A SOLUTION OF THE MINIMUM WAVE RESISTANCE PROBLEM

R. Timman and G. Vossers
Netherlands Ship Model Basin

INTRODUCTION

The problem of the determination of ship hulls with minimum wave resistance has been the subject of numerous investigations. The usual way of approach, based on a classical paper by Weinblum (1930) is to consider a ship hull in the form of an infinitely deep cylinder and to solve the variational problem of minimizing the resistance integral with prescribed horizontal cross section of the ship. In Weinblum's original paper a Ritz method is used; the waterline is represented by a polynomial with unknown coefficients which are determined from a minimum problem for a function of a finite number of variables. Computation of the Weinblum-functions (1955) facilitates the procedure.

The method, however, is open to some criticism, since it is not a priori clear, that it yields a good approximation to the variational problem.

It is known that Pavlenko (1934) reduced the variational problem to the solution of an integral equation of the first kind. It is known that its solution has certain singularities, which cannot properly be represented by a polynomial.

In this paper the problem is reduced to an integral equation of the second kind, obtained by the following consideration. The resistance integral in the thin ship approximation is known to be a quadratic functional of the hull function. If the additional condition is also expressed by a quadratic functional, the problem is simply equivalent to a principal axis problem and the solution exists, as is known from the general theory of quadratic functionals. Essentially, it is assumed that the cross sections of the ship are similar in shape, which (with a slight modification) produces the required result. This means, that the ship considered has a finite draft.

In order to obtain a simple integral equation, the influence of the bottom in the evaluation of Michell's integral is neglected, although a more correct treatment lies within the power of numerical methods.

MICHELL'S FORMULA FOR THE RESISTANCE INTEGRAL

We introduce a coordinate system with the x axis along the axis of the ship, the y axis athwartships, and the z axis downward. The ship hull is represented by a function

$$y = \pm f(x,z) , \qquad -L/2 < x < L/2 , \qquad 0 < z < T$$

where L is the length of the ship and T the draft. The ship is moving in the x direction with constant velocity c. The disturbance potential φ, from which the disturbance velocities $u = -\partial\varphi/\partial x$, $v = -\partial\varphi/\partial y$, $w = -\partial\varphi/\partial z$ are derived, is determined by the condition that at the hull of the ship

$$\frac{\partial \varphi}{\partial x} = c \frac{\partial f}{\partial x}.$$

The linearization, introduced by Michell consists in applying this condition at the plane of symmetry $y = 0$, in which case the potential, satisfying the Laplace equation $\nabla^2\varphi = 0$ and the free surface condition at the plane $z = 0$:

$$\varphi_{xx} - \kappa\varphi_z = 0$$

where

$$\kappa = \frac{g}{c^2}.$$

The solution of this boundary value problem is then

$$\varphi(x, y, z)$$

$$= \frac{2c}{\pi} \int_{-L/2}^{+L/2} dx_1 \int_0^T dz_1 \frac{\partial f(x_1, z_1)}{\partial x_1} \int_\kappa^\infty \frac{e^{-\frac{\alpha^2}{\kappa}(z+z_1)} \sin\left[\alpha(x-x_1) + y\frac{\alpha}{\kappa}\sqrt{\alpha^2-\kappa^2}\right]}{\sqrt{\alpha^2-\kappa^2}} \alpha\, d\alpha$$

$$- \frac{2c}{\pi} \int_{-L/2}^{L/2} dx_1 \int_0^T dz_1 \frac{\partial f(x_1, z_1)}{\partial x_1} \int_0^\kappa \frac{e^{-\frac{\alpha^2}{\kappa}(z+z_1) - \frac{\alpha y_1}{\kappa}\sqrt{\kappa^2-\alpha^2}}}{\sqrt{\kappa^2-\alpha^2}} \cos\alpha(x-x_1)\, d\alpha$$

$$+ \frac{2c}{\pi^2} \int_{-L/2}^{+L/2} dx_1 \int_0^T dz_1 \frac{\partial f(x_1, z_1)}{\partial x_1} \int_0^\infty \cos\alpha(x-x_1)\, d\alpha$$

$$\times \int_0^\infty \frac{e^{-\sqrt{\alpha^2+n^2}y}}{\sqrt{\alpha^2+n^2}} \cos(nz-\varepsilon) \cos(nz_1-\varepsilon)\, dn$$

where

$$\tan\varepsilon = -\frac{\alpha^2}{\kappa n}.$$

The pressure is determined by Bernoulli's equation

$$-\varphi_t + \frac{1}{2}\left[(-c+\varphi_x)^2 + \varphi_y^2 + \varphi_z^2\right] + \frac{p}{\rho} - gz = \frac{1}{2}c^2 + \frac{p_\infty}{\rho}$$

or, in linearized approximation for steady flow,

$$p - p_\infty = -\rho c \varphi_x + \rho g z .$$

The force in the x direction, exerted on a surface element of the hull is

$$dK = -p \frac{\partial f}{\partial x} dxdz$$

where $dxdz$ is the projection of the element on the x,z plane. The total resistance is

$$R = -2 \int_{-L/2}^{+L/2} \int_0^\infty p \frac{\partial f(x,z)}{\partial x} dxdz = 2\rho c \int_{-L/2}^{+L/2} \int_0^\infty \frac{\partial \varphi}{\partial x} \frac{\partial f}{\partial x} dxdz.$$

Substitution of the expression for φ gives for the resistance integral

$$R = \frac{4\rho c^2}{\pi} \int_{-L/2}^{+L/2} dx \int_{-L/2}^{+L/2} dx_1 \int_0^\infty dz \int_0^\infty dz_1 \frac{\partial f(x,z)}{\partial x} \frac{\partial f(x_1,z_1)}{\partial x_1}$$

$$\times \int_\kappa^\infty e^{-\frac{\alpha^2}{\kappa}(z+z_1)} \frac{\alpha^2 \cos \alpha (x - x_1)}{\sqrt{\alpha^2 - \kappa^2}} d\alpha .$$

We introduce dimensionless coordinates

$$\xi = \frac{2x}{L} , \qquad \eta = \frac{2y}{B} , \qquad \zeta = \frac{z}{T}$$

where B is the beam and T the draft of the ship. Then

$$\frac{\partial f(x,z)}{\partial x} = \frac{B}{L} \frac{\partial \eta(\xi,\zeta)}{\partial \xi} = \frac{B}{L} h(\xi,\zeta)$$

and the resistance integral takes the form

$$R = \frac{\rho c^2 B^2 T^2 \kappa^2}{\pi} \int_{-1}^{+1} d\xi \int_{-1}^{+1} d\xi_1 \int_0^{+1} d\zeta \int_0^{+1} d\zeta_1 \, h(\xi,\zeta) \, h(\xi_1,\zeta_1)$$

$$\times \int_1^\infty e^{-\lambda^2 \kappa T(\zeta+\zeta_1)} \cos \frac{\lambda L \kappa}{2} (\xi - \xi_1) \frac{\lambda^2 \, d\lambda}{\sqrt{\lambda^2 - 1}} .$$

The volume of the ship is given by

$$V = \int_{-L/2}^{+L/2} dx \int_0^T f(x,z)\ dz\ .$$

The problem then consists in the determination of the function $f(x,z)$, which minimizes the resistance at a prescribed volume.

SIMPLIFIED TREATMENT OF THE VARIATIONAL PROBLEM

We put the condition that $f(x,z) = 0$ for $x = \pm L/2$ and transform the resistance integral by integration by parts:

$$R = \frac{\rho c^2 B^2 T^2 L^2 \kappa^4}{4\pi} \int_{-1}^{+1} d\xi \int_{-1}^{+1} d\xi_1 \int_0^\infty d\zeta \int_0^\infty d\zeta_1\ \eta(\xi,\zeta)\ \eta(\xi_1,\zeta_1)$$

$$\times \int_1^\infty e^{-\lambda^2 \kappa T(\zeta+\zeta_1)} \cos\frac{\lambda L \kappa}{2}(\xi-\xi_1)\frac{\lambda^4\, d\lambda}{\sqrt{\lambda^2-1}}\ .$$

In order to simplify the problem, a special class of ship hulls is considered which is so determined that both R and V appear as quadratic functionals of a function $f(\xi)$. We assume

$$f(\xi,\zeta) = f(\xi)\ g\left[\frac{\zeta\sqrt{1-\xi^2}}{f(\xi)}\right].$$

where $f(\xi)$, the waterline function, is to be determined and $g(\mu)$ is a fixed function of μ which is zero for μ greater than a certain value. If the factor $\sqrt{1-\xi^2}$ did not exist, the assumption would mean that all cross sections of the ship were similar. If $f(\xi)$ vanishes at the front and the stern of the ship, also the draft would vanish there. This is prevented by the present assumption, and moreover it will appear later that the addition of this factor actually yields functions $f(\xi)$ which vanish at these places. The volume of the ship is given by

$$V = \frac{LT}{2}\int_{-1}^{+1} d\xi \int_0^\infty d\zeta\ f(\xi)\ g\left[\frac{\zeta\sqrt{1-\xi^2}}{f(\xi)}\right] = \frac{LT}{2}\int_{-1}^{+1}\frac{f^2(\xi)}{\sqrt{1-\xi^2}}\,d\xi\int_0^\infty g(\mu)\,d\mu\ .$$

The resistance is

$$R = \frac{\rho' c^2 B^2 T^2 L^2 \kappa^4}{4\pi} \int_{-1}^{+1} d\xi \int_{-1}^{+1} d\xi_1 \int_0^\infty d\zeta \int_0^\infty d\zeta_1 \; f(\xi) \; f(\xi_1) \; g\left[\frac{\zeta\sqrt{1-\xi^2}}{f(\xi_1)}\right]$$

$$\times \int_1^\infty e^{-\lambda^2 \kappa T(\zeta+\zeta_1)} \cos\frac{\lambda L \kappa}{2}(\xi - \xi_1) \frac{\lambda^4 \, d\lambda}{\sqrt{\lambda^2 - 1}}.$$

In order to obtain a simple integral expression, we neglect the dependence of g on ζ and assume infinite depth. An estimate of the validity of this procedure is obtained by considering the case, where the cross sections are rectangular:

$$\begin{cases} g(\mu) = 1, & 0 < \mu < 1 \\ g(\mu) = 0, & \mu > 1. \end{cases}$$

Then

$$\int_0^\infty d\zeta \; g\left(\frac{\zeta\sqrt{1-\xi^2}}{f(\xi)}\right) e^{-\lambda^2 \kappa T \zeta} \, d\zeta = \int_0^{\frac{f(\xi)}{\sqrt{1-\xi^2}}} e^{-\lambda^2 \kappa T \zeta} \, d\zeta$$

$$= \frac{1}{\lambda^2 \kappa T}\left[1 - e^{\frac{-\lambda^2 \kappa T f(\xi)}{\sqrt{1-\xi^2}}}\right]$$

which integral is simply replaced by $1/\lambda^2 \kappa T$.

If necessary, the method can be refined, yielding more complicated integrals which can be treated by numerical methods. The simplified expression for R is

$$R = \frac{\rho' c^2 B^2 L^2 \kappa^2}{4\pi} \int_{-1}^{+1} d\xi \int_{-1}^{+1} d\xi_1 \; f(\xi) \; f(\xi_1) \int_1^\infty \frac{\cos \gamma_o(\xi - \xi_1)\lambda}{\sqrt{\lambda^2 - 1}} \, d\lambda$$

or, using the formula

$$Y_o\left[\gamma_o(\xi - \xi_1)\right] = -\frac{2}{\pi} \int_1^\infty \frac{\cos \gamma_o(\xi - \xi_1)\lambda}{\sqrt{\lambda^2 - 1}} \, d\lambda,$$

$$R = -\frac{\rho' c^2 B^2 L^2 \kappa^2}{8} \int_{-1}^{+1} d\xi \int_{-1}^{+1} d\xi_1 \; f(\xi) \; f(\xi_1) \; Y_o\left[\gamma_o(\xi - \xi_1)\right],$$

where

$$\gamma_o = \frac{\kappa L}{2} = \frac{gL}{2c^2} = \frac{1}{2F^2}$$

depends on the Froude number. We introduce variables ϑ and ϑ_1 by $\cos\vartheta = \xi$, $\cos\vartheta_1 = \xi_1$ and have to minimize the integral

$$J = \int_0^\pi d\vartheta \int_0^\pi d\vartheta_1 \, f(\vartheta) \, f(\vartheta_1) \, \sin\vartheta \, \sin\vartheta_1 \, Y_o\left(\gamma_o |\cos\vartheta - \cos\vartheta_1|\right) = J(f,f)$$

with the condition

$$\int_0^\pi f(\vartheta^2) \, d\vartheta = C \, .$$

REDUCTION TO AN INTEGRAL EQUATION OF THE SECOND KIND

Suppose that $f(\vartheta)$ is the minimal function and consider a neighbouring function

$$f(\vartheta) + \varepsilon h(\vartheta)$$

where ε is a small parameter. Then

$$J(f+\varepsilon h) = J(f,f) + 2\varepsilon J(f,h) + \varepsilon^2 J(h,h)$$

and

$$V(f+\varepsilon h) = V(f,f) + 2\varepsilon V(f,h) + \varepsilon^2 V(h,h).$$

Since $J(f,f)$ must be a minimum for f, with constant V, we consider

$$J(f+\varepsilon h) - \lambda V(f+\varepsilon h)$$

where λ is a Lagrangian multiplier. The expression

$$J(f+\varepsilon h) - \lambda V(f+\varepsilon h) = J(f,f) - \lambda V(f,f) + 2\varepsilon [J(f,h) - \lambda V(f,h)] + \varepsilon^2 [J(h,h) - \lambda V(h,h)]$$

can only be an extremum if

$$J(f,h) - \lambda V(f,h)$$

vanishes for any choice of h. This means that f must be such that

$$\int_0^\pi d\vartheta \int_0^\pi d\vartheta_1 \; f(\vartheta)\; h(\vartheta_1)\; \sin\vartheta\; \sin\vartheta_1\; Y_o\Big[\gamma_o(\cos\vartheta - \cos\vartheta_1)\Big] - \lambda \int_0^\pi f(\vartheta)\; h(\vartheta)\; d\vartheta = 0$$

for all functions $h(\vartheta)$, or

$$\int_0^\pi h(\vartheta_1)\; d\vartheta_1 \left\{ \sin\vartheta_1 \int_0^\pi f(\vartheta) \sin\vartheta\; Y_o\Big[\gamma_o(\cos\vartheta - \cos\vartheta_1)\Big]\; d\vartheta - \lambda f(\vartheta_1) \right\} = 0.$$

This can only be true, if $f(\vartheta)$ satisfies the integral equation of the second kind:

$$\lambda f(\vartheta_1) = \int_0^\pi f(\vartheta)\; \sin\vartheta\; \sin\vartheta_1\; Y_o\Big[\gamma_o(\cos\vartheta - \cos\vartheta_1)\Big]\; d\vartheta .$$

Since the kernel is integrable, this is an ordinary Fredholm equation and the functions $f(\vartheta)$ are the eigenfunctions of the corresponding equation, while λ is the corresponding eigenfunction. Apparently the solutions vanish for $\vartheta = 0$ and $\vartheta = \pi$, as was required. Moreover, the eigenvalue λ is the value of the quotient $J(f,f)/V(f,f)$ for the eigenfunction. For the ship hull only the first eigenfunction, which has no zeros, will come into consideration.

SOLUTION OF THE INTEGRAL EQUATION

The integral equation

$$\lambda f(\vartheta_1) = \int_0^\pi f(\vartheta)\; \sin\vartheta\; \sin\vartheta_1\; Y_o\Big(\gamma_o\big|\cos\vartheta - \cos\vartheta_1\big|\Big)\; d\vartheta$$

resembles an integral equation satisfied by the Mathieu functions. Therefore, at first a solution in terms of a Fourier series in ϑ, with coefficients depending on γ_o was looked for. This method, however, is only feasible for very small values of γ_o. For this reason, a direct numerical approach was used.

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