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SOME CURRENT PROBLEMS IN TURBULENT SHEAR FLOWS

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INTRODUCTION

This paper is intended to be a heuristic account of several problems in the dynamics of turbulent shear flow. More to formulate questions than to answer them, it is in no sense a survey of the current research status in the field; the admirable new monograph of Townsend [1] will fill this role for some time to come. Unfortunately, it has not been possible to incorporate here the relevant contributions of his book, but I have attempted to make some references to it.

The turbulent transfer of a passive scalar contaminant in a mean gradient has become partly clarified through the use of Lagrangian dispersion analysis (in the ideal case, just by following indelibly tagged fluid) [2] instead of the Eulerian representation more convenient for turbulence dynamics. In approaching the problem of turbulent momentum transfer, it is therefore unavoidable that we ask the naive question of whether an idealized problem analogous to the scalar case can be sensibly formulated. Can we treat kinematically the turbulent dispersion of momentum from a "point source", at least in some kind of "small perturbation" limit?

An immediate difficulty is the conceptual one of visualizing a dispersing fluid (or solid) particle indelibly tagged with a mean momentum increment. In the very act of being convected by turbulent fluid the particle partakes of the general flow dynamics. On the other hand, if this action can be described by a linear differential equation, and if it does not affect the basic turbulence, there remains some hope for the utility of this approach. If the interaction is essentially non-linear, not only will the point-source dispersion be complex, but also the entire notion of synthesizing a mean gradient by superposition of sources will be invalid.

In a flowing isotropic turbulence [mean velocity $\overline{U}$, turbulent velocity $u_i \ll \overline{U}$], the possibility of an equation linear in v_i, the perturbation velocity of a particle, depends upon the negligibility of effects quadratic in v_i.

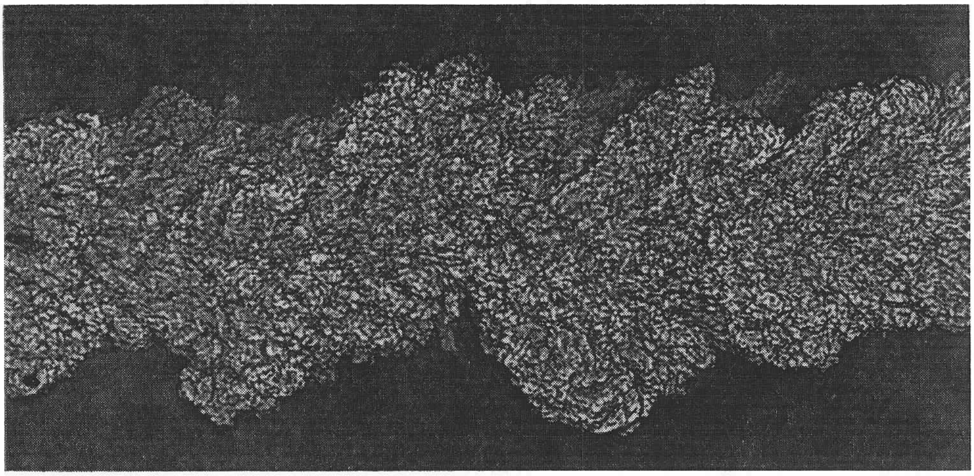


Figure 1. Turbulent wake of bullet. [courtesy of Ballistic Research Laboratory, Aberdeen Proving Grounds]

Inequalities like

$$v_j \frac{\partial v_i}{\partial x_j} \ll u_k \frac{\partial v_i}{\partial x_k} \tag{1}$$

can be gotten in principal by setting

$$v_k' \ll u_k', \tag{2}$$

but the static pressure contribution of v_i to the Navier-Stokes equation will probably behave like

$$p_v \sim \rho \bar{U} v_j \tag{3}$$

so that the inequality (2) may be insufficient to permit neglecting static pressure effects.* Nevertheless, if we can devise a convenient quantitative representation for static pressure perturbation, there is some hope of exploiting a point source method.

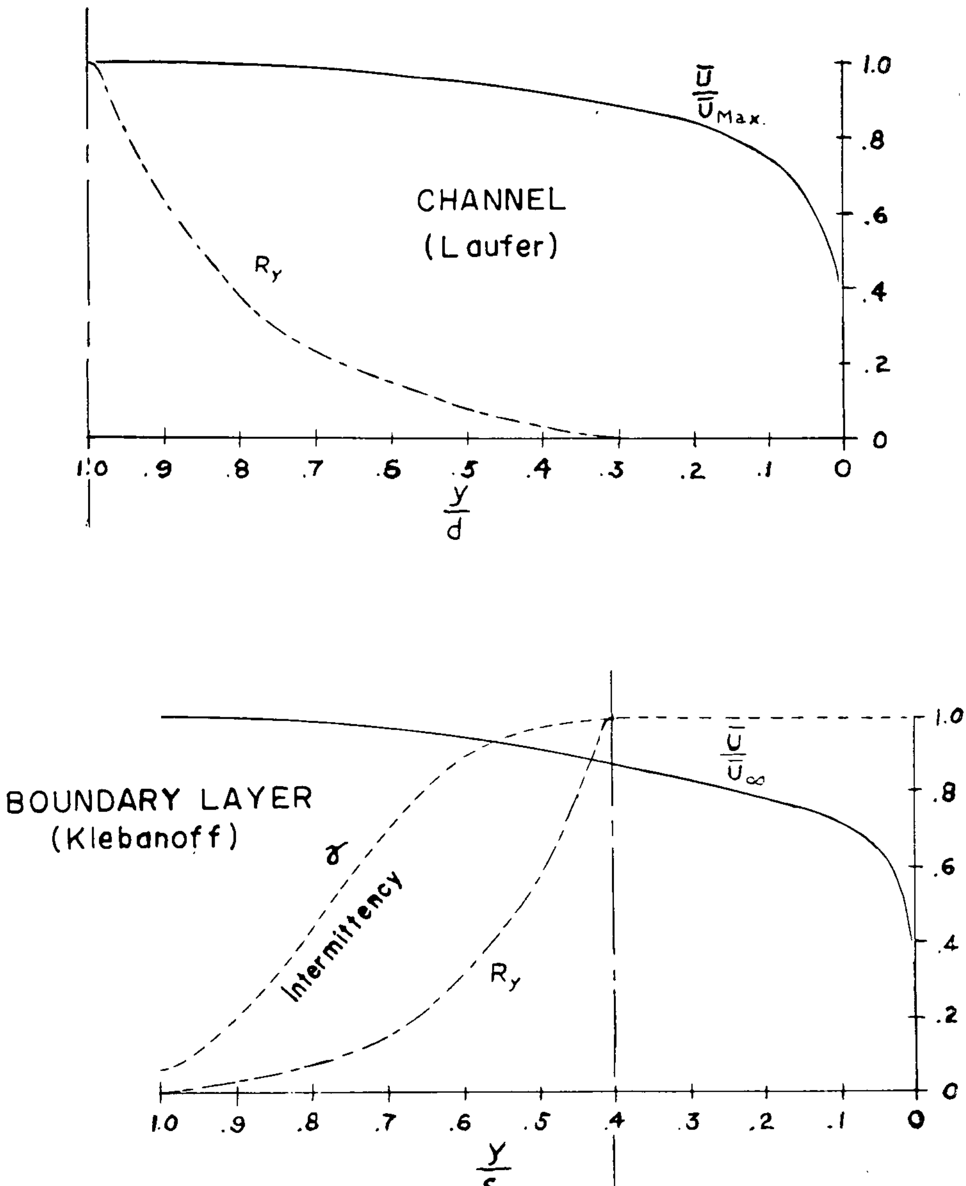
The foregoing matter is carried no farther here. This paper deals instead with some traditional shear flows such as boundary layer, channel, wake and jet—especially in terms of shear stress, strain-rate and their fluctuations.

The Proximity of Boundaries in Turbulent Shear Flows

In theoretical analysis of turbulence, restriction to homogeneity brings considerable simplification, and such an approach is being made to shear flow by Reis and Lin [3] and by Burgers and Mitchner [4]. Homogeneity, the invariance to translation of all statistical properties of the turbulence, requires that the boundaries be at infinity, or at least far apart compared with the maximum correlation distance of the fluctuations. To evaluate the possible role of a theory of homogeneous shear flow, it is instructive to recall some pertinent experimental results.

Figure 1 is a shadowgraph of the wake of a supersonic bullet, several hundred diameters downstream. In the ballistic range the camera is fixed with respect to the undisturbed air, and at this station all velocities are far below sonic. The wake is visible because of residual temperature fluctuations in the turbulence. Of interest here

* Townsend appears to disagree: [1], p. 57.

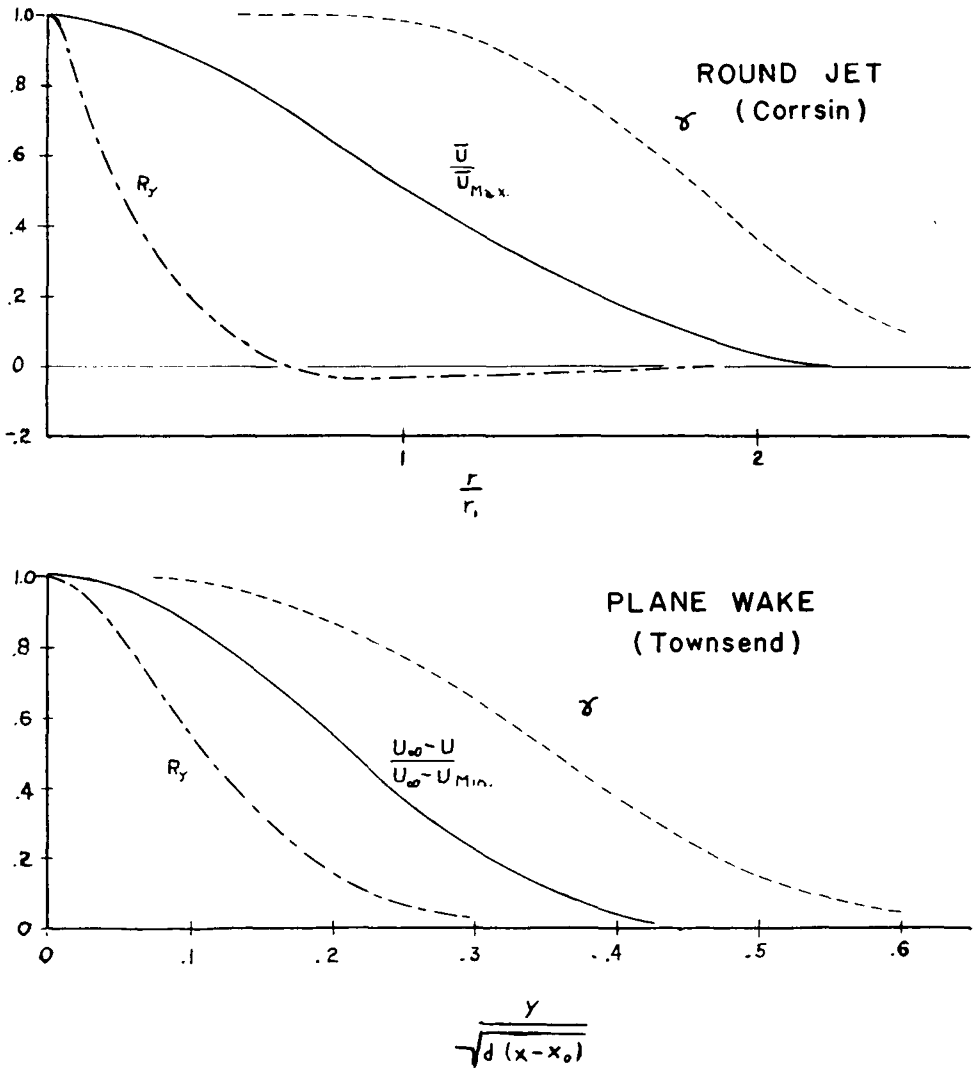


Extent of Turbulent Correlation Across Shear Flow

Fig. 2

is the fact that even though the shadowgraph technique responds to the second derivative of the index of refraction field, thus giving heavy emphasis on the fine structure, there is still detectable structure of a size comparable with the wake diameter.

For a quantitative display, we look at the comparison between mean velocity profiles and transverse correlation functions of turbulent velocities in representative shear flows. Figures 2 and 3 show that in channel, boundary layer, wake and jet [5, 6, 7, 1, 8, 9], there is measurable correlation over distances comparable with the shear



Extent of Turbulent Correlation Across Shear Flow

Fig. 3

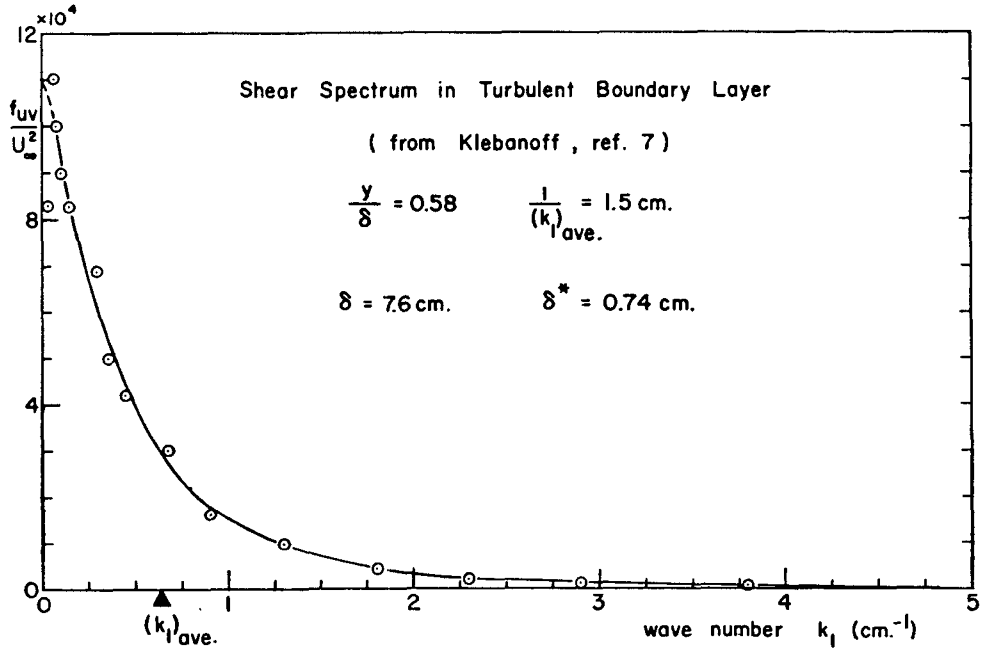


Fig. 4

zone width. Since the shear stress is carried principally by the large eddies, and these may be influenced by the boundaries, there exists a question of whether a turbulent shear theory postulating homogeneity can be expected to give quantitative results for momentum transfer.

Figure 4 demonstrates the momentum transfer burden carried by the large structure, and shows that the average wave number for transfer ${}_{uv}(k_1)_{\text{ave}}$, corresponds to a length greater than the displacement thickness of the boundary layer. The result would look even more emphatic in terms of wavelength $\dfrac{2\pi}{{}_{uv}(k_1)_{\text{ave}}}$.

It seems likely, however, that homogeneous shear flow theories could give semi-quantitative explanation of the momentum transfer, and should certainly be quantitatively applicable to the large wave number region, including that in which the shear correlation coefficient spectrum ${}_nR_{uv}$ is small but not zero.

Mean Stress and Strain-Rate

The early theories of turbulent shear flow focussed attention upon the mean momentum equations (the Reynolds equations), seeking to render them determinate by various postulates expressing the turbulent shear stress $(-\rho\overline{uv})$ in terms of the local mean strain rate $\left(\sim\dfrac{\partial \overline{U}}{\partial y}\right)$ or its derivatives. Reserving criticism of the local feature for later, it is instructive to look at the magnitudes *and directions* of the mean principal stresses and strain-rates in some turbulent flows.

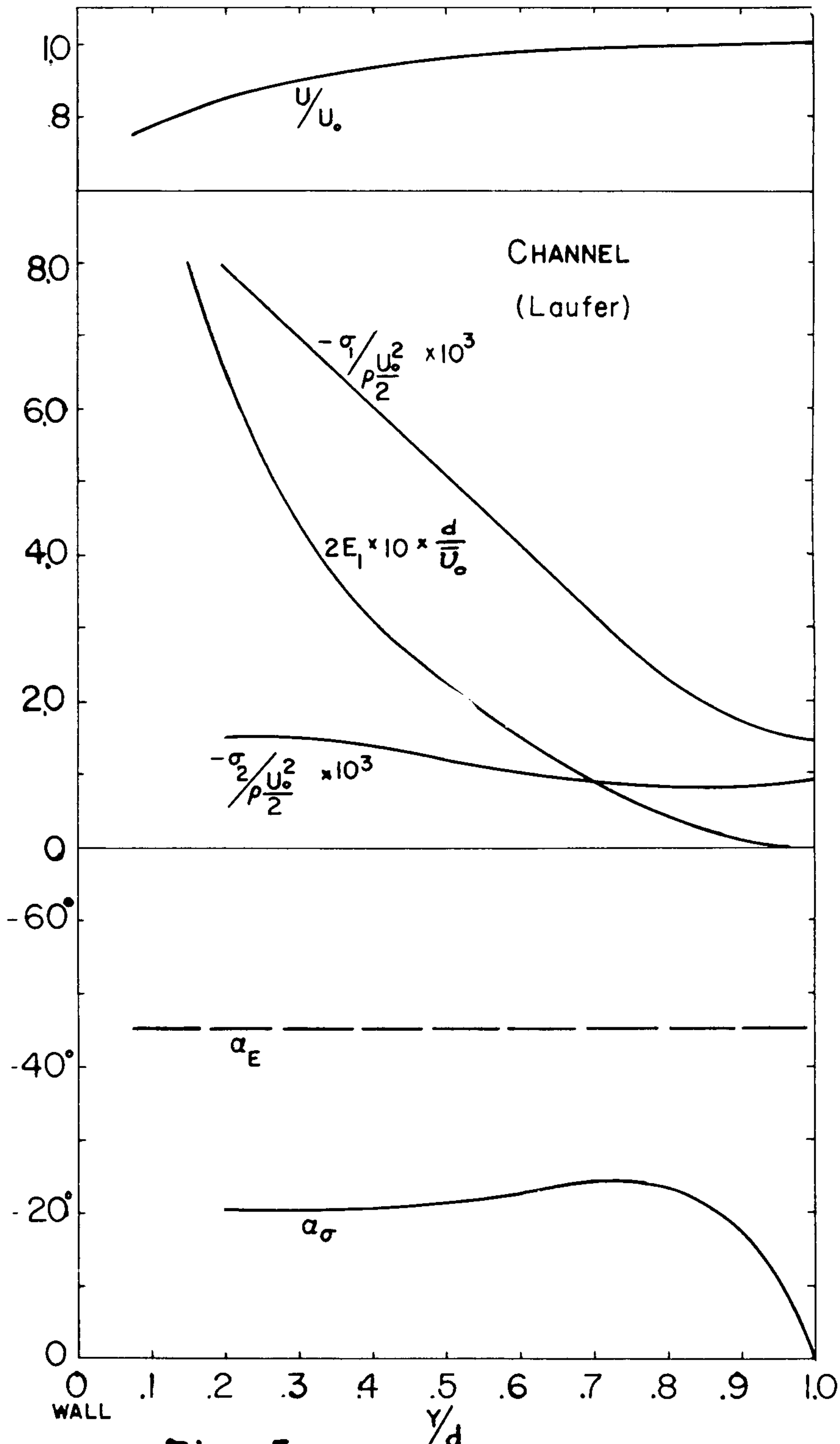

1.0
.8
U/U₀
CHANNEL
(Laufer)
8.0
6.0
4.0
2.0
0
-60°
-40°
-20°
0
−σ₁/ρ(U₀²/2) ×10³
2E₁ ×10 × d/Ū₀
−σ₂/ρ(U₀²/2) ×10³
αE
ασ
0
WALL
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
Y/d

Fig. 5

Figures 5, 6 and 7 give this information for plane channel, boundary layer and wake:

$$E_{1,2} \approx \pm \tfrac{1}{2} \frac{\partial U}{\partial y},$$

$$-\frac{1}{\rho}\sigma_{1,2} = \frac{\overline{u^2}+\overline{v^2}}{2} \pm \sqrt{\left(\frac{\overline{u^2}-\overline{v^2}}{2}\right)^2 + (\overline{uv})^2},$$

$$\alpha_{E_1} \approx \tfrac{1}{2}\tan^{-1}\left\{\frac{\tfrac{1}{2}\dfrac{\partial \overline{U}}{\partial y}}{\dfrac{\partial \overline{U}}{\partial x}}\right\} \approx 45^0, \tag{4}$$

$$\alpha_{\sigma_1} = \tfrac{1}{2}\tan^{-1}\left\{\frac{2\overline{uv}}{\overline{u^2}-\overline{v^2}}\right\}.$$

It seems noteworthy that the magnitudes of principal stress and strain-rate tend to vary together, loosely speaking. However, the directions of the corresponding principal axes are quite different in the two bounded flows, and this should recall the well known fact that assumptions of simple gradient transport of momentum in these flows have been rather unsuccessful.

In contrast, the wake shows a wide zone of rough coincidence of principal directions, a result consistent with the fact that its velocity profile is close to that obtained with the assumption of simple gradient transport with constant exchange coefficient. Some consequences of this directional coincidence are explored by Townsend ([1], pp. 117 et seq.).

Local Isotropy

Recent measurements on the anisotropy produced by irrotational, homogeneous strain-rate on the (roughly) isotropic turbulence downstream of grids [10, 11] (Figure 8) suggest re-examination of Kolmogoroff's postulate of local isotropy in shear flow [12]. The ubiquitous strain-rate of shear flow acts upon "eddies" of all sizes, tending to make the turbulence anisotropic at all wave numbers.

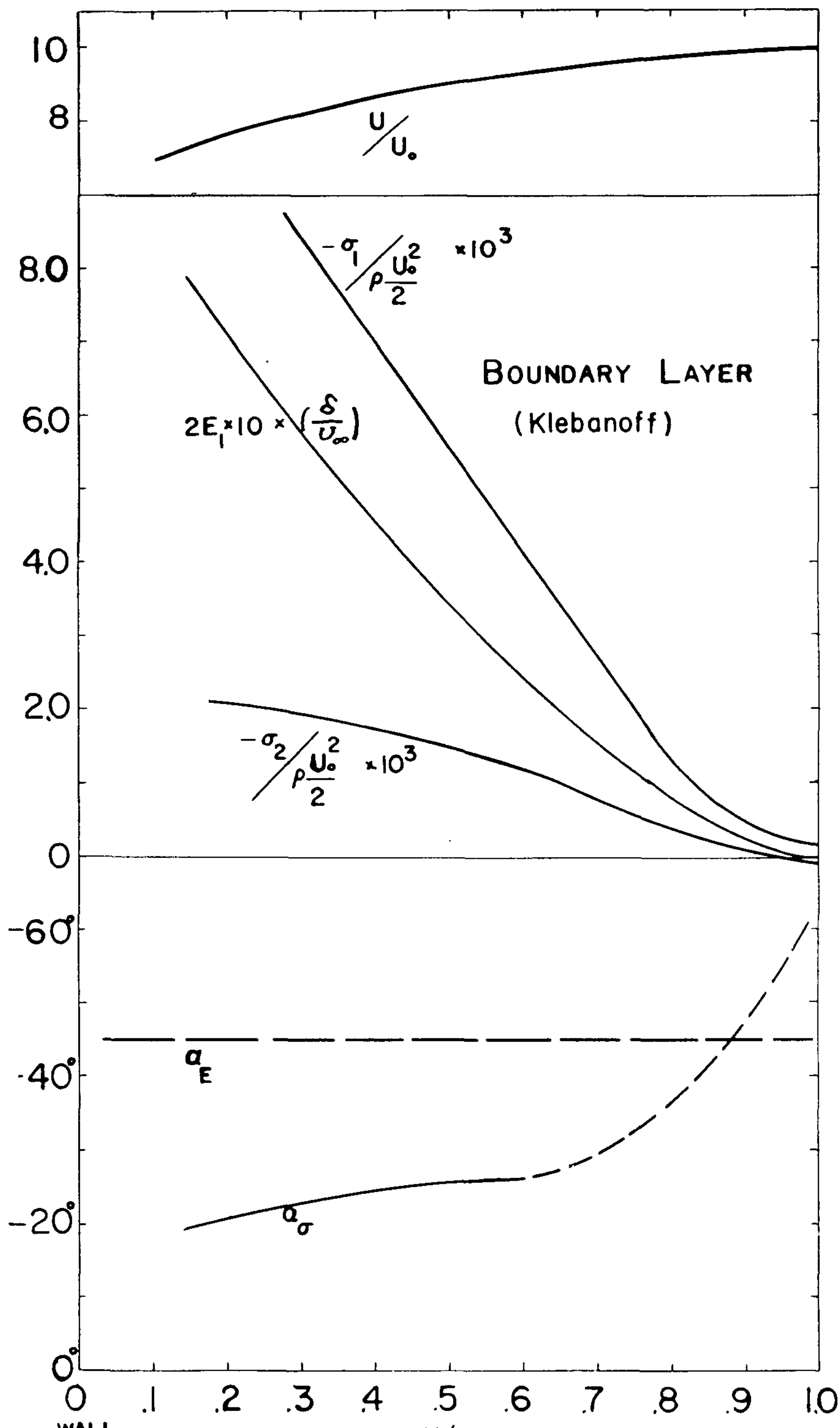
In shear flow the situation is complicated by the fact that energy is received by the turbulence from the mean flow in a highly anisotropic condition, as can be seen from component energy equations. In plane turbulent channel flow, for example,

$$\rho\overline{uv}\frac{d\overline{U}}{dy} + \frac{\rho}{2}\frac{d}{dy}(\overline{vu^2}) + \overline{u\frac{\partial p}{\partial x}} + (\text{viscous terms}) = 0, \tag{5}$$

$$\frac{\rho}{2}\frac{d}{dy}(\overline{v^3}) + \overline{v\frac{\partial p}{\partial y}} + (\text{viscous terms}) = 0, \tag{6}$$

$$\frac{\rho}{2}\frac{d}{dy}(\overline{vw^2}) + \overline{w\frac{\partial p}{\partial z}} + (\text{viscous terms}) = 0, \tag{7}$$

which shows that *all* turbulent energy appears first as $\overline{u^2}$. This is very nearly true for all flows in which the boundary layer approximation is valid.



10
8
U/U₀
8.0
6.0
4.0
2.0
0
-60°
-40°
-20°
0°
$-\sigma_1 / \rho \frac{U_0^2}{2} \times 10^3$
BOUNDARY LAYER
(Klebanoff)
$2E_1 \times 10 \times (\delta / U_\infty)$
$-\sigma_2 / \rho \frac{U_0^2}{2} \times 10^3$
α_E
α_σ
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
WALL
Y/δ

Fig. 6

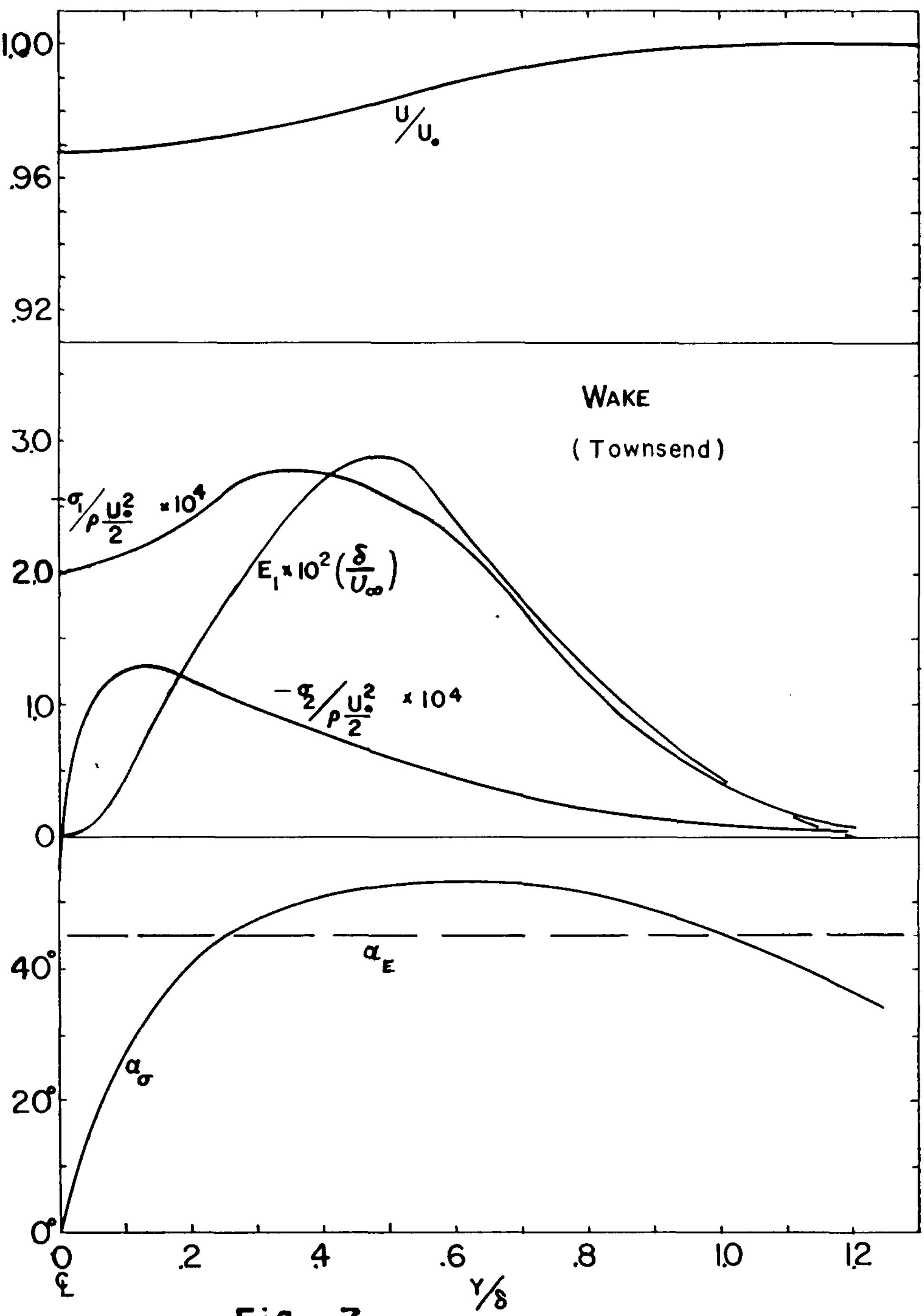

1.00
.96
.92
U/U₀
WAKE
(Townsend)
3.0
2.0
1.0
0
$\sigma_1 / \rho \frac{U_0^2}{2} \times 10^4$
$E_1 \times 10^2 \left(\frac{\delta}{U_\infty}\right)$
$-\sigma_2 / \rho \frac{U_0^2}{2} \times 10^4$
40°
20°
0°
α_E
α_σ
0
.2
.4
.6
.8
1.0
1.2
℄
Y/δ

Fig. 7

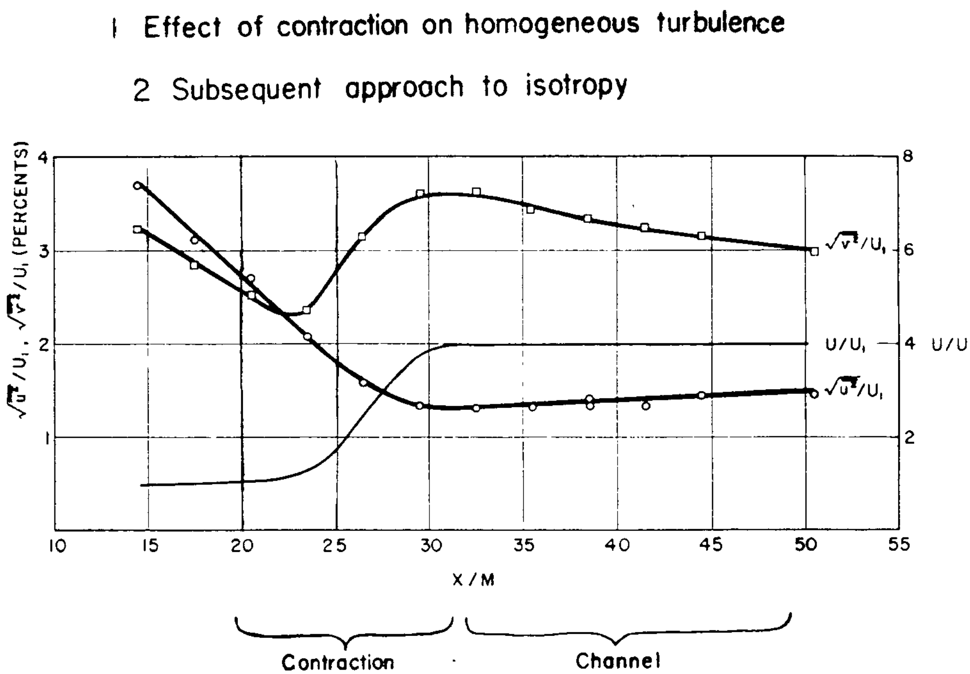


Figure 8. **Curves from Uberoi (11), by permission of the Journal of Aeronautical Sciences.**

There must also be a general inertial tendency toward isotropy through the pressure-velocity correlations at all wave numbers.

A necessary condition for local isotropy is that there exist a wave number beyond which the spectral times characterizing inertial and viscous mechanisms favoring isotropy [$\vartheta_i(k)$ and $\vartheta_v(k)$] are much smaller than the time characterizing the gross straining process, $\frac{1}{E_1}$:

$$\vartheta_i(k) \quad \text{and} \quad \vartheta_v(k) \ll \frac{2}{\left|\frac{\partial U}{\partial y}\right|}. \tag{8}$$

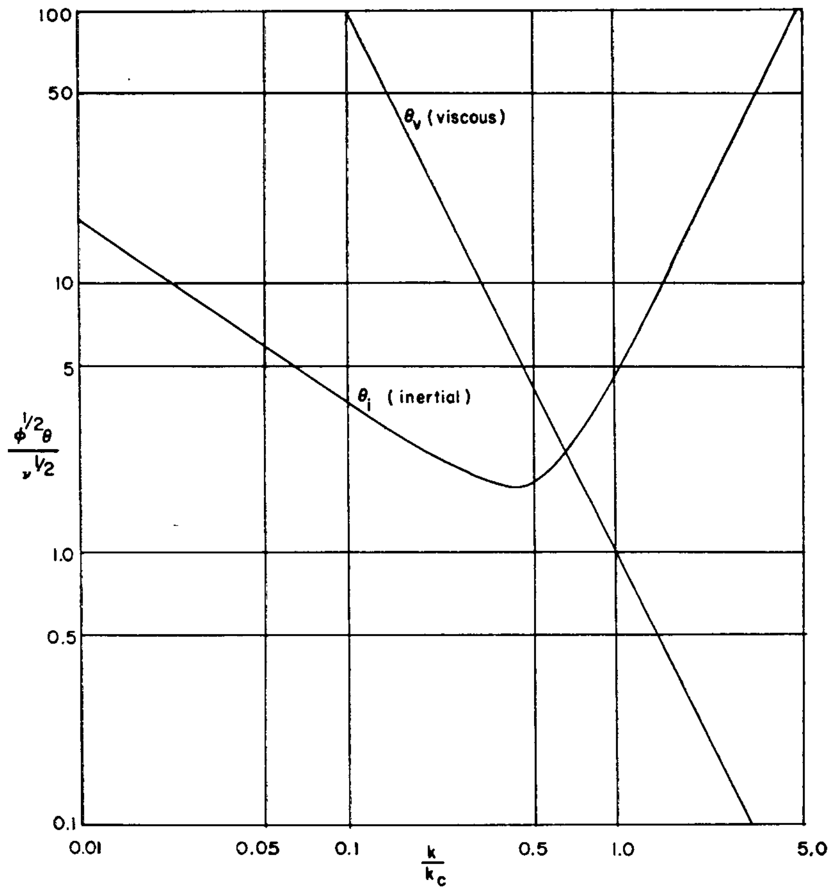
Of course, in a purely inertial range only the ϑ_i inequality is necessary, while in the purely viscous range (if a thing exists in stationary turbulence), only the ϑ_v inequality is necessary.

With $E(k)$ as turbulent energy spectrum, we may follow Onsager's definition [13] of inertial time,

$$\vartheta_i(k) \equiv \frac{1}{\sqrt{k^3\epsilon(k)}}. \tag{9}$$

Similar reasoning leads to

$$\vartheta_v(k) \equiv \frac{1}{\nu k^2}. \tag{10}$$



Characteristic Spectral Times
(Assuming a "Heisenberg" Spectrum)

Figure 9

As a basis for rough numerical estimates, it is convenient to replace $E(k)$ by the solution to Heisenberg's transfer equation [13]. Then, in terms of the Kolmogoroff wave number, which locates the viscous region of the spectrum,

$$k_c \equiv \left(\frac{\phi}{\nu^3}\right)^{\frac{1}{4}}, \tag{11}$$

(where ϕ is viscous dissipation rate and ν is kinematic viscosity), and

$$\vartheta_i\left(\frac{k}{k_c}\right) \approx \left(\frac{\nu}{\phi}\right)^{\frac{1}{2}}\left[1 + \frac{27}{2}\left(\frac{k}{k_c}\right)^4\right]^{\frac{2}{3}}\left(\frac{k}{k_c}\right)^{-\frac{2}{3}}. \tag{12}$$

In this estimate, Heisenberg's empirical constant has been chosen as $\frac{4}{9}$.

Both (10) and

$$\vartheta_v\left(\frac{k}{k_c}\right) = \left(\frac{\nu}{\phi}\right)^{\frac{1}{2}}\left(\frac{k}{k_c}\right)^{-2} \tag{13}$$

are plotted in Figure 9 [14].

The minimum value of ϑ_i turns out to be

$$\vartheta_i(0.40) = 1.8\left(\frac{\nu}{\phi}\right)^{\frac{1}{2}}, \tag{14}$$

so that one necessary condition for local isotropy is

$$\left(\frac{\phi}{\nu}\right)^{\frac{1}{2}} \gg \frac{\partial \overline{U}}{\partial y}. \tag{15}$$

Under still more simplifying assumptions (14), this can be translated into a Reynolds number inequality.

Both inertial and viscous forces may be appreciable in the range $0.2 < \dfrac{k}{k_c} < 5$. Hence, rather severe conditions for local isotropy may exist since

$$\vartheta_v(0.2) \approx 30\left(\frac{\nu}{\phi}\right)^{\frac{1}{2}}, \quad \text{and} \quad \vartheta_i(5.0) \approx 100\left(\frac{\nu}{\phi}\right)^{\frac{1}{2}}. \tag{16}$$

Turning to actual numbers, the homogeneous strain rate of Figure 8 has a maximum of 11 per second, and Uberoi's highest [11] (a 16-1 contraction) reached 63 per second. Townsend's homogeneous plane strain-rate was 9.4 per second. These numbers may be compared with principal strain rates varying from zero to 10^4 per second in Laufer's plane channel [5] at a Reynolds number of 61,000. A general order of magnitude in this example is $\dfrac{1}{2}\dfrac{\overline{U}_{\max}}{d} \approx 120$ per sec. Klebanoff's boundary layer [7] has strain-rates of the same order, while Townsend's wake strain-rates are appreciably smaller [15], e.g. roughly 2.0 per second at 800 diameters behind a 0.159-inch cylinder at Reynolds number 1360. Except near solid boundaries, these shear flow strain-rates are comparable with the homogeneous ones imposed by Townsend and Uberoi, so it is possible that the data of the latter may be employed semi-quantitatively to explain some features of shear turbulence. Townsend [1] has apparently done considerable work along this line.

In a discussion of local isotropy in shear flow, it is pertinent to note the strong departures from local isotropy in the homogeneous strain experiments, at least out as far as the spectral region contributing to mean square first derivatives. In Figure 10, for example, we see Uberoi's evidence at a strain-rate of about 30 per second. From the decay rate just before the contraction, $\left(\dfrac{\phi}{\nu}\right)^{1/2}$ may be estimated as 16 per second, evidently not an order of magnitude larger than 30. This is consistent with the lack of local isotropy.

Taking Klebanoff's boundary layer data [7] as typical and considering a point $\dfrac{y}{\delta} = 0.2$, in the fully turbulent region, we may check inequality (15) by estimating

$$\frac{\left(\dfrac{\phi}{\nu}\right)^{\frac{1}{2}}}{\dfrac{\partial \overline{U}}{\partial y}} \approx 10. \tag{17}$$

This implies that local isotropy is more likely here than in the homogeneous cases reported.

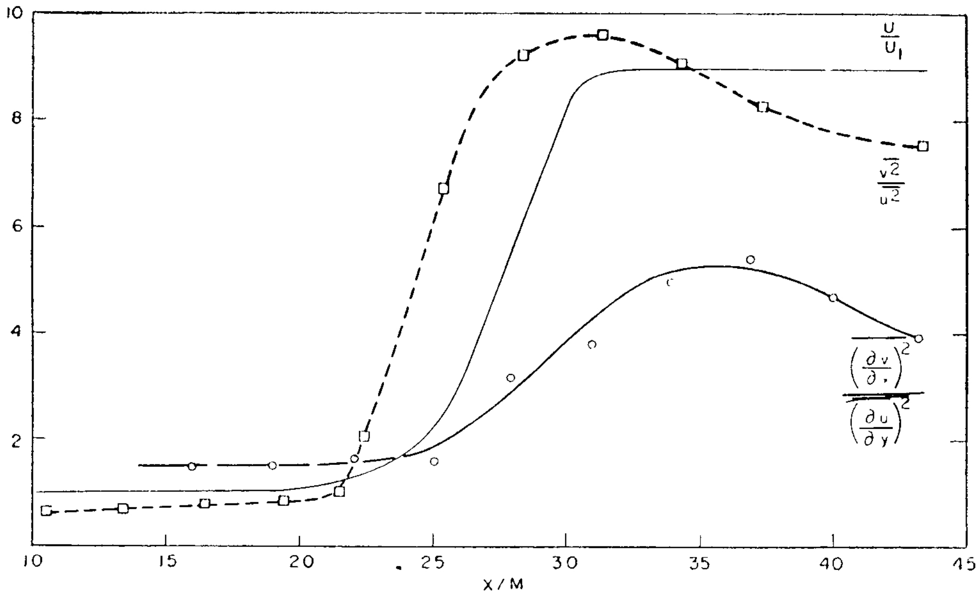


Figure 10. **From Uberoi (11), by permission of the Journal of Aeronautical Sciences.**

Superficially distinct from the characteristic time criteria is the obvious requirement that the locally isotropic spectral region be at wave numbers much larger than those at which energy is fed into the turbulence. With a mixing-length type of behavior qualitatively in mind, we may estimate the latter as

$$k_p \approx \frac{1}{v'} \frac{\partial \bar{U}}{\partial y}, \tag{18}$$

so that we cannot expect local isotropy at wave numbers which do not satisfy the inequality

$$k \gg \frac{1}{v'} \frac{\partial \bar{U}}{\partial y}. \tag{19}$$

Hence, for dissipative local isotropy, a necessary condition is that

$$k_c = \left(\frac{\phi}{\nu^3}\right)^{\frac{1}{4}} \gg \frac{1}{v'} \frac{\partial \bar{U}}{\partial y}. \tag{20}$$

At the same boundary layer position,

$$\frac{\left(\dfrac{\phi}{\nu^3}\right)^{\frac{1}{4}} \cdot v'}{\dfrac{\partial \bar{U}}{\partial y}} \approx 45, \tag{21}$$

which is enough greater than unity to be consistent with locally isotropic dissipation.

Klebanoff reports roughly isotropic dissipation for $.05 < \frac{y}{\delta} < 1.0$, which includes this position.

Shear Spectrum

Since elementary dimensional reasoning has had a bit of success in predicting the character of local regions in the energy spectrum, it seems natural to attempt the simplest kind of dimensional argument on the shear spectrum, $\tau_1(k_1)$ defined by

$$-\overline{uv} = \int_0^\infty \tau_1(k_1)\, dk_1. \tag{22}$$

We are interested in predicting from gross turbulence data the wave number above which the coefficient of shear correlation ${}_kR_{uv}$ is much less than unity. If $F_1(k_1)$ and $F_2(k_1)$ are the one-dimensional spectra of $\overline{u^2}$ and $\overline{v^2}$,

$${}_{k_1}R_{uv}(k_1) \equiv \frac{\tau_1}{\sqrt{F_1 \cdot F_2}}, \tag{23}$$

and this has been measured in several flows [5, 7, 16, 17].

For large Reynolds number we naively assume that at the "small eddy" end of the shear spectrum, $\tau_1(k_1)$ depends primarily upon the local mean velocity gradient $\frac{\partial \overline{U}}{\partial y}$. Then, dimensionally,

$$\tau_1(k_1) \sim \left(\frac{\partial \overline{U}}{\partial y}\right)^2 k_1^{-3}, \tag{24}$$

SPECTRUM OF SHEAR CORRELATION COEFFICIENT

[Straight line slope = −4/3]

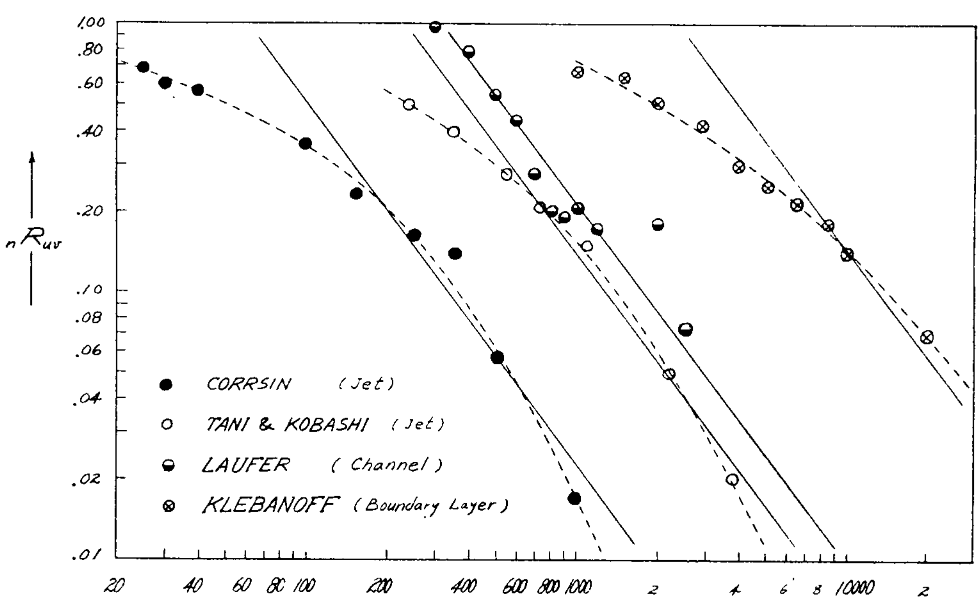


Fig. 11

which could be compared with experiment. However, the actual comparison here has been made with ${}_{k_1}R_{uv}$ under the additional assumption that $F_1 \sim F_2 \sim k_1^{-5/3}$, since the high wave number end borders on the region of local isotropy. Then

$$ {}_{k_1}R_{uv} \sim k_1^{-\frac{4}{3}} \tag{25} $$

which does not look particularly successful in Figure 11. The abscissa there is frequency rather than wave number.

By a considerably more sophisticated approach, Tchen [18] arrived essentially at

$$ \tau_1 \sim k_1^{-\frac{7}{3}} \tag{26} $$

for "small $\dfrac{\partial \bar{U}}{\partial y}$".

The Gradient Transport Postulate

It is a familiar fact that the validity of any simple gradient transport description requires the transfer mechanism to have a characteristic length much smaller than the gross dimensions of the field. With a random "free path" type of mechanism, for example, the transfer rate Q of some property $\psi(x)$ may often be expanded in a power series in the "mean free path" l:

$$ Q(x_1) \sim \frac{l}{2}\left(\frac{d\psi}{dx}\right)_{x_1} + \frac{\left(\frac{l}{2}\right)^3}{3!}\left(\frac{d^3\psi}{dx^3}\right)_{x_1} + \text{-----}\,. \tag{27*} $$

Provided ψ' and ψ''' are properly behaved, (27) reduces to simple gradient transport when

$$ l \ll \sqrt{6\frac{\psi'}{\psi'''}}, \tag{28} $$

a characteristic length of the gross field.

In Figures 1, 2, 3, 4 we were reminded of the well-known fact that shear turbulence has structure comparable with the shear zone width. Furthermore, it turns out that if we apply (27) to heat diffusion measurements in isotropic turbulence, where the Lagrangian integral scale serves in place of a free path, the second term is about 30% of the first.

Of course, (27) indicates only constant coefficients, and it seems reasonable to see whether

$$ Q(x) \approx A\psi'(x) + B\psi'''(x) \tag{29} $$

can be represented as gradient transport with coefficient dependent on x**, i.e. we define $D(x)$ such that

$$ Q(x) = D(x)\psi'(x) = A\psi'(x) + B\psi'''(x). \tag{30} $$

---

* The even powers vanish if we choose x_1 at the center of the mean free path, and consider particles traveling only between $(x_1 - \frac{l}{2})$ and $(x_1 + \frac{l}{2})$.

**This mathematical trick is not intended to include inhomogeneous flows, in which the transport mechanism actually does vary from place to place.

Then

$$D(x) = A + B\frac{\psi'''(x)}{\psi'(x)}, \tag{31}$$

which displays an unfortunate singularity wherever the gross field happens to have an extremum.

The surprising fact is that some turbulent shear flows, notably the so-called "free shear flows" without solid boundaries (wakes and jets), seem to have mean velocity distributions rather close to those given by assuming that turbulent shear stress is Newtonian in character. Clearly this cannot be correct in principle.

Although boundary layers and channels cannot be approximated even roughly as quasi-Newtonian, it turns out that two-fluid models having a wall layer $0 \leqslant \frac{yU_*}{\nu} \leqslant 12$ with the true viscosity and an outer fluid with a "turbulent viscosity" can be instructive. I have used such a pedagogical device for channel flow; it was independently introduced into turbulent boundary layer analysis by F. H. Clauser [20].

Here y is distance from the wall, $U_* \equiv \sqrt{\frac{\overline{T}_0}{\rho}}$, the "fraction velocity", $\overline{T}_0 \equiv \mu(\frac{\partial \overline{U}}{\partial y})_{y=0}$, the mean skin friction stress.

The superficial success of a "turbulent viscosity" in describing the mean velocity pattern of jets and wakes suggests evaluation of the corresponding "effective Reynolds number" $R_e \equiv \frac{U_1 L}{\nu_T}$, in order to see how it varies with the true Reynolds number, $R \equiv \frac{U_1 L}{\nu}$. U_1 and L are characteristic velocity and shear zone width.

*For jets and wakes, it turns out that R_e is independent of R.** For the round jet entering fluid at rest,

$$R_e = \frac{\overline{U}_m \cdot \Delta}{\nu_T} \approx 15, \tag{32}$$

where $\overline{U}_m$ is the maximum velocity at a cross section and Δ is the "momentum diameter," defined by

$$\frac{\pi\Delta^2}{4}\overline{U}_m{}^2 \equiv 2\pi \int_0^\infty U^2(r)\, r\, dr, \tag{33}$$

for isopycnic flow. For the plane wake,

$$R_e \equiv \frac{(\overline{U}_\infty - \overline{U}_{\min})h}{\nu_T} \approx 12, \tag{34}$$

where $\overline{U}_\infty$ is free stream velocity, $\overline{U}_{\min}$ is the velocity on the axis at a section and h is the lateral coordinate at which $(\overline{U}_\infty - \overline{U}) = \frac{1}{2}(\overline{U}_\infty - \overline{U}_{\min})$.

It is noteworthy that these values of R_e are both of the same order as the "lower critical Reynolds numbers" for the stability of laminar free shear flows.

When a two fluid model is used for turbulent flow through a round tube, for example, the "effective Reynolds number" defined with the ν_T of the turbulent core

* This observation has also been made by Townsend [1].

flow, the maximum velocity $\overline{U}_m$ and the tube radius a, turns out to vary with the actual Reynolds number:

$$R_e \equiv \frac{\overline{U}_m \cdot a}{\nu_T} \approx \frac{4}{c_f} - \frac{24\sqrt{2}}{\sqrt{c_f}}, \tag{35}$$

where $c_f \equiv \dfrac{\overline{T}_0}{\frac{1}{2}\rho \overline{U}_m{}^2}$ is a slowly varying function of $R = \dfrac{\overline{U}_m a}{\nu}$. (36)

As numerical example, we may take Laufer's two cases:

$$\begin{aligned} R_e &\approx 525 \quad \text{for} \quad R = 25{,}000, \\ R_e &\approx 900 \quad \text{for} \quad R = 250{,}000. \end{aligned} \tag{37}$$

It is possible that there exists a more suitable definition of R_e for tube flow, i.e. a definition which does not display this slow variation with R. However, in contrasting the actual Reynolds number dependence of turbulent jet and tube flows we recall that the mean velocity profile of the former is virtually invariant, while that of the latter is appreciably sensitive to Reynolds number.

In terms of the turbulence parameters, we may regard the "turbulent diffusivity" as behaving like

$$\nu_T \sim v'\mathscr{L}, \tag{38}$$

where v' is the r.m.s. velocity fluctuation in the transport direction, and $\mathscr{L}$ is a corresponding Lagrangian integral length scale. It is found experimentally that $\dfrac{v'}{\overline{U}_m}$ is relatively insensitive to R in jets, but decreases with R in channels and tubes. The R dependence of $\mathscr{L}$ is not established, but as a property of the large eddies it is likely to be more or less proportional to shear zone width. The foregoing assumptions give $\nu_T \sim \overline{U}_m L$ for jets [$\therefore R_e = Const$] and ν_T slowly decreasing with increasing R for tubes.

The low empirical values of R_e for jet and wake, and the superficial agreement of the corresponding constant exchange coefficient postulate, suggest that an engineering approximation to such flows may be gotten via a quasi-Oseen approximation to the Reynolds equations, incorporating a constant ν_T as well.*

Shear Stress Fluctuations

For some applications it may be interesting to know the shear stress fluctuation level in a turbulent flow. In a fully turbulent region the mean (Reynolds) turbulent stress $-\rho\overline{u_i u_k}$ dominates the viscous momentum transport, and we may begin by a look at the fluctuation in turbulent stress. For simplicity, we restrict the discussion to a boundary layer type flow, in which only one kind of mean shear component, $\overline{T}_T \equiv -\rho\overline{uv}$, is important.

Since the so-called turbulent shear stress is actually the average *y-convection* of *x-momentum* attributable to the turbulence, it seems plausible to identify

$$\tau_T \equiv -\rho[\overline{U}v + \overline{V}u + (uv - \overline{uv})] \tag{39}$$

as the instantaneous fluctuation in turbulent shear stress. Then, for $\overline{V} \ll \overline{U}$ and turbulence levels not too high,

$$\frac{\tau'_T}{\overline{T}_T} \approx \frac{\overline{U}v'}{\overline{uv}}, \tag{40}$$

a number considerably greater than unity. Figure 12 shows how τ'_T varies across the turbulent boundary layer. $\overline{T}_0$ is the mean shear stress at the wall. It should be

* This has been applied to the wake by I. Imai (lecture at Fluid Mechanics Colloq., The Johns Hopkins University, November, 1955).

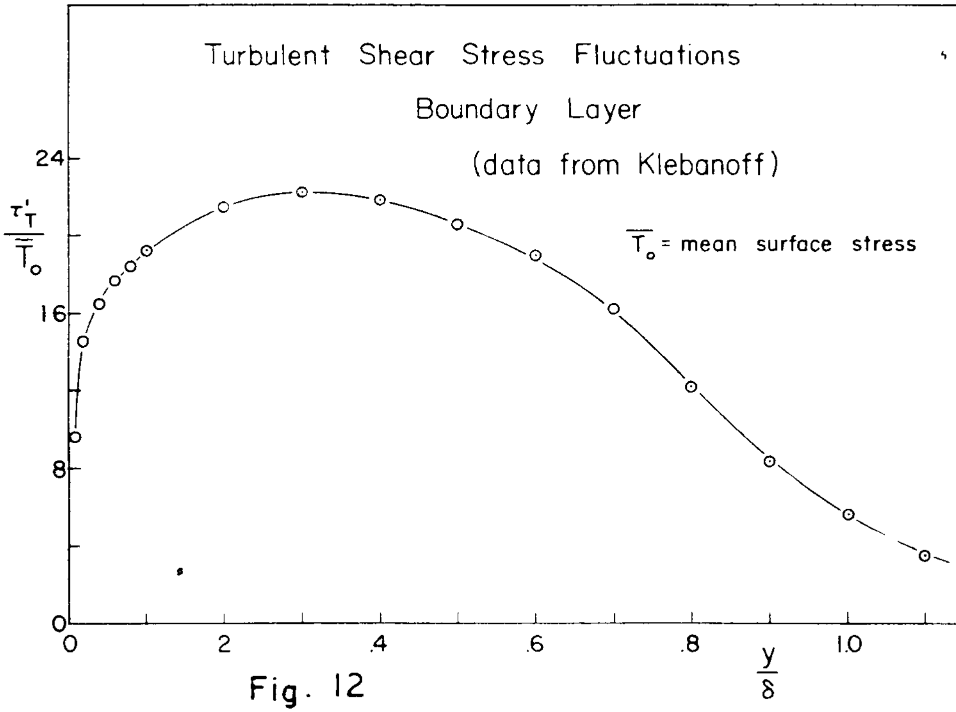


Fig. 12

noted that in the fully turbulent zone, $\overline{T}_T$ decreases monotonically with increasing y/δ, so that $\frac{\tau'_T}{\overline{T}_T}$ goes to even larger values than $\frac{\tau'_T}{\overline{T}_0}$.

If (39) is indeed a proper identification, then the large values of $\frac{\tau'_T}{\overline{T}_T}$ describes a process in which instantaneous transfer against the mean gradient is only a little less frequent than instantaneous transfer down the mean gradient.

Parenthetically we may look at the fluctuations in the molecular transport of laminar flow. In the gas kinetic formulation of momentum transport at low Mach numbers, the expression analogous to (39) is dominated by the $(uv - \overline{uv})$ term since molecular velocities are much higher than "drift" velocities. Hence the gas kinetic quantity corresponding to our $\frac{\tau'_T}{\overline{T}_T}$ is

$$\frac{\tau'}{\overline{T}} = \sqrt{\frac{\overline{u^2v^2}}{\overline{uv}^2} - 1}, \tag{41}$$

where u and v are, for this discussion only, molecular velocity components. If we assume u and v to be jointly normal,

$$\overline{u^2v^2} = \overline{u^2}\,\overline{v^2} + 2\overline{uv}^2, \text{ and}$$

$$\frac{\tau'}{\overline{T}} = \sqrt{\frac{\overline{u^2}\,\overline{v^2}}{\overline{uv}^2} + 1} \approx \frac{u'v'}{\overline{uv}}, \tag{42}$$

which is much larger than unity if the inverse of the mean strain rate is much larger than the mean free time of the molecules.

Toward a solid boundary, the mean and turbulent velocities approach zero, so τ'_T decreases rapidly with decreasing y/δ. Figure 13 displays this behavior in terms of dimensionless length coordinate

$$y_* \equiv \frac{yU_*}{\nu}, \tag{43}$$

where

$$U_* \equiv \sqrt{\frac{\overline{T}_0}{\rho}}.$$

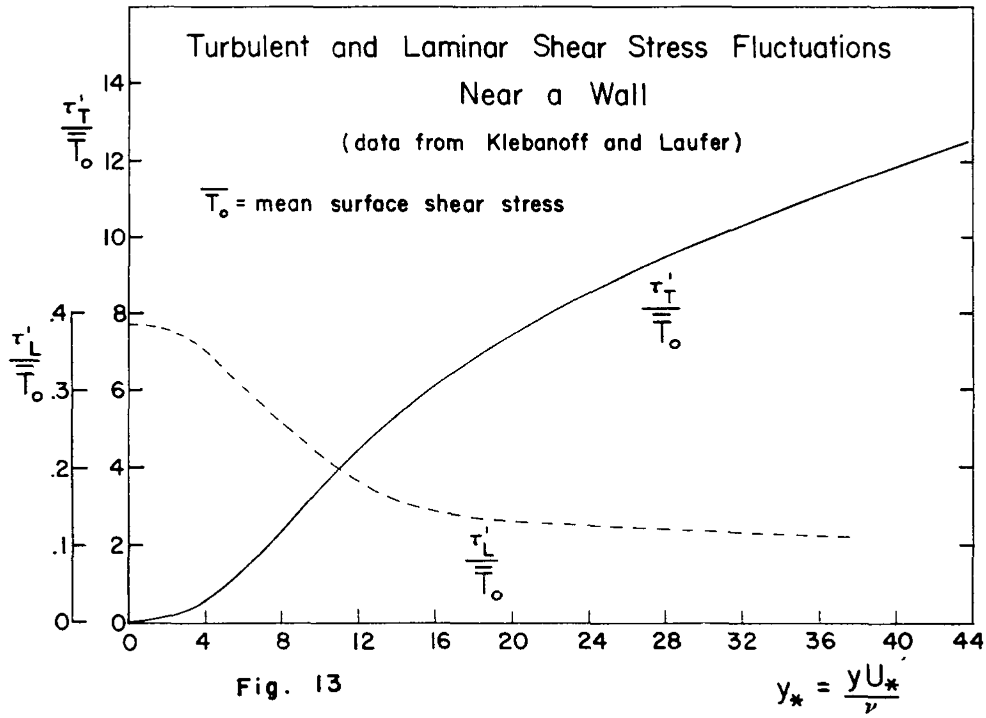


Fig. 13

On this same figure is a very rough estimate of the laminar shear stress fluctuation level,

$$\frac{\tau'_L}{\overline{T}_o} = \frac{\mu \sqrt{\overline{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2}}}{\mu \left(\frac{\partial \overline{U}}{\partial y}\right)_{y=0}}. \tag{44}$$

This dashed curve was sketched by its estimated value at $y = 0$, the skin friction fluctuation, and the measured mean square derivatives at large values of y_*. The indicated monotonic decrease of τ'_L with increasing y is an assumption based upon its apparent flatness at the wall: both u' and $\overline{U}$ are linear in y as $y \rightarrow 0$, and $\frac{\partial \overline{U}}{\partial y} \approx \frac{U}{y}$ in this vicinity, so $\tau_0 \approx \tau_{L0} \approx \mu \frac{u}{y}$ *and* $\tau'_0 \approx \mu \left(\frac{u'}{y}\right)_{y\to 0}$. It is, of course, conceivable that τ'_L could have a maximum away from the wall.

We note from figure 13 that $\tau'_L < \tau'_T$ except for $y_* < 3$. This is a bit different than the case of many other physical properties in this region; it is found that laminar and turbulent contributions to various transport and dissipation phenomena are equal at $y_* \approx 12$, a common choice of boundary between a *"wall layer"* and the *"outer flow."* It seems preferable to restrict the term *"laminar sublayer"* as designation for the zone in which turbulent transport is actually negligible.

WALL LAYER AND LAMINAR SUBLAYER

The physical picture of a wall neighborhood in which viscous effects are not negligible has remained virtually unchanged since its description by Taylor in 1916 [22]. He described it essentially as a Couette flow violently disturbed by the turbulence in the main body of the shear zone, suggesting that it should therefore have a Reynolds number $R_\varepsilon \equiv \dfrac{\bar{U}(\varepsilon)\varepsilon}{\nu}$ roughly equal to the theoretical lower critical value for stability of such a Couette flow, 150 to 200.

This assumption permits engineering estimates of turbulent boundary layer and channel flow (via the Kármán integral relation, for example). As is well known, it has since turned out to be quantitatively correct if we identify Taylor's layer as extending from the solid wall out to where laminar and turbulent shear are exactly equal. As mentioned in the previous section, this occurs at $y^* \approx 12$ [23]. This is the *"wall layer."*

Because of the no-slip condition, there must be a non-inertial layer at the wall, presumably characterized by a length made up of the characteristic ("friction") velocity $U_* \equiv \sqrt{\dfrac{\bar{T}_0}{\rho}}$ and the kinematic viscosity:

$$l_* = \frac{\nu}{U_*}. \tag{45}$$

In this *"laminar sublayer"* [$y_* < 1$], the direct turbulent transport is negligible.

Recent hot-wire studies, like [7] and [23], have given information on r.m.s. velocity fluctuation levels well inside the wall layer, and Klebanoff [7] has presented a u' spectrum at $y_* = 3$, virtually in the laminar sublayer. Still it seems likely that more information will be necessary before a suitable theoretical picture can be developed.

By passing to the $y = 0$ limits of the equations of motion and their derivatives, it is possible to deduce some restricted relations. For example, it is well known that the Reynolds equations reduce to Stokes flow, with only one appreciable component:

$$\left(\frac{\partial^2 \bar{U}}{\partial y^2}\right)_0 = \frac{1}{\mu}\left(\frac{\partial \bar{P}}{\partial x}\right)_0. \tag{46}$$

The fluctuation momentum equations reduce to a simplified two-dimensional Stokes flow:

$$\frac{\partial^2 u}{\partial y^2} \cdot = \frac{1}{\mu}\frac{\partial p}{\partial x}, \tag{47}$$

$$\frac{\partial^2 w}{\partial y^2} \cdot = \frac{1}{\mu}\frac{\partial p}{\partial z}, \tag{48}$$

since continuity requires that $\overline{V}(y)$ and $v(y)$ start out parabolically from $y = 0$. Further, since

$$u(y) \sim y + - - - - -, \tag{49}$$

$$w(y) \sim y + - - - - -, \tag{50}$$

$$v(y) \sim y^2 + - - - - -, \tag{51}$$

we see that the turbulent shear stress can rise no faster than a cubic:

$$- \overline{uv} \sim y^3 + - - - - -. \tag{52}$$

In fact, there is some evidence that $\left| \dfrac{\overline{uv}}{u'v'} \right|$ sometimes decreases toward the wall [24], in which case $- \overline{uv}$ may build up even more slowly.

Some of these conditions have been exploited, for example, in a report by Phillips [25].

A different but qualitative look at the wall region can be gotten by dye injection methods. The "leaching" of colored water from a glass tube by flowing clear water through it in a turbulent state leaves dye only near the wall after a short time. In figure 14 is such a situation as photographed by Beatty, Ferrell and Richardson.* The Reynolds number is about 7000. The significant property seems to be the strong orientation into stream-wise filaments of the residual dye. Presumably this indicates a predominance of axial vorticity near the wall, "sweeping" the (dyed) wall fluid into these long narrow strips. Of course the very high mean velocity gradient in this zone is expected to rotate and strain all fluid lines axially, but the parabolic decrease of v'

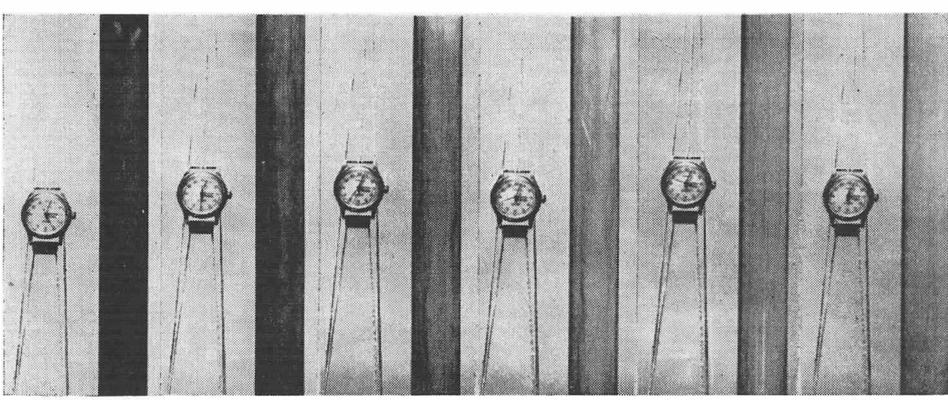


Figure 14. **Flushing of dye by a turbulent pip flow, R 7000. [Courtesy of Chemical Engineering Department, North Carolina State College]**

toward the wall would seem to prevent the dominance of this vorticity component in the laminar sublayer itself.

Figure 15 is a somewhat analogous picture taken by Hama, with dye injected from a flush cross-stream slit into the wall layer of a turbulent boundary layer. In this case the interpretation is made more difficult by the apparent non-uniformity of dye

* I should like to thank Drs. Beatty, Ferrell and Richardson of the Chemical Engineering Dept. at North Carolina State College for the loan of their manuscript, and also Dr. Hama of the University of Maryland for supplying figure 15.

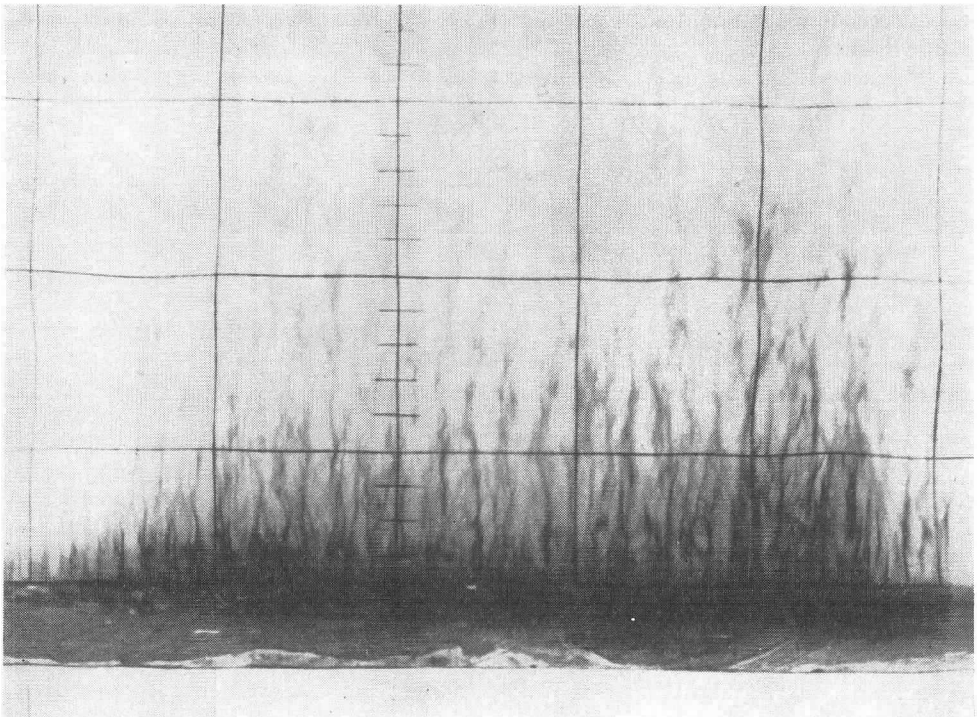


Figure 15. Dye streaks near wall in a turbulent boundary layer. [Courtesy of Dr. F. R. Hama, University of Maryland]

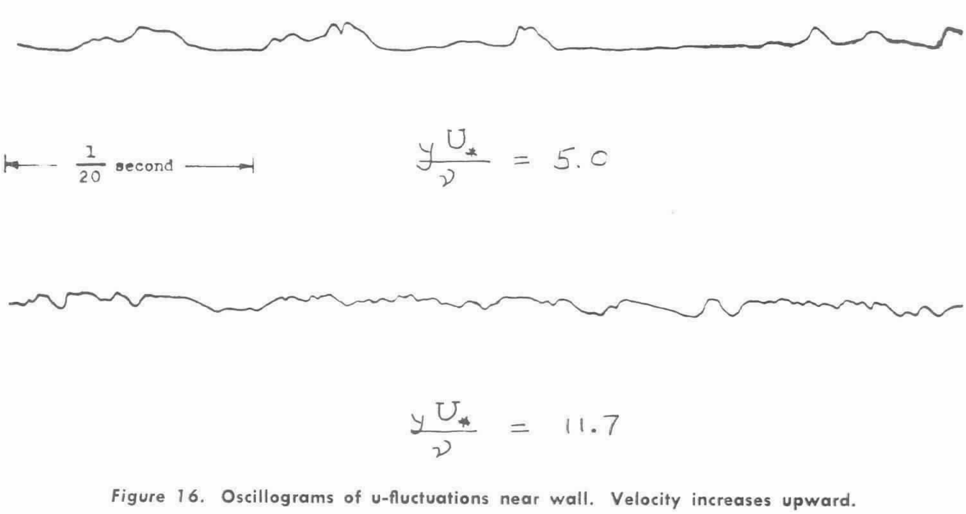


Figure 16. Oscillograms of u-fluctuations near wall. Velocity increases upward.

ejection along the slit. Since this differed in different photographs it is probably due to the local fluctuations in skin friction and static pressure forces.

A rather intermittent structure of the wall layer has been shown clearly by an oscillogram of Ruetenik [24], which is here reproduced as figure 16. As is to be expected, the "bursts" containing higher frequencies are always accompanied by an increase in velocity; this is "turbulent" fluid coming in to $y_* = 5$ from the outer part of the wall layer.

Independent observations on the intermittent character of this region have encouraged Einstein and Li [26] to propose a periodically growing and collapsing laminar "Rayleigh model." Although proper adjustment of constants enables this model to give encouraging agreement with measurements of some flow properties, it appears to suffer from the basic ailment of an infinite r.m.s. skin friction fluctuation.

It seems likely that a "Stokes model" of the sublayer, with sinusoidal rather than sawtooth periodicity, would overcome this difficulty. However, a model of the entire wall layer must doubtless be three-dimensional, as we see from the photographs.

FLUCTUATIONS IN SKIN FRICTION AND SURFACE STATIC PRESSURE

Since $u \sim y$ near $y = 0$, it may be possible to interpret Klebanoff's u'-spectrum at $y^* = 3$ [7] as a spectrum of skin friction fluctuations. This possibility is encouraged by the properties of a Couette flow with fixed wall at $y = 0$ and moving wall (at $y = h$), having mean velocity U_0 plus sinusodal oscillation $u_0 \sin \omega t$ in its own plane. The solution of this problem gives

$$\frac{\left(\dfrac{u'}{\bar{U}}\right)}{\left(\dfrac{u'}{\bar{U}}\right)_h} = \frac{h}{y}\sqrt{\frac{\cosh^2 \beta y - \cos^2 \beta y}{\cosh^2 \beta h - \cos^2 \beta h}}, \tag{53}$$

where

$$\left(\frac{u'}{\bar{U}}\right)_h = \frac{1}{\sqrt{2}}\frac{u_0}{U_0} \quad \text{and} \quad \beta \equiv \sqrt{\frac{\omega}{2\nu}}. \tag{54}$$

This gives at the fixed wall

$$\frac{\left(\dfrac{u'}{\bar{U}}\right)_0}{\left(\dfrac{u'}{\bar{U}}\right)_h} = \frac{\sqrt{2}\,\alpha h}{\sqrt{\cosh^2 \alpha h - \cos^2 \alpha h}}. \tag{55}$$

If ω is identified as an average frequency of the "shear-carrying eddies," i.e.

$$\omega = {}_{uv}(k_1)_{av} \cdot \bar{U}_{y_*=3} \approx 100 \text{ sec}^{-1}, \tag{56}$$

and if h is chosen such that $\dfrac{hU_*}{\nu} = 3$, i.e. $h \approx .008$ cm, we find $\alpha h \approx 0.15$. Hence $\left(\dfrac{u'}{\bar{U}}\right)_0 \approx \left(\dfrac{u'}{\bar{U}}\right)_h$, and we conclude that at $y_* = 3$ the relative longitudinal velocity fluctuations have reached their boundary value.

Klebanoff's power spectrum, though plotted in terms of wave number, was actually measured in time, and in figure 17 is the Fourier transform, essentially the

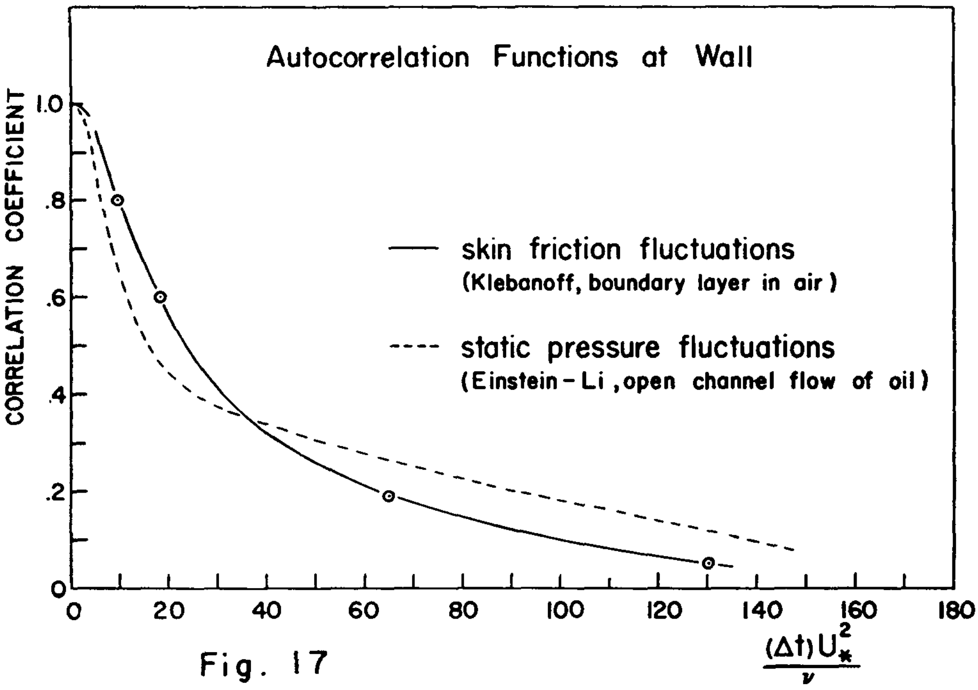


Fig. 17

time autocorrelation of the skin friction fluctuations. The time interval is made dimensionless by the characteristic sublayer time,

$$\frac{U_*^2}{\nu} = \left(\frac{\partial \bar{U}}{\partial y}\right)_0^{-1}. \tag{57}$$

Einstein and Li [26] have measured the autocorrelation of surface static pressure fluctuations on the bottom of a turbulent open channel flow of oil. Their curve, somewhat smoothed, is plotted on the same sheet as the skin friction correlation. Considering the differences between the two flow systems, the general consistency is remarkable. The situations are doubtless too diverse to permit drawing detailed conclusions, but it does appear that the two kinds of surface force fluctuations have about the same duration of autocorrelation.

Boundary layer thickness fluctuations

Recent attempts at prediction of sound radiation by a turbulent boundary layer have taken two apparently diverse paths. Working from the viewpoint of Lighthill's formulation, Phillips [25] has related sound radiation to skin friction fluctuations. Working from a more explicit picture of how the free stream "sees" a boundary layer, Liepmann [27] has focussed attention upon the fluctuations in boundary layer displacement thickness, presumably acting as a random distribution of pistons.

Of course there must be some analytical connection between skin friction fluctuations and boundary layer thickness fluctuations, although we may expect to find the connection with momentum thickness rather than displacement thickness.

Taking a very simplified *two-dimensional* viewpoint, we may define local displacement and momentum thickness by

$$\triangle^* \equiv \int_0^\infty \left(1 - \frac{U}{U_\infty}\right) dy, \tag{58}$$

$$\Theta = \int_0^\infty \frac{U}{U_\infty}\left(1 - \frac{U}{U_\infty}\right) dy. \tag{59}$$

Introducing $U = \overline{U} + u$, we get mean thicknesses

$$\overline{\triangle^*} = \int_0^\infty \left(1 - \frac{\overline{U}}{U_\infty}\right) dy, \tag{60}$$

$$\overline{\Theta} = \int_0^\infty \frac{\overline{U}}{U_\infty}\left(1 - \frac{\overline{U}}{U_\infty}\right) dy - \int_0^\infty \frac{\overline{u^2}}{U_\infty^2}\, dy, \tag{61}$$

and mean square fluctuations

$$\overline{\delta^{*2}} = \frac{1}{U_\infty^2}\int_0^\infty \int_0^\infty \overline{u(y_1)u(y_2)}\, dy_2\, dy_1, \tag{62}$$

and

$$\overline{\vartheta^2} = \frac{1}{U_\infty^2}\int_0^\infty \int_0^\infty \left\{\left[1 - \frac{\overline{U}\ \ 2(y_1)}{U_\infty}\right]\left[1 - \frac{2\overline{U}(y_2)}{U_\infty}\right] \cdot \overline{u(y_1)u(y_2)} + 0(\overline{u^3})\right\} dy_2\, dy_1, \tag{63}$$

where terms of order $\overline{u^3}$ and higher may be neglected.

Using very rough approximations for mean velocity profile and lateral correlation function from reference 6, we estimate the following values for $\delta^{*\prime}$ and ϑ':

$$\frac{\delta^{*\prime}}{\delta} \approx .027\ ; \qquad \frac{\delta^{*\prime}}{\overline{\triangle^*}} \approx 0.23 \tag{64}$$

$$\frac{\vartheta'}{\delta} \approx .014\ ; \qquad \frac{\vartheta'}{\overline{\Theta}} \approx 0.15 \tag{65}$$

δ is a "geometrical boundary layer thickness," the distance at which $\overline{U} \approx U_\infty$

It would, of course, be most desirable to be able to predict these values from a rough estimate of skin friction fluctuations. The proper approach must involve a two-dimensional random field of skin friction and the proper three-dimensional generalization of the von Kármán integral relation.

A rough estimate was attempted, however, by using an over-simplified form of two-dimensional Kármán integral relation:

$$\frac{d\Theta}{dx} = \frac{T_0}{\rho U_\infty^2}, \tag{66}$$

so that

$$\frac{d\vartheta}{dx} = \frac{\tau_0}{\rho U_\infty^2}, \tag{67}$$

or

$$\vartheta(x) = \frac{1}{\rho U_\infty^2} \int_0^x \tau_0(x_1)\, dx_1. \tag{68}$$

Hence

$$\overline{\vartheta^2}(x) = \frac{1}{\rho^2 U_\infty^4} \int_0^x \int_0^x \overline{\tau_0(x_1)\tau_0(x_2)}\, dx_2 dx_1. \tag{69}$$

This integral was evaluated by assuming the skin friction spatial autocorrelation to be equivalent to the time correlation by $(\Delta x) \approx \overline{U}\,(\Delta t)$, a very poor assumption near the wall as shown experimentally by Klebanoff [7]. With the further assumptions of similarity and $\delta \sim x^{4/5}$ mean growth rate of the boundary layer, the momentum thickness fluctuation was estimated from (69) as

$$\frac{\vartheta'}{\delta} \approx .001, \tag{70}$$

at the boundary layer position where the computed value is .014. The failure of this estimate may be due in part to the erroneous shear stress correlation and in part to the general omission of z-variations both in the differential equation and the surface stress.

ACKNOWLEDGEMENT

A number of people have contributed to the calculations in this report. I should particularly like to thank F. Lien, J. L. Lumley, W. G. Rose, and R. R. Mills. Dr. R. W. Stewart of the University of British Columbia was kind enough to point out some errors in the first draft.

SYMBOLS

A, B	constants
c_f	skin friction coefficient
d	pipe diameter
D	diffusivity
$E_{1,2}$	principal strain rates
ε	three-dimensional spectrum
$F_{1,2}$	one-dimensional spectra
k	wave number
k_1	wave number in x_1 direction
k_c	"cutoff" wave number
k_p	wave number for turbulent energy "production"
l	characteristic diffusive length
l_*	thickness of "laminar sublayer"
L	characteristic length
$\mathscr{L}$	Lagrangian integral length scale
P	static pressure
$\bar{P}$	mean static pressure
p	static pressure fluctuation $[= P - \bar{P}]$

p_v	perturbation static pressure fluctuation due to v_i
Q	transport rate
${}_kR_{uv}$	turbulent shear stress spectrum $\left(\text{dimensionless form} = \sqrt{\dfrac{\tau_1}{F_1 \cdot F_2}}\right)$
R	Reynolds number
R_e	Reynolds number based on "turbulent kinematic viscosity,"
T	total shear stress
$\overline{T}_0$	mean skin friction stress
$\overline{T}_T$	mean turbulent shear stress
$\overline{T}_L$	mean viscous shear stress
U, V, W	cartesian velocities in x, y, z directions respectively
$\overline{U}, \overline{V}, \overline{W}$	mean velocities
u, v, w	velocity fluctuations $[= (U - \overline{U}), (V - \overline{V}), (W - \overline{W})]$
u_j	velocity fluctuation in x_j direction
U_*	"friction velocity" $\left[\equiv \sqrt{\dfrac{\overline{T_0}}{\rho}}\right]$
v_i	perturbation velocity fluctuation in x_i direction
x, y, z	cartesian coordinates
x_i	cartesian coordinates $(i = 1, 2, 3)$
$y_* =$	$\dfrac{yU_*}{\nu}$
α_{E_1, σ_1}	principal directions
β	wave number
δ	geometrical boundary layer thickness
Δ	momentum diameter of round jet
Δ^*	displacement thickness of boundary layer
δ^*	displacement thickness fluctuation
ε	thickness of wall layer in turbulent shear flow
$\vartheta_{i \cdot v}$	characteristic spectral times
Θ	momentum thickness of boundary layer
ϑ	momentum thickness fluctuation
ν	kinematic viscosity
ν_T	"turbulent kinematic viscosity"
ρ	density
$\sigma_{1,2}$	principal stresses
$\tau_{0,T,L}$	shear stress fluctuations
ϕ	rate of dissipation of kinetic energy
ψ	a scalar property

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DISCUSSION

K. Wieghardt

The following remark refers to a complication which might occur in the boundary layer along a three-dimensional body without incidence. A study of the flow around the run of double-models of simplified ships in a wind tunnel has shown a longitudinal vortex-pair near the keel plane produced by the flow around the sharp bilges (see Fig. 1).

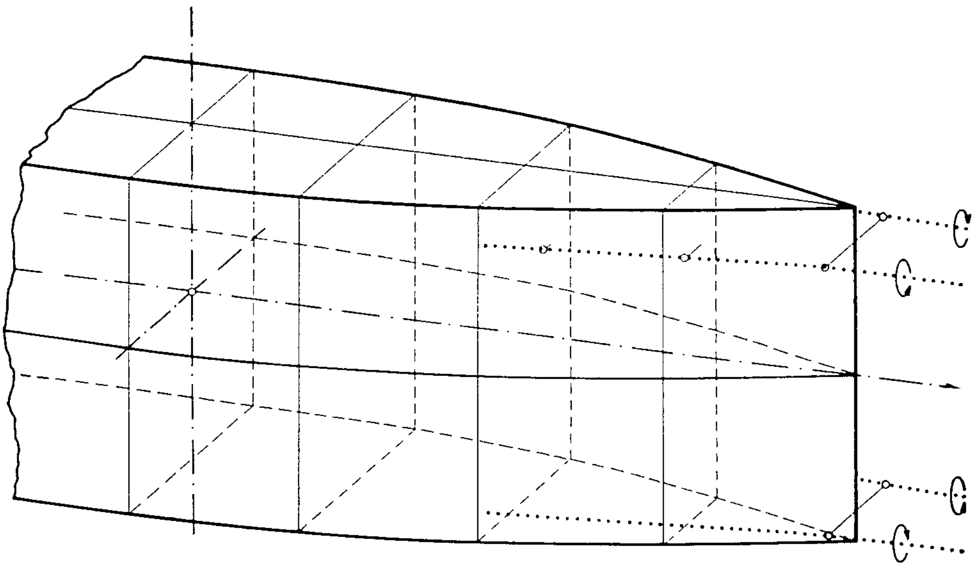


Figure 1.

The strength of the vortices was so that they induced a downward velocity of almost half the main speed in the center line of the wake. This might be of practical interest for double screw ships. In most cases outward turning propellers are best, i.e. screws running in the opposite sense to these vortices. This would seem plausible since in this case the residual vorticity is as small as possible. Of course, on a real ship, the influence of bossings and the interference with the wave drag will also be of great importance. Further on, this effect will certainly be much smaller with bilges well rounded off. E.g. with a second extreme model, a circular cylinder and a run with elliptical sections leading to the same vertical edge, no such vortices were observed. However, since some ships are more similar to the first model such a vortex-pair might play a certain role with real ships.

L. Landweber

First I would like to ask Professor Corrsin's opinion of the relative importance in the boundary-layer momentum equation of the term

$$\frac{d}{dx}\int_0^\delta (\overline{u^2} - \overline{v^2})\, dy. \tag{71}$$

It was shown in several papers, in about 1950, that this term could become large in adverse pressure gradients, but recently it has been indicated that the term is negligible. This apparent contradiction should be clarified.

Next let us consider the validity of the so-called universal logarithmic laws of turbulent boundary layers which is assumed in even the most recent work of Clauser, Coles and Townsend. These laws are based upon two similarity laws, the law of the wall and the velocity-defect law. Of these, the former may be derived by dimensional reasoning and has been well-confirmed by experiment; the latter can be derived theoretically only for pipe and channel flows, and its adoption for the case of flat-plate boundary layers is based upon its correlation of velocity profiles over the narrow range of Reynolds numbers for which data are available.

It has recently been shown by Townsend, on the basis of the concept of approximate self-preservation of the structure of the turbulence in the outer part of a boundary layer on a flat plate, that the velocity defect law may be retained, provided the previous history of the flow is taken into account by replacing the local velocity scale by that corresponding to some upstream position. Townsend's subsequent analysis is approximate and retains the logarithmic relations by assumption. A more precise analysis of Townsend's theory yields a family of possible universal laws of which the logarithmic law is only one member, and among which that one which best fits the boundary-layer data is to be selected.

The theory assumes the validity of the law of the wall

$$\frac{u}{u_\tau} = f(y^*), \qquad y^* = \frac{yu_\tau}{\nu} \tag{72}$$

where $u_\tau = \sqrt{\tau/\rho}$, τ is the shear stress at the wall, and ρ and ν are the density and kinematic viscosity of the fluid; and in the outer part of the boundary layer, the validity of a modified velocity-defect law

$$\frac{U_0 - u}{u_\tau} = \frac{F(\zeta)}{g(\sigma)}; \qquad \zeta = \frac{y}{L}, \qquad \sigma = \frac{U_0}{u_\tau} \tag{73}$$

where U_0 is the free stream velocity, L is a length scale proportional to the boundary-layer thickness, and $g(\sigma)$ is an unspecified function. Townsend assumes a specific form for $g(\sigma)$, but if it is further assumed that there is a range of values of y in which the inner and outer similarity laws overlap, it is found that there is no freedom of choice concerning either the form of $g(\sigma)$ or the similarity laws.

By an exact functional analysis of the foregoing assumptions, one obtains the result

$$y^* \frac{f''(y^*)}{f'(y^*)} = n - 1 \tag{74}$$

If $n = 0$, this yields the well-known logarithmic relations for part of the velocity distribution and the variation of the coefficient of shear stress with Reynolds number. If $n \neq 0$, an entirely new set of relations may be derived. The results for the two cases are compared in the following table:

(75)

	$n = 0$	$n \neq 0$
$f(y^*)$	$c \log y^* + d$	$ay^{*}_{n} + b$
$F(\zeta)$	$-c \log \zeta + d'$	$b'(1 - a\zeta^n)$
$g(\sigma)$	1	$b'/(\sigma - b)$
c_r	$2/\sigma^2$	$2/\sigma^2$
R_δ	$c\sigma e^{\sigma/k}$	$\zeta_0\sigma(\sigma - b)^{1/n}$
R_{δ_1}	$c\beta e^{\sigma/k} + \lambda$	$\alpha_0 + \alpha_1(\sigma - b)^{1+1/n}$
R_{δ_2}	$ce^{\sigma/k}\left(\beta - \dfrac{\gamma}{\sigma}\right) - \lambda - \dfrac{\alpha}{\sigma}$	$\dfrac{1}{\sigma}[\beta_0 + \beta_1(\sigma - b) + \beta_2(\sigma - b)^{1+1/n} + \beta_3(\sigma - b)^{2+1/n}]$
R_x	$ce^{\sigma/k}[\beta\sigma^2 - (\gamma + 2\beta k)\sigma + 2k(\gamma + \beta k)] + \alpha\sigma + c$	$\gamma_0 + \gamma_1(\sigma - b) + \gamma_2(\sigma - b)^{1+1/n} + \gamma_3(\sigma - b)^{2+1/n} + \gamma_4(\sigma - b)^{3+1/n}$

F. R. Hama

I am not quite prepared to discuss Dr. Corrsin's interesting and stimulating lecture, and I am rather anxious to read his manuscript carefully.

However, in connection with his Fig. 15, which was taken in a fully-developed turbulent boundary layer at Dr. Corrsin's request, I would like to show one other picture (Fig. 1) taken in a laminar boundary layer disturbed by vortices due to a trip wire.

In the absence of the wire, the dye comes out uniformly and no filamentation of the dye occurs. Once the wire is applied, the flow of the dye shows a concentration as shown in Fig. 1, indicating the existence of secondary spiral motion near the boundary surface, although the boundary layer still remains laminar or, more properly, is in the stage toward transition.

There is naturally a distinctive difference between the two cases. Whereas the attached eddies are smaller and randomly distributed in turbulent flow, the wisps are larger and regular in the other flow. Nevertheless, there seems to be a resemblance for the two flow conditions, i.e., secondary spiral motion.

I don't know, first, how we can explain this secondary flow formation which takes place when a laminar boundary layer is disturbed by strong-two-dimensional vortices. I don't know, second, whether there is any relation between the predominant secondary flow in turbulent boundary layers and that in laminar layers in the transition region.

R. W. Stewart

We seem to get very close to the boundary. There is one phenomenon that must be connected with this, which I thought I would mention. We have examined very closely the bottom of some ships in dry dock. Whenever these ships were painted in Hongkong, apparently they used very soft paint, and along the bottom of the hull, along the direction of the mean flow—this has no relation whatsoever to the way the paint brush strokes went, or the direction of the plates on the ships—but in the direction of the mean flow one finds streaks of raised paint. Sometimes a single streak will be

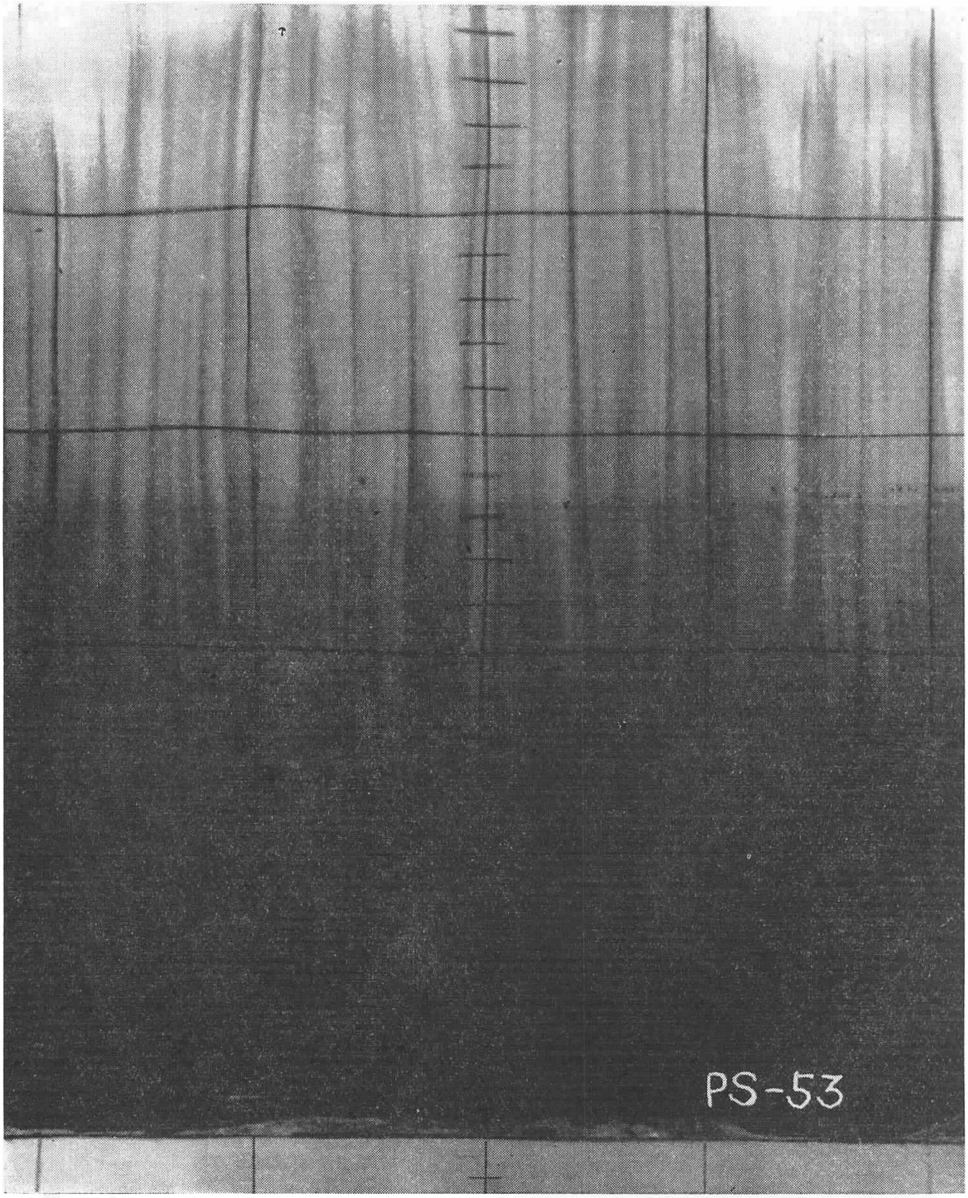


Figure 1. (F. R. Hama)

as many as several yards long. These streaks are about an 8th of an inch or a millimeter apart.

The effect must be somewhat related to this type of phenomena.

The thing interesting about it, since these have actually deformed a fairly solid body like this paint, is that there must be some self-stabilizing feature, so that once a little seed is developed, this process continues from then on, in essentially the same place.

Also another interesting thing about this, of course, is that the boundary layer on the ship varies a great deal in thickness from the bow to the stern, close to the propellers, but the streaks are all about the same distance apart, whether at the front or the back.

It must be closely associated with something close to the boundary, because, for example, there is a wake behind each rivet head. I think this streaky phenomenon is something that occurs very characteristically close to the boundary even in fully developed boundary flows.

E. Silberman

The remarks just made (by the previous discusser) bring to mind similar results obtained on the walls of a curved duct. The duct cross section was 6 in. square and the walls were of Lucite. The inner surfaces were coated with a thin paste of aluminum powder in machine oil. The fine streaks mentioned appeared in the oil and apparently took the directions of the streamlines at the walls. (Directions given by the streaks coincided with the limiting directions of fine yarns moved from the interior to the boundary of the fluid.) Observations of the erosion process producing the streaks gave the impression that the paste was first dislodged in a direction other than the streak direction, but almost immediately turned in the streak direction. Reynolds number based on duct hydraulic diameter was about 5×10^4 for the experiments.

G. K. Batchelor

I don't think these remarks have any bearing on the explanation of the observed streaks of dye. The same phenomenon of longitudinal streaks or ridges on thin films has been observed on aircraft wings smeared with oil, and in fact has been used to make visible the shape of the streamlines of the air flow. I don't think it has been analyzed, but I feel pretty sure it is related to the characteristics of the film itself, and is the result of some kind of instability tending to make the oil move sideways and gather itself into longitudinal ridges and troughs. I think one can see this as fairly natural if one thinks of the way in which the oil ought to arrange itself in order to have the least amount of viscous dissipation as a result of the shearing action produced by the airflow outside the film, and the fixed wall underneath it.

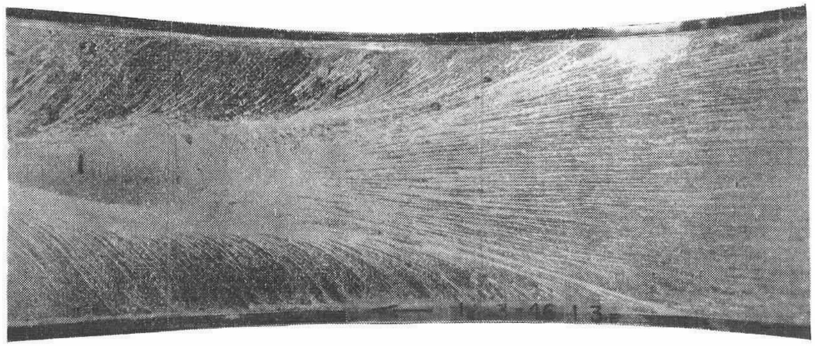
In other words, this seems to me to be a film phenomenon, whereas the problem we are talking about is concerned with the flow in the boundary layer itself. Surely the streaks of dye are related to the idea of longitudinal diffusion, which Taylor talked about a few years ago in connection with the dispersion of salt solution in a pipe. He pointed out that if you produce a small cloud of salt solution in turbulent flow in pipe, the variation of mean velocity across the pipe tends to draw out the cloud longitudinally. Wouldn't this also happen here, when Professor Hama's dye is ejected across the flow? Will not the part of the dye that lies farther from the wall also be a part which moves forward more rapidly, thus producing a streak of dye?

F. R. Hama

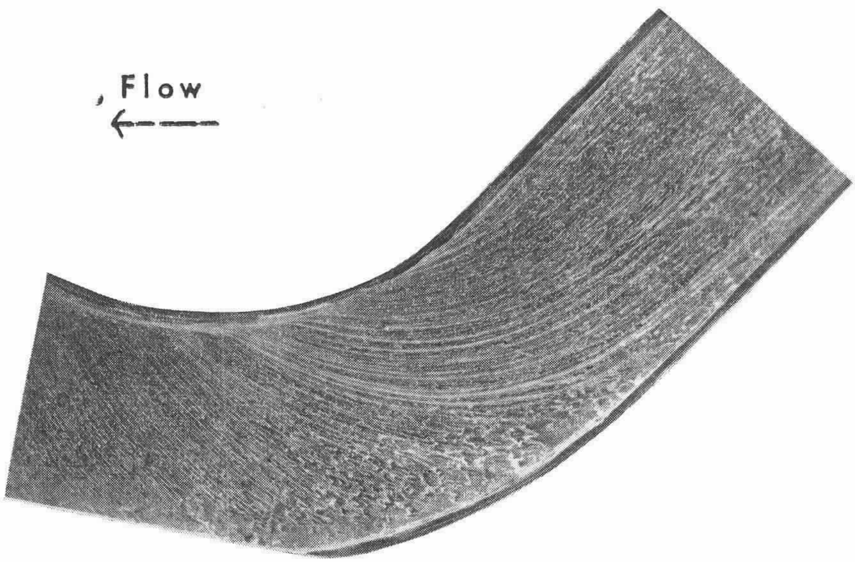
The distinct difference between the streak formation and the non-streaky flow is that, if you don't have the trip, the dye comes out uniformly and no filamentation occurs, and, only when the trip is applied, the wisp formation takes place. Therefore, the wisp formation must have resulted from the secondary spiral flow.

M. P. Tulin

Just before Dr. Batchelor got up I was thinking of experiments we (Mr. Ralph Cooper and myself) had made at the Model Basin several years ago, investigating the effects of studs on transition. These measurements were made on a bi-convex, thin plate which was towed in a model basin. Transition was determined by coating the



Inside of Bend



Bottom of Bend

Streaks on interior surfaces of 6-in. square duct at a 90° bend. Walls are Lucite. Photographed from outside. Coating is paste of aluminum powder in SAE 30 oil.

Figure 1. (E. Silberman)

plate with a solid substance which would wash away in the turbulent region and would remain on the plate in the laminar region.

In regions of transition, and just preceding transition, we noticed a regular and very fine pattern of streamwise striations, perhaps of the order of a millimeter distance apart, these distances considerably less than the distances between the studs, and apparently not much depending on the stud distance.

This experience suggests that in order to determine whether the boundary layer streaks are really connected with film behavior it might be instructive to perform further observations in water using the aforementioned technique.