5. Illustrating the two kinds of breaking of water waves. Curve (a): wave of greatest height in deep water; further increase of wave energy produces spray at the crests. Curve (b): amplitude dispersion converts this sinusoidal wave on shallow water in a very short distance of travel into the breaking form (c) and then into a succession of bores.

full advantage. The observer must be stationed nearly in a line with the ridges of the waves where they begin to break." Stokes calculated also the form of waves of intermediate height, which are usually called Stokes waves (see Fig. 4).

10. Amplitude dispersion

By contrast, at the long-wave end of the scale, say for $\lambda > 8h$, the existence of periodic waves is limited by amplitude dispersion. By this I mean the tendency for each value η of surface elevation to be propagated with the speed $\sqrt{g\eta}$ appropriate to the local depth rather than with the speed \sqrt{gh} corresponding to the mean depth. It is well known how this* leads to steepening of the front of the wave until it may become a "hydraulic jump" or "bore."

11. Cnoidal waves and the solitary wave

However, if the amplitude is not too great, this amplitude dispersion may be exactly balanced by frequency dispersion in such a way that periodic waves become possible. These are the "cnoidal" waves of Korteweg and de Vries [17], so called because their waveform is the square of the Jacobian elliptic function known as "cn." Their existence depends on the fact that the higher harmonics in the wave travel slightly slower than the fundamental, which for particular waveforms can just balance the steepening just mentioned. Changes due to this frequency dispersion are proportional to h^2/λ^2 , but changes due to amplitude dispersion are proportional to a/h ; as a result, any given cnoidal shape is possible for a fixed ratio $(a/h) : (h^2/\lambda^2)$. The limiting case here is the solitary wave with $a\lambda^2/h^3 \approx 25$. For amplitudes above that of the solitary wave we must have steepening and bore formation. For amplitudes just below it the cnoidal wave train is beginning to degenerate into a sequence of isolated solitary waves.

* Together with the fact that the particle velocity is forwards where η is greatest and backwards where η is least.

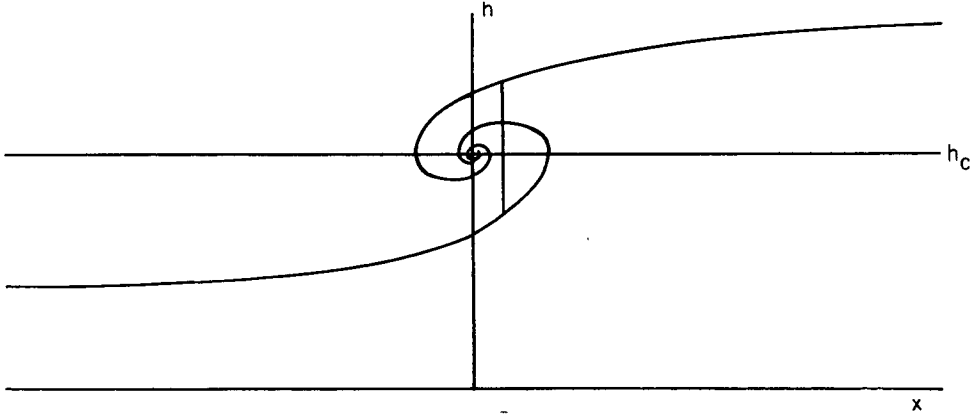


Figure 6. Integral curves of equation (9), showing spiral singularity and transition between curves by means of a hydraulic jump.

The two curves (Fig. 4) for the different kinds of breaking meet at a point representing the solitary wave of greatest height computed by McCowan [24]. The linearised theory leading to sinusoidal waveforms is valid only if the amplitude is small compared with that giving breaking. Conversely, it is only if the amplitude is large compared with that of the solitary wave of given λ/h that the so-called shallow-water theory which neglects frequency dispersion and predicts steepening and the formation of a hydraulic jump or bore is valid.

12. Occurrence of hydraulic jumps in steady flow

We must now discuss these bores in more detail. They occur also, of course, as stationary jumps in steady streams wherever they decelerate from supercritical to subcritical velocity. Then the equations of steady flow

$$vh = Q, \quad v \frac{dv}{dx} = g \left(\alpha - \frac{dh}{dx} \right) - \frac{f\rho v^2}{h}, \quad (8)$$

where Q is the volume flow per unit breadth, α is the slope of the bottom and f a coefficient of friction, can be written

$$\frac{dh}{dx} = \frac{\alpha h^3 - fh_c^3}{h^3 - h_c^3} \quad (9)$$

where h_c is the critical depth $\sqrt[3]{Q^2/g}$ for which $v = \sqrt{gh}$, and the flow is subcritical when $h > h_c$ (which for constant h means $\alpha < f$, a gradual slope) and supercritical when $h < h_c$ (which for constant h means $\alpha > f$, a steeper slope giving the so-called "torrent flow"). At a point where α decreases through the critical value f , the integral curves of (9) have a spiral singularity (Fig. 6) and no continuous solution exists.* The curves specified by the upstream and downstream conditions (the latter being the so-called backwater curve determined by the water-level in the down-

* On the other hand, at a point where α increases through the critical value f , the integral curves have a saddle-point singularity, and a solution representing continuous acceleration through the critical speed exists.

stream reservoir [27]) must be joined by a hydraulic jump at which h changes discontinuously but the momentum flow

$$S = v^2h + \frac{1}{2}gh^2 = \frac{Q^2}{h} + \frac{1}{2}gh^2 \quad (10)$$

is continuous. Such a jump is possible because S has a minimum value for $h = h_c$.

13. Mechanics of energy loss in bores

Thus, classical bore theory (due to Rayleigh [26]) assumes conservation of volume flow Q across the bore, and conservation of momentum flow S (more precisely, the decrease in momentum flow v^2h is balanced by the difference in horizontal pressure force $\frac{1}{2}gh^2$ upstream and downstream). Energy, however, is lost at the bore; the total head $\frac{1}{2}v^2 + gh$ decreases by

$$\frac{1}{4}g \frac{(h_2 - h_1)^3}{h_1 h_2}, \quad (11)$$

and Rayleigh suggested this was due to friction.

Now it is certainly true that vigorous turbulent dissipation and churning-up of the flow is the main mechanism of energy loss in strong bores, but weaker bores have a different structure. They carry a train of waves behind them, whose position is stationary relative to the bore, and Favre [10] found that these waves exhibit no breaking, and that the flow appears perfectly smooth (Fig. 7) for depth ratios $h_2/h_1 < 1.28$.

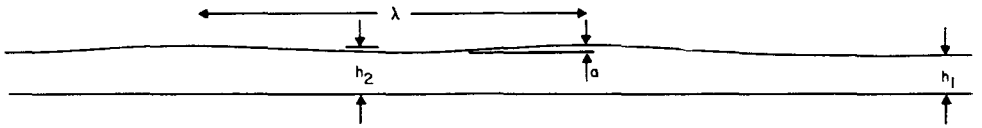


Figure 7. Weak bore with train of waves behind.

As an example, the tidal bore formed on the British river Severn often carries such a train of waves along behind it. It is clear that the energy dissipation then takes place principally by radiation through this stationary wave train at a speed equal to the difference between the wave velocity and the group velocity—exactly as in the waves generated by a two-dimensional obstacle in a sub-critical stream. A calculation of amplitude by Lemoine [20], assuming sinusoidal waves, was based on the amount of energy lost per unit time (which is ρQ times (11) per unit span), and gave

$$\frac{a}{h_2} = 1.15 \left(\frac{h_2 - h_1}{h_1} \right), \quad \frac{\lambda}{h_2} = 3.0 \left(\frac{h_2 - h_1}{h_1} \right)^{-1/2} \quad (12)$$

The reason why fairly long waves are produced is that their velocity must equal the flow speed behind the bore, which is only just subcritical. As a result, a substantial amplitude is needed to carry away a fairly small amount of energy, simply because the difference between group velocity and wave velocity is small for such long waves. In support of this theory of Lemoine there is the observation that the waves form one by one behind the bore after it is first created—as the energy transmitted by the group of waves travels backwards.

14. Cnoidal wave trains behind bores

However, Mr. Brooke Benjamin and I thought it desirable to reexamine this theory on a more accurate basis [2], partly because* the ratio $a\lambda^2/h_2^3$ is too great for sinusoidal-wave theory to be applicable (it is about 10) and so the long waves behind the bore should really be treated as cnoidal waves. (Another reason was that although the 1.15 in (12) is close to the mean of the observations of many workers, it has been common to find on occasion values either much larger or much smaller.) To understand the subject better we re-worked the theory of cnoidal waves as follows. The momentum flow per unit span (divided by the density) is in general

$$S = \int_0^\eta \left(\frac{p}{\rho} + u^2 \right) dy \quad (13)$$

and for irrotational flow this can be calculated, for given volume flow Q per unit span and total head $R = \frac{1}{2}(u^2 + v^2) + gy$, as

$$S = R\eta - \frac{1}{2}g\eta^2 + \frac{1}{2}\frac{Q^2}{\eta} \left(1 - \frac{1}{3}\eta'^2 + O\left(\frac{h^4}{\lambda^4}\right) \right) \quad (14)$$

Thus, taking in frequency dispersion to the first approximation only, we have a differential equation

$$\frac{1}{3}Q^2 \left(\frac{d\eta}{dx} \right)^2 + g\eta^3 - 2R\eta^2 + 2S\eta - Q^2 = 0 \quad (15)$$

as the equation for all stationary long waves.

The character of the solutions is easily seen by thinking of x as the time, and the equation as that of a particle of mass $\frac{2}{3}Q^2$, velocity $d\eta/dx$, potential energy equal to the cubic, and zero total energy. The particle would oscillate if the cubic has two zeros between which it is negative as in Fig. 8, curve B; this behavior oscillatory in x

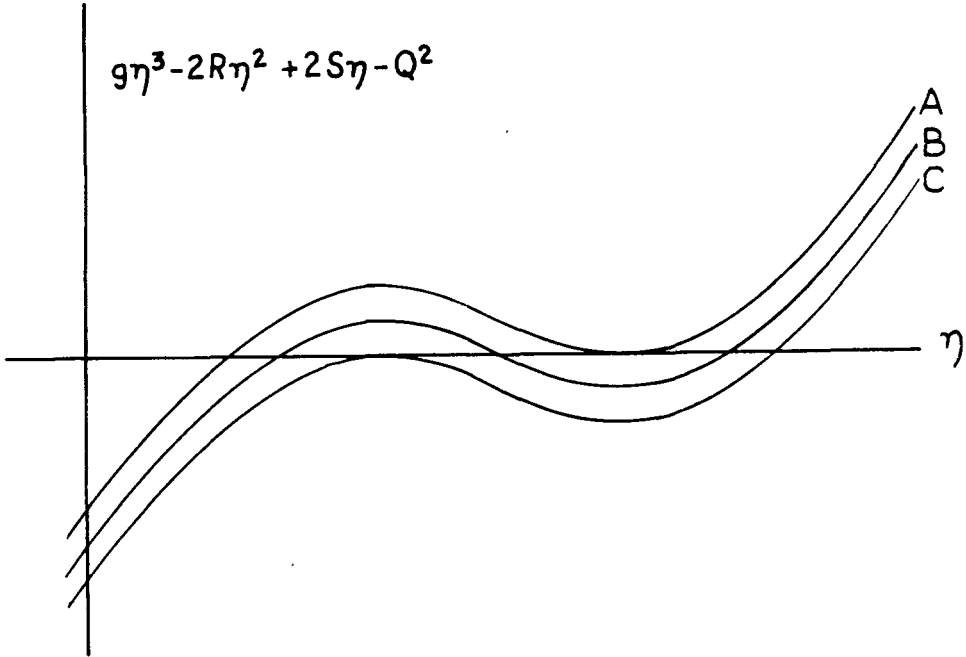


Figure 8. Different possible positions of the graph of the cubic occurring in equation (15).

* Note that our a is twice the a of reference 2.

corresponds of course to waves (cnoidal waves, in fact). In the extreme case of curve *A* the amplitude becomes zero and we have a steady subcritical stream. In the other extreme case of curve *C* we have two solutions, either a steady supercritical stream or an oscillation taking an infinite time—this corresponds to the solitary wave. For positions of the curve beyond *A* or *C* there is no solution; at least, the depth would become zero in a finite distance.

Now in the bore problem we start upstream with a uniform supercritical flow, so we see that if *R* and *S* remain unchanged the only possible downstream flow is the solitary wave. To this extent Lemoine is proved wrong. However, you observe that it is necessary to reduce the total head *R* by viscous dissipation by only the slightest amount to raise the curve to one of type *B*. A reduction by the full value given by Rayleigh would of course raise it to one of type *A*, representing a uniform subcritical flow; no greater loss of head is possible without simultaneous loss of momentum. But if energy intermediate between zero and the classical Rayleigh value is dissipated, then the curve becomes one of type *B* and cnoidal waves will be present behind the bore. We find that these will have the amplitude calculated by Lemoine if 20% of the classical energy dissipation is accomplished by friction, but that amplitudes from zero to 1.3 times Lemoine's value are possible for other values of this percentage. These considerations give a reason for the wide scatter of the experimental results.

15. Energy-momentum diagram for steady flows

Next, following up these general ideas, Brooke Benjamin and I and Mr. S. C. De [2, 6] constructed a diagram (Fig. 9) giving the values of *R* and *S* for different kinds of steady flow with given volume flow *Q*. The abscissa and ordinate on this diagram are R/R_c and S/S_c , where R_c and S_c are the total head and momentum flow for a critical stream of volume flow *Q*; actually, both *R* and *S* have minimum values for such a uniform stream with $v = \sqrt{gh}$. On this diagram the cusped curve represents uniform streams with the Froude numbers v/\sqrt{gh} as marked, subcritical on the left, supercritical on the right. The broken-line barrier represents "waves of greatest height." Waves are possible in the thin region between this and the other two barriers. Weak, sinusoidal waves occur only near the left-hand barrier, cnoidal waves occur in the region below *Z*, Stokes waves in the region above *Z*. The solitary wave lies on the right hand branch of the cusped curve below *Z*, since this can appear out of a uniform supercritical stream without any loss of momentum or energy. The point *Z* represents the solitary wave of greatest height. For every point in the diagram the amplitude and wavelength have been computed (those for cnoidal waves by Benjamin and myself and those for Stokes waves by De). Hence the diagram can be used to find the characteristics of the waves generated by adding or subtracting any given amounts of energy or momentum from a given uniform flow.

In the problem of the weak bore, for example, one starts from a point below *Z* in Fig. 9, representing uniform supercritical flow, and moves to the left in proportion to the energy removed. If some momentum is lost also by friction at the bottom one moves also slightly down. Again, in the problem of wave resistance to a two-dimensional obstacle, one can use the diagram to obtain a theory for large-amplitude waves by starting at a point on the left-hand curve representing uniform subcritical flow and moving down (if the obstacle is streamlined so that only momentum is lost), or down and to the left (if dissipation occurs so that energy also is lost). In any problem, if one lands beyond the broken-line barrier the waves have exceeded their greatest possible height and must lose energy by breaking, which carries the point away to the left. Strong bores represent a transition from the isolated solutions above *Z*, representing uniform supercritical streams, to the range of physically possible flows to the left of the

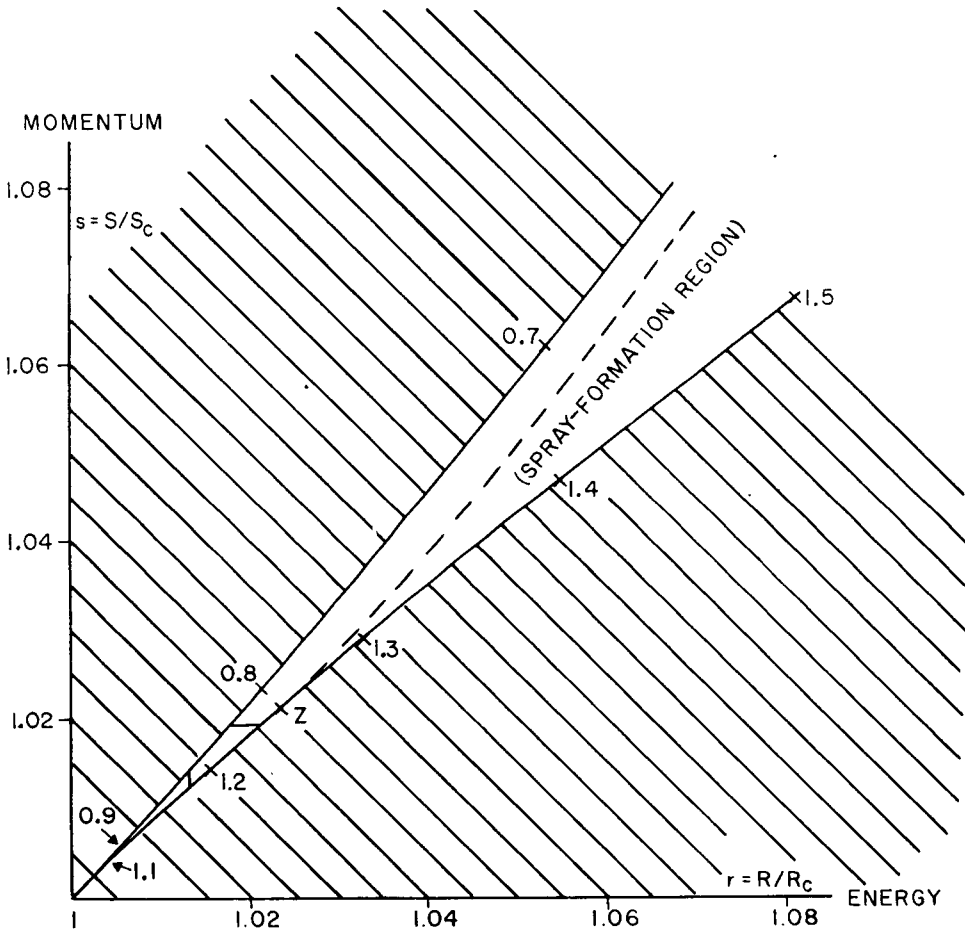


Figure 9. Energy-momentum diagram for steady flows, including stationary-wave motions.

broken-line barrier. In these, obviously, much energy must be lost in spray and turbulence. Brooke Benjamin has recently shown that many transitions to wavy flow regimes resulting from changes in depth or breadth of the stream can also be understood by use of this diagram.

16. Attenuation of long waves by turbulent friction.

I must pass now to flows which extend over much greater lengths of river than those we have considered so far—over lengths so great, in fact, that wave attenuation by turbulent friction becomes important. If one begins by writing down the equation for long waves of small amplitude on a stream of subcritical velocity, V , resisted by a frictional force $f \cdot \rho V^2$ per unit area, one finds that for either direction of propagation they are damped approximately like

$$e^{-fVt/h} \tag{16}$$

so that for $f = 0.005$, say, their amplitude is reduced by e^{-1} in the time taken for the stream to travel 200 times its depth. This result appears for upstream propagation in Abbott's treatment of the formation of tidal bores [1] and for downstream propagation

in Whitham's treatment of flood waves [22]. I shall describe the further ramifications of this result separately in the two cases.

17. Conditions for the formation of tidal bores

One of the problems of river waves which most need an improved mathematical treatment, with the object of throwing light on what are the essential physical processes involved, is that of the propagation of the influence of tidal motions in an estuary into the upstream part of the river. Despite all the beautiful experimentation which has been done with tidal models, the physical mechanisms governing what part of the tidal energy passes upstream are still not fully clear. An interesting beginning, however, has been made by Mr. M. R. Abbott [1], and I should like to describe his work.

He uses a simplified method due to Dr. G. B. Whitham [32] for calculating wave propagation with amplitude dispersion. In this method, one first calculates the flow using the linearised equations for small disturbances. One then allows for amplitude dispersion by taking the paths of individual waves as calculated, and modifying each path by giving it at every point a wave velocity appropriate to the local depth and flow speed in the presence of the wave.

In this method, therefore, one must begin by considering the variation of wave amplitude on linearised theory. This is a combination of attenuation due to friction (equation (16)), and amplification due to reductions in the breadth and depth of the river as the wave passes upstream. If only the attenuation were present the amplitude would decay too rapidly for steepening to the extent of bore formation to occur. Therefore the reductions in breadth and depth are essential. Their effect can be calculated only very roughly. If no energy is reflected back down the river as the channel becomes narrower and shallower we have Green's law [12, 19]

$$\frac{a_2}{a_1} = \frac{b_1^{1/2} h_1^{1/4}}{b_2^{1/2} h_2^{1/4}}. \quad (17)$$

This can give very substantial amplification. At the other extreme, if the narrowing occurs suddenly, so that the maximum energy is reflected, then we have the much smaller amplitude increase

$$\frac{a_2}{a_1} = \frac{2}{1 + \frac{b_2 \sqrt{h_2}}{b_1 \sqrt{h_1}}} < 2. \quad (18)$$

Intermediate values occur in intermediate conditions. The condition for Green's law is that the length scale of the tidal wave is small compared with the length scale in which the breadth or depth changes by a large fraction. Now, the basic time scale (period/(2 π)) for tidal waves is 2 hours, so that for a typical wave in a river of moderate depth the length scale would be around 20 miles. This is unlikely to be small compared with the length scale for changes in breadth and depth. For greater rivers both scales would be larger but the conclusion would be much the same. However, for want of a better approximation, Abbott used Green's law of amplification combined with the attenuation due to turbulent friction and deduced a condition for a bore to form. This requires that the amplitude shall remain sufficiently high for sufficiently long, assisted by narrowing of the river and in spite of turbulent energy loss—sufficiently. I say, for neighboring waves to run into one another due to amplitude dispersion. The condition may be expressed in terms of the total variation of tide level in the estuary, and for the Severn Abbott predicted bore formation if the tidal variation exceeds 39 feet. This is close to what is observed; the tidal variation is 41 feet at the spring tides and 22 feet at the neap tides and bores appear only during a few days before and after the highest tide. Again, Abbott's theory gives the position where the

bore is formed correctly as 12 miles upstream of the estuary; for comparison, one obtains only 4 miles in a theory neglecting friction. The agreement on both points is really far better than it should be in view of the fact that the tidal length scale is too great for Green's law to be applicable.

However, there is another observed fact which may explain this mystery. The theory indicates that where the bore forms the local depth η has a vertical tangent but has already been rising for some time. What is observed is that the bore when it arrives is the initial increase of depth from the low-water mark. This indicates that perhaps the wave as a whole has not obeyed Green's law, but that only the steepened part, where the effective length scale has been greatly reduced by amplitude dispersion, has been capable of extracting the full Green's-law amplification out of the reductions in breadth and depth. The rest of the wave has been left behind, its energy being reflected downstream (as well as dissipated by turbulence), and only the concentrated part of the disturbance has reached the upper river. This hypothesis would explain why the use of Green's law for this concentrated part gives a good result for the criterion, and predicted position, for bore formation. However, it is clear that a more exact analysis of this problem is greatly needed.

18. *Effect of friction on the downstream propagation of flood waves*

I come now to the effect of the turbulent frictional dissipation of long gravity waves on the downstream propagation of flood waves; with this problem of how the effect of variations over a period of days in the runoff into the river is felt far downstream we reach perhaps the extreme long-wave end of the spectrum of river waves.

Now, it is clear that any local increase in flow will cause a signal to be propagated downstream at the velocity \sqrt{gh} of long gravity waves. However, we have seen that this signal is damped by a factor of e^{-1} in the time that it takes the river to travel a distance of 200 times the depth, say in 10 minutes or so. Here there will in general be no compensating amplification, by reductions in breadth or depth, and therefore in about an hour the signal moving downstream at the speed of gravity waves will be completely damped out. The question then arises: what has happened to the additional fluid put in at the beginning? It must be found somewhere in the river! (Note in passing that such an appeal to continuity could not be made in the upstream propagation problem, where a rise in the estuary water-level can perfectly well be accommodated through a steady flow in the river, the so-called "backwater curve." No steady flow, however, can pass down the river a flow quantity whose value at one upstream point is varying).

The answer to the question can be obtained in any particular case by getting a complete solution of the equations for small disturbances. This has been done for a channel of uniform breadth and slope and a velocity-squared resistance law by Whitham, who included it as section 3 of our joint paper on kinematic waves [22]. This proves that the maximum of the disturbance travels downstream at the velocity $\frac{3}{2} V$ (where V is the stream velocity); this of course is in general far less than the velocity of gravity waves $V + \sqrt{gh}$. Such a theory shows that, although the characteristics of the partial differential equations represent propagation at the speeds $V + \sqrt{gh}$ and $V - \sqrt{gh}$, nevertheless signals propagated along those characteristics are so damped that for changes extending over really long periods they are unimportant, and only the propagation speed $\frac{3}{2} V$, with which the bulk of the energy travels, is vital to the problem.

This propagation speed is that of the Kleitz-Seddon flood waves, which can however be treated separately under far less restrictive conditions both as to the character of the river and the size of disturbance which is treated, and which therefore,

after the investigation showing how in a typical case they come in time to dominate the scene, must now be described in detail.

19. Kinematic waves

Although such flood waves were discovered 98 years ago by Kleitz (unpublished), also treated very ably by Boussinesq [3] in 1877 and greatly illuminated by James Seddon [29] in 1900 with his studies of the Mississippi and the Missouri, Dr. Whitham and I felt that the subject was still sufficiently little-known, and contained a sufficient number of doubts and ambiguities, to warrant a new treatment [22]. In this treatment we considered such flood waves as a special case of a general class of wave motions which we ventured to call “kinematic waves.”

Kinematic waves exist in any one-dimensional flow system if, to a sufficient approximation, there is a functional relationship between

- (i) the flow q (quantity passing a given point in unit time),
- (ii) the concentration k (quantity per unit distance), and
- (iii) the position x .

Thus we assume that

$$q = q(k, x) \text{ or } k = k(q, x). \quad (19)$$

Note that q/k is the mean velocity of flow, V say, so that the assumption also fixes V as a function of flow or of concentration at every place in the system.

In the application to river waves, “quantity” is taken as volume of water; q is volume flow rate, and k is the cross-sectional area of the water. For any river there is a series of possible steady regimes, each with uniform flow rate q all along the river, and k (or V) is a definite observed function of x for each regime; if this is known for every value of q we have our function $k(q, x)$. I shall have more to say about the mechanisms that govern the relationship between k and q at a given position x , but a common one is of course the balance between gravity and friction. In a flow in which q is varying there is obviously a time lag before the changes in k at a point catch up with the changes in q , but for sufficient slowly-varying flows it is reasonable to neglect this time lag.

On this basic assumption, the properties of kinematic waves follow from the equation of continuity alone,

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (20)$$

It is for this reason that the name “kinematic” is suggested, in contrast to the classical wave motions which we would call “dynamic,” depending as they do on Newton’s second law of motion. Another contrast is that kinematic waves possess only one wave velocity at each point, while dynamic waves possess at least two (forwards and backwards relative to the medium). This is because equation (20) is a first-order equation. On multiplication by

$$C = \left(\frac{\partial q}{\partial k} \right)_{x = \text{constant}} \quad (21)$$

it becomes

$$\frac{\partial q}{\partial t} + C \frac{\partial q}{\partial x} = 0 \quad (22)$$

This means that the flow q is constant on waves travelling past the point with velocity C . Mathematically, the equation has one system of “characteristics” (given by $dx = C dt$), and along each of these q is constant.

The kinematic wave velocity C is the slope of the flow-concentration curve for fixed x . This fact is known in the literature of flood waves as the Kleitz-Seddon law.

20. Other fields of application of kinematic wave theory

Other fields of application in which extensive use of kinematic wave theory has proved possible are the theory of traffic flow on long crowded roads [23], where one starts from the assumption (see Fig. 10) that at each point of the road the mean velocity is a function of vehicle concentration (falling from a maximum velocity at zero concentration to a zero velocity for a certain critical concentration $k_J - J$ for Jam); and, again, the theory of sedimentation of small particles in a liquid, where also the velocity falls as the concentration increases, this time as a consequence of Einstein's law for the viscosity of suspensions. This theory was worked out in 1952 by Professor G. J. Kynch [18], and it was only after Dr. Whitham and I had published our paper on the general theory that we realized that Professor Kynch had anticipated many of our results, at least as far as this application is concerned.

I might remark also that at an early stage of our work we felt that the fact that, in flood waves, the energy travels not at the gravity-wave velocity but at the slower

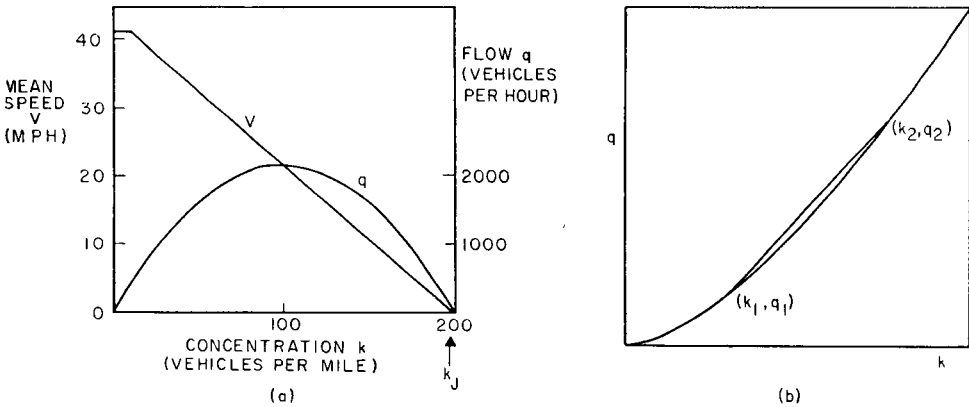


Figure 10. (a) Mean velocity V and flow q plotted as a function of concentration k for rural roads in U.S.A.: data due to B.D. Greenshields. Proc. Highw. Res. Bd. 14, 448 (Washington, 1935). (b) Flow-concentration curve for flood waves, showing construction to determine speed of kinematic shock wave.

speed of the kinematic waves has so much in common with the theory of group velocity (in systems with what I have called frequency dispersion) that it might in some obscure way be a special case of that theory. We could not sustain this view for long, because no frequency analysis comes into the floodwave problem, but later we saw that the two problems were differently related, both being in fact special cases of the general kinematic wave theory.

To see this for the case of wave propagation with frequency dispersion, one has to apply a law of conservation of number of wave crests. This gives that the flow q is the number of wave crests passing per unit time, which is the frequency (c/λ), and that k is the number per unit distance, namely $1/\lambda$, so that the frequency remains constant along kinematic waves travelling at the speed

$$\frac{dq}{dk} = \frac{d(c/\lambda)}{d(1/\lambda)} = c - \lambda \frac{dc}{d\lambda}, \tag{23}$$

which is in fact the group velocity (2). Conservation of number of wave crests is not,

of course, accurately true; but, once sufficient dispersion has occurred to render small the frequency change in a single wavelength, then the appearance or disappearance of crests through the occurrence of horizontal points of inflexion must become very rare.

21. Kinematic shock waves

Now kinematic waves themselves do not suffer frequency dispersion, but they do suffer amplitude dispersion, and it is the consequences of this fact that Dr. Whitham and I have felt were inadequately covered in the literature, making a new treatment necessary. Kinematic waves are, as we have seen, totally different from long gravity waves, and are important for problems with time scales over which long gravity waves would be completely damped out, but they have this property of amplitude dispersion in common. Thus, the wave velocity C is a function of the flow q carried by the wave, as well as of the position x . Since the mean velocity V increases with concentration, the graph of $q = kV$ against k is concave upwards, and $C = \partial q / \partial k$ increases with k , and also with q . Hence waves carrying high values of the flow travel faster than waves carrying low values, and it is possible for the former to overtake the latter, giving discontinuities in the flow q . We suggested the name "kinematic shock waves" for these discontinuities, since their process of formation is exactly like that of shock waves in a gas. It is similar also to the process of bore formation in long gravity waves which we discussed earlier.

Now, if a discontinuous wave moves downstream with velocity W , it is easily seen that the flow across the moving wave is $q - Wk$, and it follows by continuity that the law of motion of such a kinematic shock wave requires that this expression be continuous across it, giving

$$W = \frac{q_2 - q_1}{k_2 - k_1}. \quad (24)$$

This speed W of the kinematic shock wave is the slope of the chord (Fig. 10) joining the points (k_1, q_1) and (k_2, q_2) on the flow-concentration curve representing the upstream and downstream conditions. In the limit of a weak shock wave it coincides with the slope of the tangent, that is, $\partial q / \partial k$, the velocity of continuous waves.

Now one must not suppose from the use of the word "discontinuity" that these kinematic shock waves are very concentrated things, similar to bores. Actually, owing to the time lag needed for adjustments of flow and concentration to catch up with one another, such a wave, though discontinuous on the simple theory, may extend over many miles. It is in fact the so-called "monoclonal flood wave" which in the special case of uniform flow upstream and downstream has been computed for special river models by Thomas [31] and Dressler [7] as well as by Whitham [22]. Our suggestion for treating more general cases is that a first approximation to the flow be obtained by kinematic wave theory, and that the results be then improved by replacing such discontinuities as arise by calculated profiles of this kind, choosing those appropriate to the computed strengths of the kinematic shock waves (see lower diagram in Fig. 11). The justification for this is that it is only in the places where the flood-wave profile has been considerably steepened by amplitude dispersion that the time lags for adjustments of flow and concentration to catch up with one another will be important.

22. A new method for calculating flows with kinematic shock waves

It is unnecessary to give here any details of special calculations involving kinematic waves, but I will describe a new method of determining the position of the kinematic shock wave as a function of time, which is simpler than the methods given in our original paper.

To fix the ideas, suppose that the flow q is measured as a function of time at a particular station $x = x_1$ in a river, and that we want to predict what the flow variation

with time will be at some other station $x=x_2$ far downstream. Then each value q of the flow at x_1 will remain constant along a wave which will take a time

$$T(q) = \int_{x_1}^{x_2} \frac{dx}{C(q,x)} = \int_{x_1}^{x_2} \frac{\partial k}{\partial q} dx = \frac{\partial}{\partial q} \int_{x_1}^{x_2} k(q,x) dx \quad (25)$$

to reach x_2 . This $T(q)$ can be calculated in various ways, but one way is to notice that $T(q)$ is the derivative with respect to q of the total volume of water between x_1 and x_2 when the river is flowing steadily at the rate q .

The deformation of the curve is now easily sketched (Fig. 11). Each point (q,t) on the flow-time curve at x_1 becomes $(q,t + T(q))$ at x_2 . We then have to answer the question: what happens if waves carrying high values of q overtake those carrying low values, leading to a curve of the form illustrated? The simple answer is

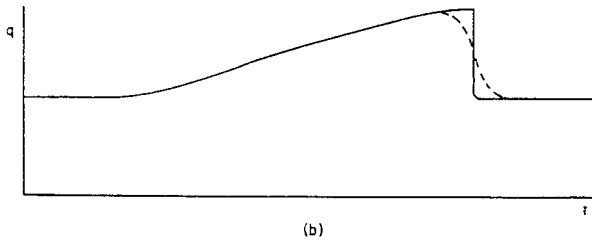
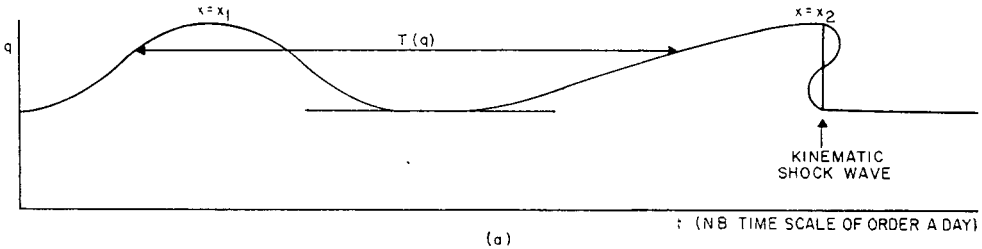


Figure 11. (a) Construction to determine position of kinematic shock wave. Lobes shown must be of equal area so that total area under (q,t) curve remains unaltered.

(b) Replacements of kinematic shock wave by calculated "monoclinal flood wave" profile.

that the kinematic shock wave (or discontinuous jump in q) must be fitted in at that value of t for which the total area under the (q,t) curve, that is the total flow across $x=x_2$, remains unaltered by the substitution.

To see this, one need observe only that the (q,t) curve has been deduced from a theory based on the equation of continuity. Therefore the area under it, or total flow across $x=x_2$ in a certain time interval must equal the total flow across $x=x_1$ in the same time interval, plus the volume of water between the stations at the beginning, minus the volume of water left at the end.* However, in the real motion the total flow

* Mathematically this says only that

$$\int_C (qdt - kdx) = \iint_S \left(\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} \right) dxdt = 0 \quad (26)$$

where C is a curve in the (x,t) plane and S its interior.

across station 2 must also equal the total flow across station 1 plus this difference in the volume between the stations at the beginning and end of the time interval. Therefore the true discontinuous flow must have the same area under its (q,t) curve as the calculated continuous flow which it replaces.

By this rule the development of the flood as it progresses downstream is easily calculated, and the kinematic shock wave is found first to grow in strength, and later on to decay (as the total length of the wave increases).

23. Mechanisms governing the flow-concentration relation

I should like to discuss now the problem of what the flow-concentration curve should be taken to be in the kinematic-wave theory of flood movements; this question is the same as asking: what is the steady regime in the river for different uniform values of the flow? Here the point I want to make is one which was already shown clearly by James Seddon in his article [29] of 1900, namely, that this question can only be answered observationally. The reasons need not be discussed at length here as the principal arguments are clearly set out in Seddon's very readable article as well as summarised in the paper by Dr. Whitham and myself.

The main point is that the equations of hydraulics [11, 27], which were developed largely for flow in man-made conduits of regular cross-section, are misleading for natural rivers. For a conduit one could certainly apply kinematic wave theory, using for example the Chézy law of frictional resistance for relating velocity to mean depth, which gives for a rectangular section

$$q \propto k^{3/2}, \quad C = \frac{dq}{dk} = \frac{3}{2} \frac{q}{k} = \frac{3}{2} V \quad (27)$$

Or one could use the Manning frictional law, which gives $C/V = \frac{5}{3}$ or with either law and a different shape of cross-section one can get different values of C/V . But the resulting theory could not be applied to the irregular bed shape of any real river. As a matter of fact, if one knew precisely the dynamics of the flow there would be little point in using a kinematic-wave theory, except as a rough guide, because one could put the complete equations of motion on to an electronic computer.

However, the variations in bed shape along the river contribute many mechanisms controlling the flow rate which are ignored by the hydraulic equations; for example, local flows similar to that through a vertical slit, or over a submerged weir. To illustrate the point further I shall give details of just one of the many difficult problems arising in steady river hydrodynamics, that of the occurrence of reversed-flow regions. My object in discussing it is mainly to emphasize the complication of the subject and to suggest that at present a dynamic theory is too difficult, but that a theory like kinematic wave theory, based on one assumed relation which is especially susceptible to direct observational determination, is the best point from which to start in the problem of flood waves.

24. Reversed-flow regions in steady river flow

Now, the standard works on hydraulics often consider flows in which the depth h varies considerably across the width of the river, but they do not usually discuss clearly what is the associated distribution of the flow velocity v across the width. Sometimes it is tacitly assumed that v is approximately uniform, but for small ratios of depth to width the bottom effectively prevents the existence of eddies which are anything like big enough to diffuse a nearly uniform velocity over the width of river (in the way that happens, say, in a pipe). An alternative hypothesis is that at each point across the

width gravity and friction are balanced, leading (say, for the Chezy resistance law) to a relation

$$v = K \sqrt{h\alpha} \quad (28)$$

where α , the slope of the free surface, is constant across the river and can be determined by integration from the given total flow rate q . This approach has a solid basis for channels of uniform section.

However, in rivers one often observes a quite different kind of flow, which is actually reversed in the regions of shallow water (Fig. 12). This is connected with the general tendency of flow to separate, not only from solid obstacles, but also from local areas of high resistance, for example a row of trees used as a wind break.

The explanation is that if separation did not occur there would still be only a slow flow in the area of high resistance. On the other hand, the velocity just outside it (on the other side of the river) needs to be greater than its values upstream and downstream if the total flow rate q is to be passed. As this high velocity slows down (farther downstream) a rise in water level must occur, by Bernoulli's equation, and this can bring to rest the slower fluid which is in the area of high resistance. Hence separation and reversed flow set in.

It would be possible to treat these flows by equations of the usual boundary-

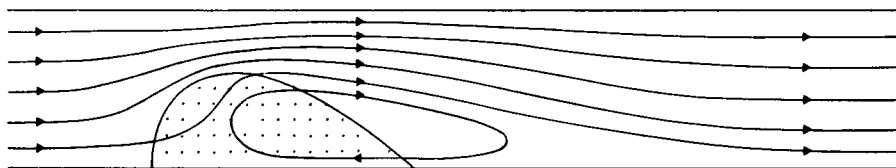


Figure 12. Separated flow resulting from presence of region (shown dotted) of shallow water.

layer type. because the condition of constant water elevation across the river (corresponding to the condition of constant pressure across a boundary layer) is very nearly fulfilled. The only difference from the usual boundary-layer equations would be the occurrence of a turbulent friction term. The condition for separation should emerge from such a treatment, but to my knowledge this has not so far been carried out.

Similar reversed flows occur in estuaries, where at high tide the presence of a submerged sandbank is often revealed by a line of foam which has collected along the locus where the forward and reversed flows meet.

25. Improvements of kinematic wave theory

At this stage it can be reasonably objected that I am advocating the use of a kinematic-wave approach to the prediction of flood movement, based on an experimental flow-concentration relationship, but am giving no suggestions on what should be done if changes are taking place somewhat too rapidly for the time lag between adjustments of flow and concentration to be accurately negligible. This serious criticism I must now try to meet.

The situation is similar to one that has arisen in the aerodynamics of rarefied gases. when one tries to apply the theory of sound waves of finite amplitude and shock waves to problems with characteristic length scales of only a few mean free paths, so that such shock waves as occur are really thick. Important advances in this subject were made recently by Prof. Eberhard Hopf [13] and Dr. Julian Cole [5], and I gave a survey of it as my contribution to the G. I. Taylor anniversary volume [21]. There the effect of lag in establishing equilibrium can in many cases be treated by taking a second approximation to the flow equations in the form of the Burgers equation, which has a

convenient exact solution for any initial conditions. It may be that a development along these lines, involving a suitable approximation to the non-linear terms in the equation of continuity, and the incorporation of a second-derivative term to represent the effect of lag, will prove practicable.

Alternatively, there is the solution given in our original paper, based on making a direct observation of how the flow varies with the rate of increase of concentration at a point for given concentration. We showed that if the ratio of the change of concentration to the change of time-derivative of concentration required to produce a given flow change is taken constant (say as an average of the observed values), then such an approach would necessarily give the kinematic shock waves their correct position and velocity.

As a final alternative, it may be that one who wants real accuracy rather than just a bird's-eye view of the picture may always need to go, as at present, to models, with the detailed characteristics of each stretch of the river reproduced in detail in the model.

26. Roll waves

As my last topic, I come to what Dr. Whitham found [22] when he applied the theory of the attenuation of gravity waves by turbulent friction to torrents. If $F = V/\sqrt{gh}$ is the Froude number of the stream, then on the model described earlier the attenuation factor

$$e^{-fVt/h} \tag{29}$$

has to be replaced by

$$e^{-\frac{fVt}{h}(1-\frac{1}{2}F^2)} \tag{30}$$

so that when F reaches 2 attenuation ceases. This is due to the fact that on the particular model used the kinematic wave velocity $C = \frac{3}{2}V$ with which the main part of the flow is carried becomes equal to the gravity-wave velocity $V + \sqrt{gh}$, so that gravity waves thereafter carry the flow without attenuation. It seems that for any system this will happen when C reaches $V + \sqrt{gh}$.

What is not so clear at a first glance is what happens when the kinematic-wave velocity actually exceeds the gravity-wave velocity. However, Whitham's analysis com-

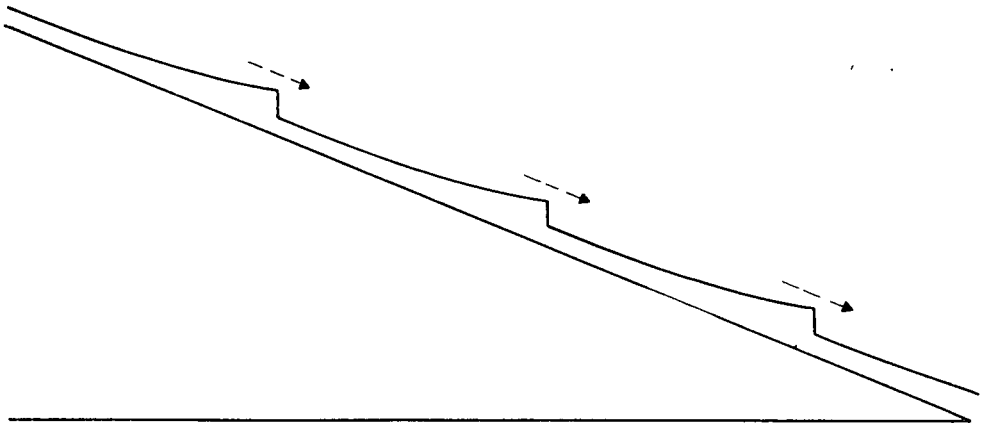


Figure 13. Roll waves (after Dressler [7]).

bined with earlier work by Jeffreys [14] and Dressler [7, 8, 9] shows that these flows with $C > V + c$ are unstable and tend to break up into the so-called "roll-waves," which can be regarded as alternations of kinematic shock waves* and bores (Fig. 13). Good two-dimensional roll-waves are observed only with a fairly flat bottom of regular slope, but a similar instability involving, if necessary, three-dimensional disturbances occurs in any sufficiently rapid torrent.

On a small scale, one can easily observe roll waves on a thin film of water flowing down a vertical glass surface; it would be interesting to try to develop the theory of these capillary roll waves, taking surface tension instead of gravity as the restoring force and laminar instead of turbulent friction.

27. Conclusion

In conclusion, I hope that the steeplechase on which I have led you over, and round, the difficulties of this extensive subject has at least suggested to you that interesting questions abound in the study of river waves. There may be some in the audience who like myself have devoted much of their working life to the quite different question of trying to unravel the mysteries revealed by the beautiful schlieren and interferometric photographs which are taken in supersonic wind tunnels and shock tubes. To them I would suggest that, fascinating though these subjects are which relate to the behaviour of gases under very extreme conditions, there is sometimes an even greater delight to be derived from fluid motions in which one can become immersed not only metaphorically but also literally.

REFERENCES

1. Abbott, M. R., Proc. Camb. Phil. Soc. 52, 344 (1956).
2. Benjamin, T. B. & Lighthill, M. J., Proc. Roy. Soc. A, 224, 448 (1954).
3. Boussinesq, J., "Essai sur la théorie des eaux courantes", Mém. prés. Acad. Sci. (Paris, 1877).
4. Burns, J. C., Proc. Camb. Phil. Soc. 49, 695 (1953); with an appendix by M. J. Lighthill.
5. Cole, J. D., Q. Appl. Math. 9, 225 (1953).
6. De, S. C., Proc. Camb. Phil. Soc. 51, 713 (1955).
7. Dressler, R. F., Comm. Pure & Appl. Math. 2, 149 (1949).
8. Dressler, R. F., Nat. Bur. Stand. Circ. 521, 237 (1952).
9. Dressler, R. F. & Pohle, F. V., Comm. Pure & Appl. Math. 6, 93 (1953).
10. Favre, H., "Etude théorique et expérimentale des ondes de translation dans les canaux découverts" (Dunod, Paris, 1935).
11. Forchheimer, P., "Hydraulik" (Teubner, Leipzig, 1930).
12. Green, G., Trans. Camb. Phil. Soc. 6, 457 (1837).
13. Hopf, E., Comm. Pure & Appl. Math. 3, 201 (1950).
14. Jeffreys, H., Phil. Mag. (Ser. 6) 49, 793 (1925).
15. Keulegan, G. H. & Patterson, G. W., J. Res. Nat. Bur. Stand. 24, 47 (1940).
16. Kishi, T., Proc. 4th Jap. Nat. Congr. Appl. Mech. p. 241 (1954).
17. Korteweg, D. J. & De Vries, G., Phil. Mag. (Ser. 5) 39, 422 (1895).
18. Kynch, G. J., Trans. Faraday Soc. 48, 166 (1952).
19. Lamb, H., "Hydrodynamics." 6th Edition (Cambridge 1932).
20. Lemoine, R., "Sur les ondes positives de translation dans les canaux et sur le ressaut ondulé de faible amplitude." La Houille Blanche, No. 2 (Grenoble, 1948).

* More strictly, kinematic shock waves in reverse; these become possible at supercritical speeds.

21. Lighthill, M. J., "Surveys in Mechanics" (ed. G. K. Batchelor & R. M. Davies), pp. 250-351 (Cambridge, 1956).
22. Lighthill, M. J. & Whitham, G. B., Proc. Roy. Soc. A, 229, 281 (1955).
23. Lighthill, M. J. & Whitham, G. B., Proc. Roy. Soc. A, 229, 317 (1955).
24. McCowan, J., Phil. Mag. (Ser. 5) 38, 351 (1894).
25. Michell, J. H., Phil. Mag. (Ser. 5) 36, 430 (1893).
26. Rayleigh, Lord, Proc. Roy. Soc. 90, 324 (1914).
27. Rouse, H., "Engineering Hydraulics" (Wiley, New York, 1950).
28. Scott Russell, J., "Report on Waves" (British Association Report, 1844).
29. Seddon, J. A., Trans. Amer. Soc. Civ. Engrs. 43, 179 (1900).
30. Stokes, G. G., "Mathematical and physical papers," Vol. 1 (Cambridge, 1880).
31. Thomas, H. A., Trans. Amer. Geophys. Un. Part II, 21, 596 (1940).
32. Whitham, G. B., J. Flu. Mech. 1, 290 (1956).

DISCUSSION

J. J. Stoker

All of you will agree, I am sure, that Professor Lighthill has given us a very stimulating discussion. While it was confined to a very small section of water wave phenomena, it still shows that there is a huge variety in even small portions of that subject, and a variety that stems not only from variety in the physical phenomena, but also from variety with respect to possible ways of mathematical treatment.

Professor Lighthill has already alluded to the fact that he and I might not agree in all respects, about some of the things that he said concerning wave motions in rivers.

I have a few slides giving results of calculations which a group of people at our Institute have been making during the last three years. We have been involved in computing flows in large rivers and reservoirs, with the object of making flood predictions for lengthy periods of time, and over long stretches of rivers. For example, we have dealt with floods in the Ohio River between Wheeling and Cincinnati; that is a stretch close to 400 miles in length. We have also computed flows through the junction of the Ohio and Mississippi at Cairo, Illinois, taking forty mile lengths of the rivers on both sides of Cairo upstream in the Ohio and Mississippi, and the same distance downstream in the lower Mississippi. Calculations have been made for flows in Kentucky Reservoir at the mouth of the Tennessee River. This reservoir lies between two dams, the Kentucky Dam on the downstream side, just about where the Tennessee runs into the Ohio River, and the Pickwick Dam, some 180 miles upstream from there.

Now it is true that kinematic waves, which Professor Lighthill described so nicely here, certainly give a correct general picture of what happens in a large river. There is no doubt that the main disturbance is one which goes much slower than the propagation speed of wavelets, and the main disturbance does indeed travel downstream in general, and not both up and down stream, in spite of the fact that the basic theory formulated by the exact differential equations leads always to wave propagation in both directions.

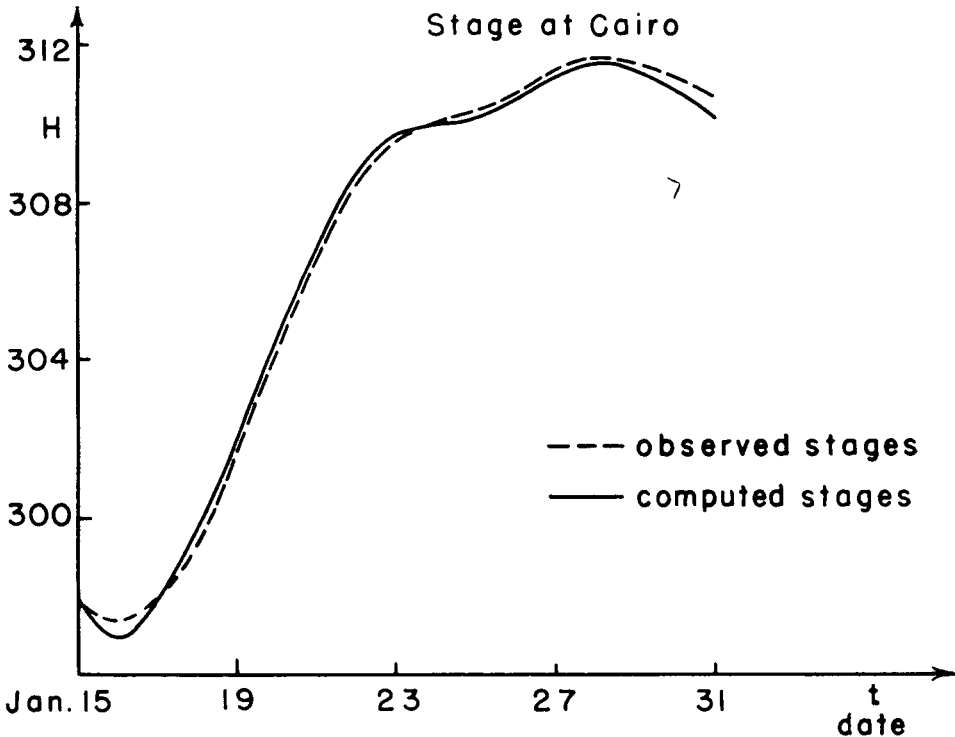
However, we would be in very great difficulties in the Ohio River if we tried to base our calculations on "kinematic waves," because even the main disturbance sometimes goes up-stream in the Ohio, because the main flow is very much influenced by the flow from tributaries and the local drainage areas. That part of the flow is so large sometimes that the main flood travels in the wrong direction, so to say. We find it necessary to use the full differential equations, both the momentum equation and the continuity equation, and consequently we must put up with the fact that the differential equations have two real characteristics which require us to use small time steps for calculation with the UNIVAC.

In the Kentucky Reservoir if one wants to make predictions for three weeks, there is ample time for many waves to propagate back and forth from the dams at both

ends, so that propagation upstream is quite vital in this case, also, and thus one needs to consider the waves that are going in both directions.

I will cut this short by showing a few slides giving the results of calculations at the junction of the Ohio and Mississippi, and in the Kentucky Reservoir.

The first slide (Fig. 1) shows stage at Cairo, at the junction of the Ohio and Mississippi, and this is a prediction for sixteen days. (Stage means height of the river.) One sees that the flood stages are predicted quite accurately. This was done on the UNIVAC, and the calculation requires a few hours of UNIVAC calculating time. That

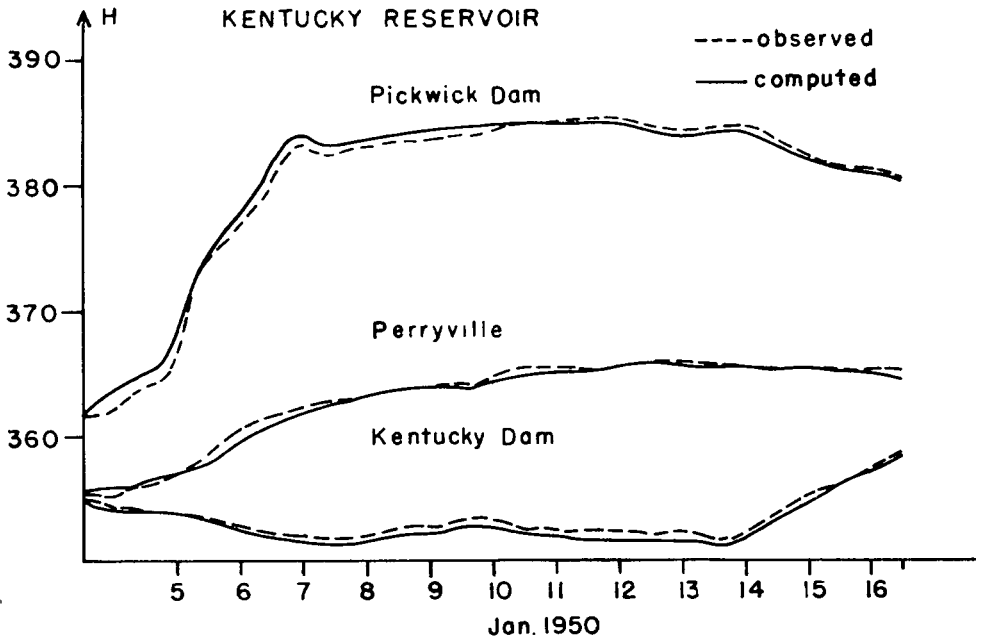
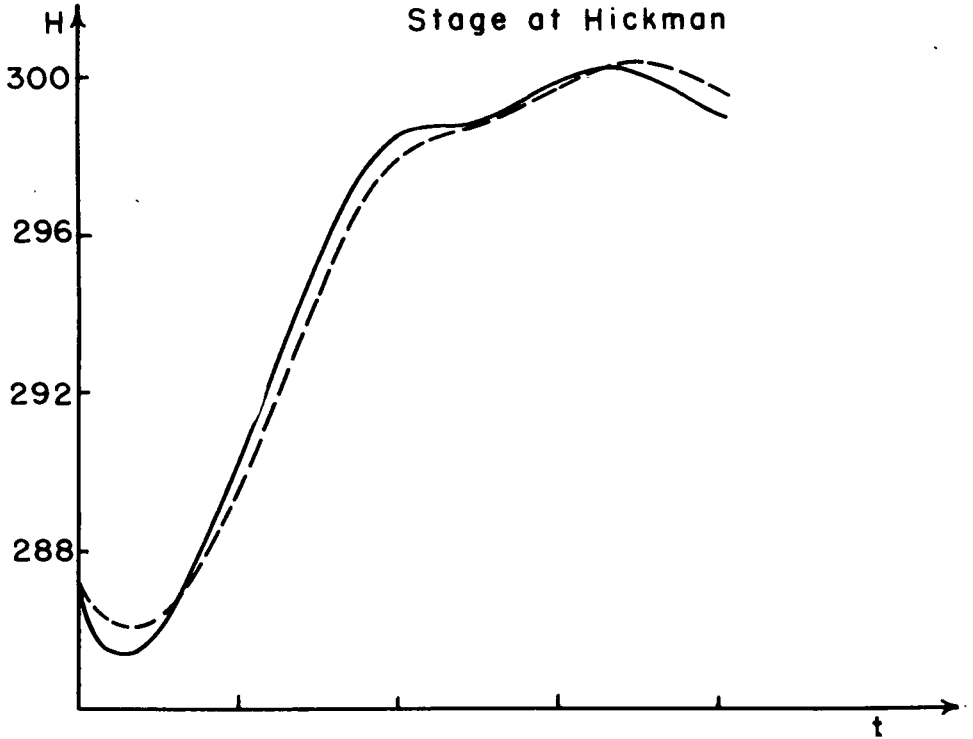


is a lot of calculating, since the UNIVAC can do a great deal in two hours time. In addition to the initial state of the system (i.e. the initial stage and velocity at all points), the flows at the upper end of the Ohio and Mississippi are assumed known over the whole time of the prediction. The second slide (Fig. 2) shows the results at Hickman, below Cairo, in the lower Mississippi, and again this shows stages for a sixteen-day forecast. At Hickman, the relation between stage and discharge was known. This is an impedance condition which simulates the effect of the rest of the Mississippi below that point. As one sees, the observed flood and the calculated one agree very well in this case. The difference is a matter of inches, even in a period of two weeks.

The third slide (Fig. 3) gives our results for the Kentucky Reservoir. The lowest curve is for Kentucky Dam, at the mouth of the Tennessee. The next one above is Perryville, somewhere near the center of the reservoir, some eighty miles upstream from Kentucky Dam. Pickwick Dam is the dam at the upper end of the reservoir. The length of this reservoir is about 186 miles.

Again we were given initially the state of the reservoir. That is, flow velocity and depth or stage were known throughout the reservoir at the initial instant. The discharges out of Kentucky Dam and into the reservoir from Pickwick Dam were given,

Stages in the Junction Problem for 16 Day Forecast



and also the flows from tributaries, which contributed as much as 30 per cent of the flow sometimes. By integrating the differential equations we find the results that you see on the slide. Again computed and observed values are very close—the difference between the observed and the calculated stages is a matter of inches.

G. K. Morikawa

I wish to make a brief comment on Prof. Lighthill's interesting paper, in particular, about his "kinematic waves" (Lighthill and Whitham, 1955). An unexpected feature of his proposed theory of flood movement is that shock (or bore) formation is included as an essential part of the theory. This approximate theory might appear rather strange at first glance since, in river flows, resistance plays such an important role. (Of course, in flow with no, or with small, resistance we know that shocks are always imminent.) For the kinematic wave approximation, all derivative terms (inertia plus gravity-surface slope) in the momentum eqn. (20) (see Ref.) are ignored, leaving only the gravity-bottom slope to balance the resistance force in the momentum equation. The neglect of the restoring force provided by the gravity-surface slope term allows bore formation.

It is well known that bores are formed only under very severe conditions such as those encountered in very steep rivers or at large tidal estuaries. In slow, long rivers, bores are not formed even under the extreme conditions of flood stage during which time heavy tributary flows into the main stream cause relatively (compared to the bottom slope) large surface gradients. These steep gradients generally are damped as they travel downstream. However, I hope that the "kinematic wave" approach, which seems quite attractive from the viewpoint of its possible simplicity, will be pursued further in the study of flood movement and other physical problems.

Reference:

M. J. Lighthill and G. B. Whitham. "On Kinematic Waves. I. Flood Movement in Long Rivers," Proceedings of the Royal Society, A, Volume 229. pp. 281-316. 1955.

M. J. Lighthill

In reply to Dr. Morikawa, there are different kinds of "shock waves." One is the dynamic shock wave, which we call a bore, and the other is the kinematic shock wave, which is very extensive, covering many miles. You could see, from one of the curves Professor Stoker showed, this rapid rise at the beginning of the flood.

Now, of course, my answer to Professor Stoker is that I agree with him. On the other hand, he has the advantage of having come to the end of his programme, and being able to show the final results. The kind of thing that Whitham and I have suggested hasn't yet been applied, and of course I wasn't able in the lecture to discuss all the modifications we propose. When run-off, for example, is present, as Professor Stoker was describing, we incorporate it in the theory in a perfectly straightforward manner (described in our paper). We also attach importance to the inclusion of second-derivative terms which in our paper we called diffusion (rather like diffusion in a gaseous shock wave, which increases its thickness) but which in my lecture I treated only in a brief section.

This theory, as it were, hasn't been fed into the data on which Professor Stoker worked.

Now, I am not suggesting that it would give such good results as the kind of thing Professor Stoker does, because obviously if you spend a lot of money, if you use very big machines, and very long and complicated models, you are bound to get better results. To a large extent it is a matter of compromise. By applying kinematic waves, and then improving the theory, you may be able to get rough results which have some

value fairly quickly. On the other hand, you can go to the construction of a model of the river, or, what is the equivalent of a model, use of a digital computer. You are obviously best-off to use the model, if you want an accurate answer.

J. J. Stoker

I would say that if you once decide to solve these problems by integrating differential equations by one scheme or another, then the biggest job is the processing of the data, obtaining the resistance coefficient, getting the cross-section areas, coding all of this, and setting up a procedure. It could be that by using simplified differential equations, the whole thing might be done on a smaller computing machine, which would be less costly in its operation. On the other hand, as I have pointed out, it doesn't take very long anyway to make these calculations.

M. J. Lighthill

Three years!