I

SOME PROBLEMS AND METHODS IN HYDRODYNAMICS

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When I received Admiral Bennett's invitation to address you tonight I felt almost as uncomfortable as General Burgoyne must have felt at Saratoga with the thought of the North behind me and hydrodynamicists massing in the South. Indeed I began to reflect on the distance from London to York which as you know, or ought to know, is a long way, and even non-stop trains take several hours to cover it. Once upon a time in one such non-stop train sat an American and an Englishman the only occupants of a compartment. For about an hour each looked out of the window. Then the American said "Do you mind if I talk to you" to which the Englishman replied "What about?" You see my difficulty. An after dinner speech which conveyed any information whatever is outside the range of my experience. An after dinner speech concerning Hydrodynamics must be unique in the history of dining and I therefore feel honored to be chosen as the first to make such history.

Hydrodynamics as an exact science started with Archimedes. It is true that he treated the particular case of zero velocity, but his work remains today a correct piece of applied mathematics and indeed a giant achievement for his time.

Scoffers have said that Archimedes' chief claim to fame is that he took a bath and then forgot to dress in spite of his principles.

It is not my purpose nor indeed am I competent to discuss hydrodynamics from the point of view of the historian, who, as you know, is an amiable gentleman advancing into eternity stern first.

Rather I should like to say a few words concerning some specific problems and methods available for tackling them. Omission of a name or a topic does not imply undervaluation by me. It merely proves that a choice has to be made. Many other choices would have been possible.

It is becoming increasingly evident that the most insight giving statement of the equations of motion of continuous media in general and of fluids in particular is by means of tensors.

There are at least two ways of regarding tensors, namely as quantities attached to a coordinate system, or intrinsically, the latter being to my way of thinking incomparably superior.

The intrinsic definition of a tensor $T^{(n)}$ of rank n is recursive.

A tensor of rank n is a linear operator which operating on an arbitrary vector $\mathbf{x}$, by scalar multiplication, gives rise to a tensor of rank n-1.

This definition, together with the statement that a tensor of rank zero is a scalar, completely characterizes tensors of all ranks.

Thus for example a vector $\mathbf{a}$ is a tensor of rank 1 since $\mathbf{a} \cdot \mathbf{x}$ is a scalar, or tensor of rank zero.

Similarly the dyadic product $\rho \boldsymbol{q};\boldsymbol{q}$ combines with $\mathbf{x}$ to give $\rho \boldsymbol{q}\ (\boldsymbol{q} \cdot \mathbf{x})$ a tensor of rank 1 and so $\rho \boldsymbol{q};\boldsymbol{q}$ is a tensor of rank 2, a 2-tensor, indeed the very important momentum transfer tensor. As an application the equation of steady motion under no body forces can be written [1]

$$\nabla \cdot [\Phi - \rho(\boldsymbol{q};\boldsymbol{q})] = 0, \tag{1}$$

where

$$\Phi = - pI - \tfrac{2}{3}\mu(\nabla \cdot \boldsymbol{q})I + \mu(\nabla;\boldsymbol{q} + \boldsymbol{q};\nabla) \tag{2}$$

is the stress tensor, I being the idemfactor or unit 2-tensor.

From this form of the equation of motion, by integration over a sphere of large radius, an expression for the force on a moving solid is readily obtained. By applying Oseen's approximation at a distance we next analyse this force into a lift and a resistance or drag. The resistance is of particular interest since its expression is VF where V is the velocity of the body and F is an inflow of liquid into the sphere, predominantly an inflow into the wake behind the body. Moreover the result is of an asymptotic character improving in accuracy as the radius of the sphere is increased. The two-dimensional form of this result was obtained by Filon [2] 30 years ago.

Tensor expression also pinpoints the Lagrangian form of the equation of motion [3]

$$\frac{\partial;\boldsymbol{r}}{\partial\boldsymbol{r}_0} \cdot \left(\frac{\partial^2\boldsymbol{r}}{\partial t^2} - \boldsymbol{F}\right) + \frac{1}{\rho}\frac{\partial p}{\partial\boldsymbol{r}_0} = 0, \tag{3}$$

where $\boldsymbol{r}$ is the position vector at time t of the particle originally at $\boldsymbol{r}_0$. The equation of continuity is

$$\rho\left(\frac{\partial;\boldsymbol{r}}{\partial\boldsymbol{r}_0}\right)_{III} = \rho_0, \tag{4}$$

where the notation indicates the third scalar invariant of the tensor derivative.

Integration from 0 to t leads directly to Weber's transformation

$$\frac{\partial;\boldsymbol{r}}{\partial\boldsymbol{r}_0} \cdot \boldsymbol{q} - \boldsymbol{q}_0 = - \frac{\partial\chi}{\partial\boldsymbol{r}_0}, \tag{5}$$

where

$$\chi = \int_0^t \left\{\int\frac{dp}{\rho} + \Omega - \tfrac{1}{2}q^2\right\}dt, \qquad \boldsymbol{F} = - \nabla\Omega. \tag{6}$$

The Lagrangian form of the equation of motion has been applied to the one-dimensional motion of a gas and more recently to free surface problems which I shall mention later.

The point that I want to make here is that the Lagrangian form is not quite so repulsive as the three equations which result from its expression in coordinates would seem to indicate. In the hydrodynamic case we have $\rho = \rho_0$ and a consequent simplification. The equation, in my opinion, should repay further study.

The two great weapons of general fluid mechanics are the theorems of Gauss and Stokes [4] and their vector forms are suggestive:

$$\int_S \boldsymbol{dS} \circ X = \int_\tau \nabla \circ X d\tau, \qquad \int_C \boldsymbol{dC} \circ X = \int_S (\boldsymbol{dS}_\wedge\nabla) \circ X, \tag{7}$$

where, in the first the closed surface S encloses the region τ, and in the second the diaphragm S spans the closed curve C. Here the small circle indicates scalar, vector or dyadic multiplication and X is a general function of position, scalar, vector or tensor.

For the rate of change of circulation in a circuit which always consists of the same fluid particles we have

$$\frac{d}{dt}\operatorname{circ} C = -\int_C \boldsymbol{dC} \cdot \frac{1}{\rho}\nabla p = \int_S \boldsymbol{dS} \cdot \boldsymbol{P} \tag{8}$$

where $\boldsymbol{P} = \nabla p_\wedge \nabla (\frac{1}{\rho})$, so that $\boldsymbol{P}$ is a vector along the intersection of surfaces of constant pressure and constant density. Also $\nabla \cdot \boldsymbol{P} = 0$ and so by Gauss's theorem $\boldsymbol{P}$ defines tubes of constant intensity. Thus we have the famous meteorological theorem of Bjerknes that the rate of change of circulation in C is measured by the number of $\boldsymbol{P}$ tubes which C embraces.

For plane flow the vector notation leads directly to the complex variable [5].

The use of the complex variable in two-dimensional problems has a long history, but it is only in recent years that full advantage has been taken of the methods of function theory as opposed to resolution into equations in x and y. What is beginning now to be more fully realized is that the variables most generally useful are not x, y but the conjugate pair $z, \bar{z}$. For example if $\varphi\ (x, y)$ is a plane harmonic function, it is the real part of a holomorphic function $f(z)$. The identity

$$\varphi(x,y) \equiv \tfrac{1}{2}[f(z) + \overline{f(z)}] \tag{9}$$

leads to

$$f(z) = 2\varphi(\tfrac{1}{2}z, -\tfrac{1}{2}iz) - \varphi(0,0) + i\gamma, \tag{10}$$

where γ is an arbitrary real constant.

Again the circle theorem [6] states that the motion of an unbounded liquid whose complex potential is $f(z)$ when disturbed by the circle $|z| = a$ is governed by the complex potential

$$f(z) + \bar{f}\left(\frac{a^2}{z}\right), \tag{11}$$

for on the boundary $\bar{z} = a^2/z$, so that the boundary is a streamline. That no new singularities are introduced is clear from the fact that of the points z and a^2/z only one lies inside the circular boundary.

Combined with conformal mapping the circle theorem enables us to deal with the perturbation produced in disturbing the flow by a cylinder of any cross-section.

In the same line of thought we know that a stream function can be defined for any two-dimensional flow whether rotational or irrotational. Using the variables $z, \bar{z}$ we can denote this stream function by $\psi(z, \bar{z})$ and then the velocity is given by

$$u - iv = -2i\frac{\partial\psi}{\partial z}. \tag{12}$$

In the case of steady streaming past a fixed cylinder, ψ is constant on the boundary and so

$$\frac{\partial\psi}{\partial z}dz + \frac{\partial\psi}{\partial\bar{z}}d\bar{z} = 0. \tag{13}$$

The Blasius theorem for the force (X, Y) then gives [7]

$$X - iY = \tfrac{1}{2}i\rho\int_C (u - iv)(u + iv)d\bar{z} = -2i\rho\int_C \left(\frac{\partial\psi}{\partial z}\right)^2 dz, \tag{14}$$

where the integral is taken round the boundary C of the cross-section. But the equation of the boundary is of the form $f(z, \bar{z}) = 0$ and so $\bar{z}$ can be eliminated and Cauchy's residue theorem used even in the case of rotational motion. For example a circular cylinder exposed to a stream V on which is superposed uniform shear flow of vorticity ω undergoes the lift $\pi\rho a^2\omega V$.

Just as the theorem of Stokes is fundamental in "solid flow," if I may use that expression, so is the form which the theorem assumes in two dimensions in terms of z and $\bar{z}$ a powerful tool. This I have named the complex Stokes' theorem [8]. The complex Stokes' theorem refers to a plane area S bounded by a closed curve C and states that

$$\int_C f(z,\bar{z})dz = 2i\int_S \frac{\partial f}{\partial \bar{z}}\, dS. \tag{15}$$

Cauchy's theorem is a particular case, namely when $\partial f/\partial\bar{z} = 0$.

One immediate and important application is the calculation of the kinetic energy [9] of liquid in irrotational motion, with its application to virtual mass. Thus

$$\begin{aligned}\text{kinetic energy} &= \tfrac{1}{2}\int_S \rho q^2 dS = \tfrac{1}{2}\rho\int_S \frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\, dS \\ &= \tfrac{1}{2}\rho\int_S \frac{\partial}{\partial z}\left(w\frac{d\bar{w}}{d\bar{z}}\right)dS = \tfrac{1}{4}i\rho\int_C w\,d\bar{w}\end{aligned} \tag{22}$$

where $w = \varphi + i\psi$ is the complex potential. If the region S is multiply-connected, suitable barriers must be introduced as part of the boundary C. Note that in virtue of the remarks made shortly ago the integral round the boundary can be evaluated by the residue theorem since $\bar{z}$ is a function of z.

The generalization of the complex variable to three-dimensions leads to Hamilton's quaternions. Alan Rose [10] defines a stream function $\psi(x, y, z; \xi, \eta, \zeta)$ as the flux across the triangle formed by the origin, the point $A(x, y, z)$ and the point $B(x + \xi, y + \eta, z + \zeta)$. If we define the vector

$$\Psi = (\psi_1, \psi_2, \psi_3) = \left(\frac{\partial\psi}{\partial\xi}, \frac{\partial\psi}{\partial\eta}, \frac{\partial\psi}{\partial\zeta}\right)_{\xi=\eta=\zeta=0}, \tag{23}$$

the velocity is

$$q = -\nabla_\wedge\Psi. \tag{24}$$

If the motion is axisymmetric and irrotational with velocity potential φ, the function

$$\varphi + i\psi_1 + j\psi_2 + k\psi_3 \tag{25}$$

satisfies the condition for it to be a right-regular quaternion function of the quaternion variable

$$w + ix + jy + kz \tag{26}$$

where w is an imagined coordinate whose axis is perpendicular to the axes of x, y, z and i, j, k are Hamilton's unit vectors.

Thus the theory of analytic quaternion functions is in principle available.

In this way, for example, it is possible to deduce the flow past a sphere in terms of the quaternion variable by a method entirely analogous to that for deducing the flow past a circle in terms of the complex variable.

Here then is a method which may well merit further investigation.

Let us now turn to a problem which is important from the naval standpoint; the problem of virtual mass. To take the simplest case when a body of mass M moves with uniform speed V in a straight line in inviscid liquid the total kinetic energy of the system is

$$\tfrac{1}{2}(M + H)V^2, \tag{27}$$

and the body moves as if the liquid were absent and the mass of the body were increased from its actual mass M to its virtual mass $M + H$. Here H is the added or hydrodynamic mass for this particular motion, and is the coefficient of $\tfrac{1}{2}V^2$ in the expression for the kinetic energy of the liquid.

It is only quite recently that a physical interpretation of hydrodynamic mass as an actual mass of liquid has been given by Darwin [11]. To understand Darwin's interpretation consider the particular case of a circular cylinder which moves along the x-axis from minus to plus infinity. Suppose that when the cylinder is at $x = -\infty$, a wall of blue dye is used to color the particles in a plane perpendicular to the direction of motion. Since the cylinder in its motion displaces a certain volume of the fluid the gap left behind must be filled up and it might appear reasonable to suppose that when the cylinder has attained the position $x = +\infty$, the wall of dye will have retreated a certain distance to the rear of its initial position.

Now the paths of the particles are elasticas, Fig. 1, so that a particle at A when the cylinder is at $x = -\infty$ will have drifted forward to B when the cylinder is at $x = +\infty$.

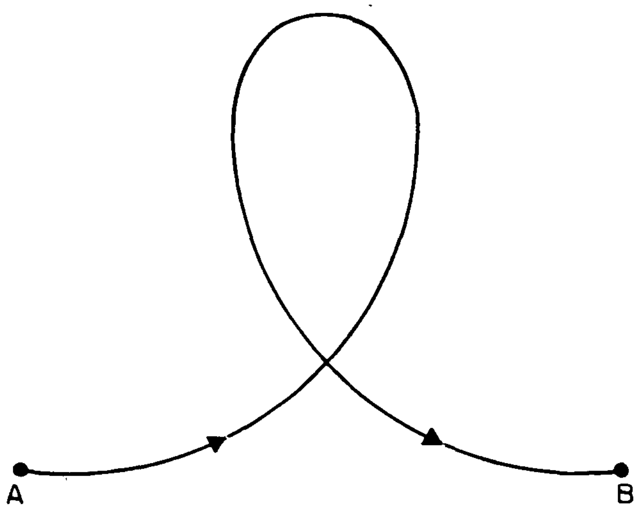

FIG 1

If then we consider those particles which at a given instant lie in an axial plane of the cylinder, perpendicular to the direction of motion, the positions of these particles when the cylinder is at $x = -\infty$ and at $x = +\infty$ define surfaces which, in the plane of the motion, are typified by curves A,A, and B,B, Fig. 2. Thus the particles on the curves A,A move forward, not backwards, to the positions B,B. The intuitive idea of reflux of a wall of dyed particles is false. Darwin's discovery is that the mass of the liquid enclosed between these initial and final positions is in fact the hydrodynamic mass of the cylinder for this particular motion.

The point can be established by integration since the coordinates of the points on the elastica are expressible rationally in terms of Jacobian elliptic functions [12]. The argument is, however, capable of general formulation independent of the particular shape of the cross-section of the cylinder. The argument can also be extended to

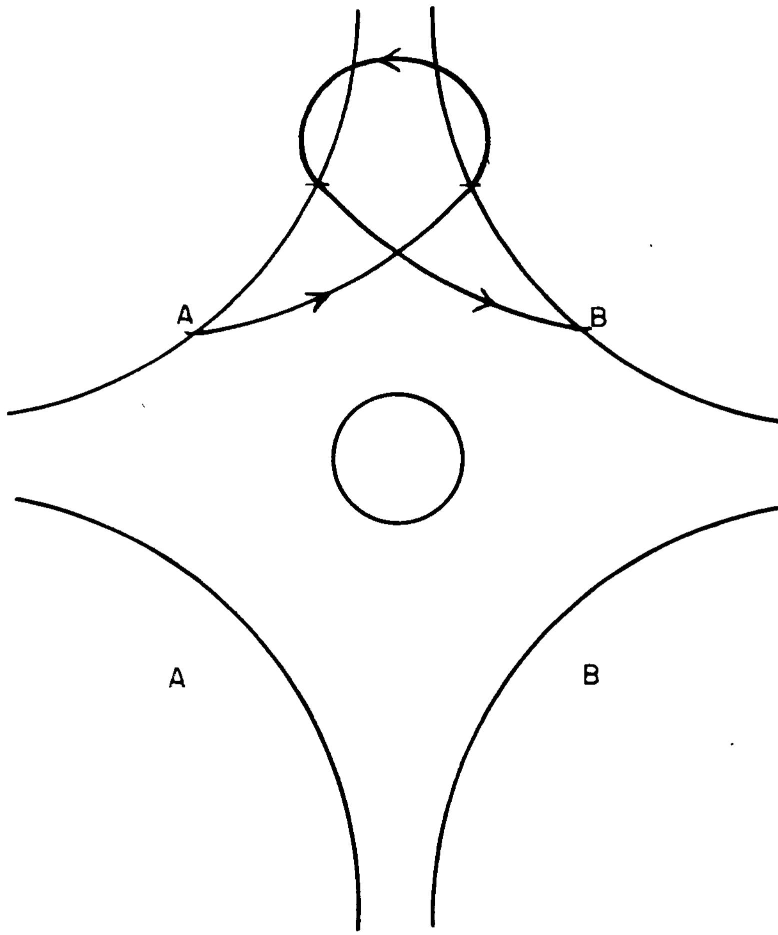

FIG 2

three-dimensional motion. Thus we have here, I believe for the first time, a genuine physical interpretation of hydrodynamic mass as a mass of liquid entrained by the body.

The corresponding problem of interpretation with a free surface still awaits investigation, although some computations of the hydrodynamic mass itself have been made by Bloh [13] for spheres and ellipsoids half immersed and totally immersed.

The problem of rotation also offers opportunities for investigation. If we consider a rotating plate, in two-dimensional motion, it is found that two regions A and B exist in which liquid is trapped and moves round with the plate, not as a rigid body but consistently with irrotational flow, Fig. 3. This points to a method of interpreting hydrodynamic mass due to rotation.

Problems concerning free streamlines have been intensively studied in recent years in relation to the cavities formed behind bodies moving at high speed and the water entry of missiles.

The two-dimensional problem has been greatly simplified by Max Shiffman's [14] method of reflection across free streamlines whereby an image of the actual flow

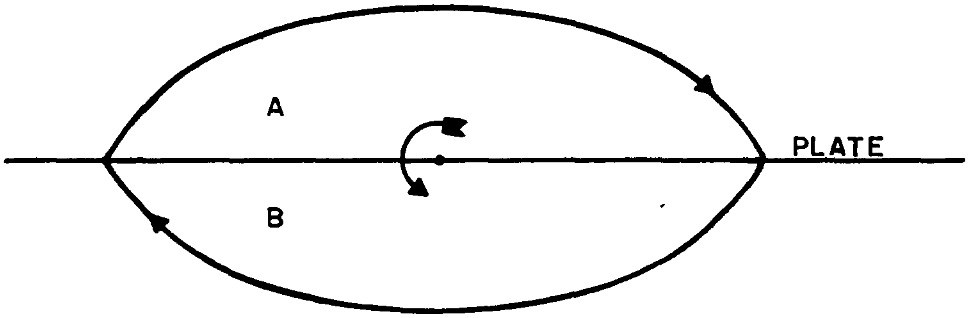

FIG 3

is obtained in the free streamline, in effect a method of analytical continuation of the flow. Apart from the insight-giving character of this method, one of the integrations is already performed by geometrical considerations, thus greatly simplifying the technique. The method also affords a direct geometrical interpretation of the drag coefficient.

To give the simplest possible illustration, it is a provable theorem that the image of flow in an angle is flow outside an equal angle, Fig. 4. Thus for a jet running along a wall ABC we reduce the problem to flow in a channel formed by the original wall and a parallel wall A'_∞ B' C_∞ got by translation along the bisector of the angle A_∞ B C_∞.

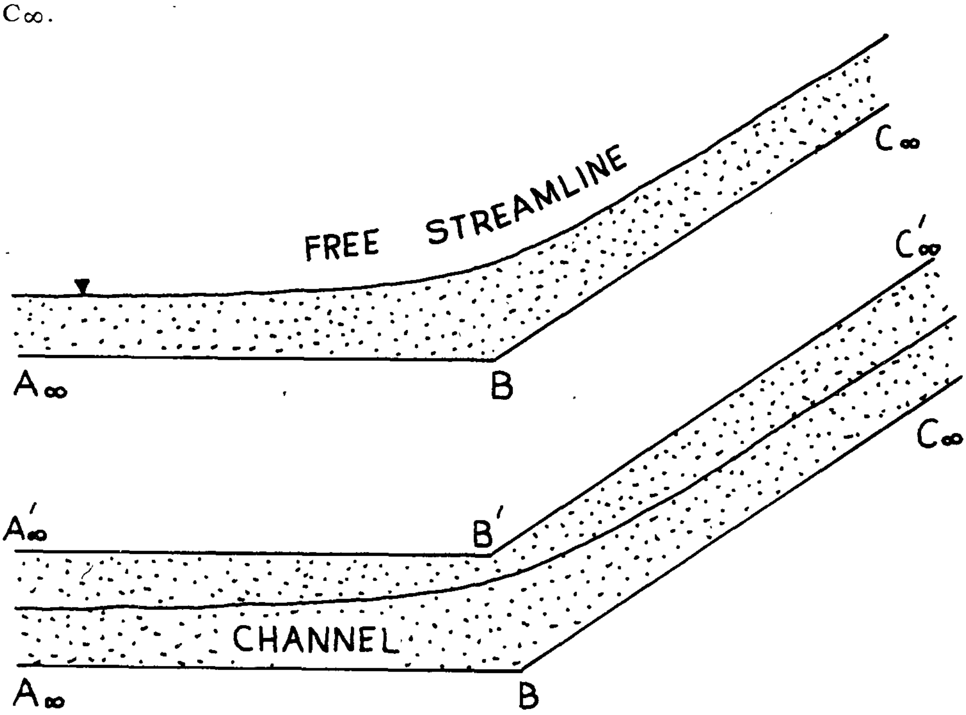

FIG 4

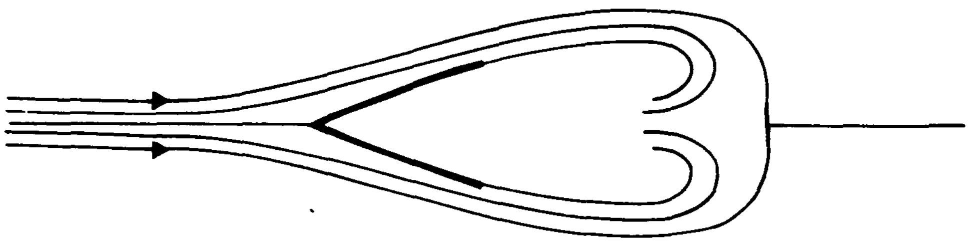

FIG 5

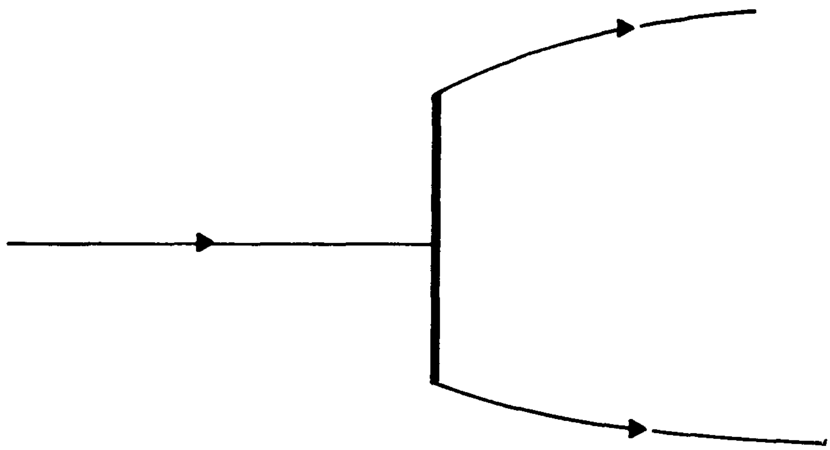

FIG 6

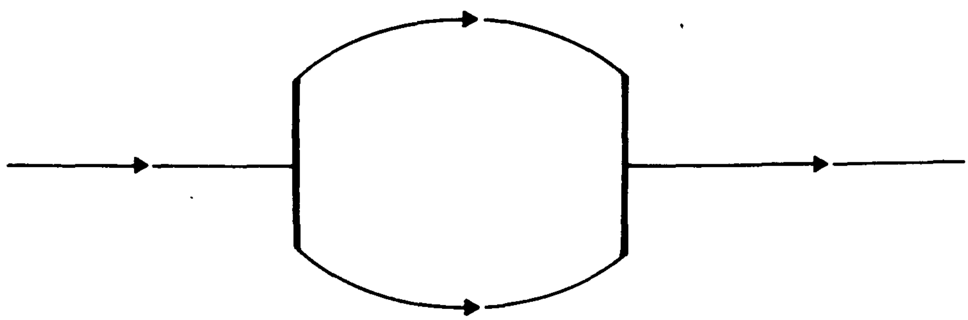

FIG 7

The method also takes charge of reentrant jet problems such as the impact of a stream on a wedge, Fig. 5.

Applied to the impact of a stream on a lamina, Fig. 6, the image flow indicates a layer of sources on the downstream face, and thus offers at least a suggestion for tackling the corresponding problem of a stream impinging on a circular disc.

Riabouchinsky [15] 30 years ago discovered a method of finding the drag on a lamina exposed to a stream, Fig. 7. The method consists in placing an image lamina downstream, the two being joined by free streamlines.

This problem is capable of exact solution [16] in terms of Jacobian elliptic functions, where Prandtl's cavitation number, $\sigma = (p_\infty - p_c)/\frac{1}{2}\rho u^2$, appears in the parameter of the elliptic functions. The case of an unlimited cavity can then be approximated by increasing the distance between image and plate.

Quite recently Garabedian [17] has undertaken the numerical study of the axisymmetrical problem, in particular that of the circular disc with an equal image disc behind it on the Riabouchinsky model. The two discs are joined by a free stream surface to enclose a region containing water vapor.

The equation satisfied by the stream function ψ is

$$\psi_{xx} + \psi_{yy} - \frac{1}{y}\psi_y = 0 \tag{28}$$

with $\psi = 0$ on the entire boundary and

$$\frac{1}{y}\frac{\partial\psi}{\partial n} = 1 \tag{29}$$

on the free stream surface, where x and y are coordinates in a meridian plane.

From these we derive

$$\frac{\partial}{\partial n}\left(\frac{1}{y}\frac{\partial\psi}{\partial n} + \frac{\kappa}{y}\psi\right) = 0 \tag{30}$$

on the free stream surface where κ is the curvature of the free streamline in the meridian plane.

The boundary conditions can be combined to give

$$\frac{1}{y}\frac{\partial\psi}{\partial n} + \frac{\kappa}{y}\psi = 1 \tag{31}$$

on the free streamline and this boundary condition is therefore stationary in the sense that by (30) a normal displacement δn of the free streamline leads to an error of order $(\delta n)^2$. A convergent iterative process can be founded on this observation. Garabedian has carried this out, taking an initial position of the free streamline based on the curve afforded by the plane flow solution of Riabouchinsky.

Let us now turn to the problem of free surfaces when the liquid moves under gravity. By a free surface we shall mean a surface which always consists of the same fluid particles and on which the pressure is constant. The grand illustration in nature is the surface of the ocean.

But few simple complete solutions of this problem are known.

The only non-trivial cases which occur to one are Gerstner's trochoidal wave and Rankine's combined vortex. In the Gerstner wave the free surface is a trochoid and the motion is rotational. In Rankine's combined vortex the motion is rotational within a vertical cylindrical core and is irrotational outside the core, Fig. 8.

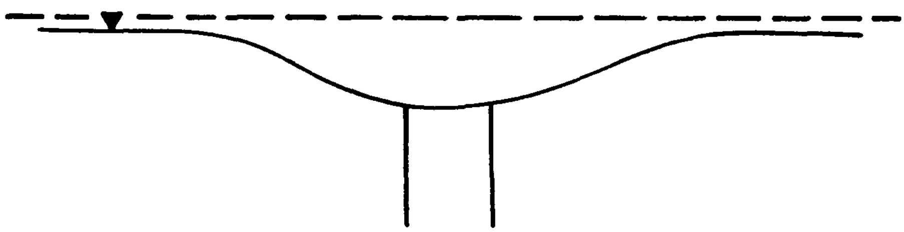

FIG 8

From the naval point of view there are three main problems whose exact solution still eludes us, namely the problems of periodic surface waves on deep water, surface waves on water of finite depth, and the solitary wave. The combination of aerial photographs with a knowledge of the relation between wave length and wave speed played an important part in determining depth and slope on the Normandy beaches.

Let us look at what is involved in the problem of the two-dimensional progressive wave of permanent type on water of finite depth. As long ago as 1925 Levi-Civita [18] stated the boundary value problem in the form

$$\frac{\partial \theta}{\partial \psi} = -\frac{g}{c^3} e^{-3\tau} \sin \theta, \qquad \psi = 0, \qquad q = ce^{\tau}. \tag{32}$$

Here θ is the inclination of the velocity vector to the horizontal.

By mapping the strip defined by a period on the unit circle and then obtaining a Taylor expansion, Levi-Civita established the existence of this type of wave.

One might inquire why such an existence theorem should be necessary when waves are to be seen any day.

But it is only fair to remember that the wave here considered is perfectly regular and is propagated in an inviscid fluid, conditions to which observed waves only approximate.

The intrinsic difficulty of the problem here envisaged is the non-linearity of the boundary condition, quite apart from the fact that the form of the boundary is part of the solution.

Putting $\theta + i\tau = \omega$, the linearized approximation is obtained by assuming

$$|\omega| = (\theta^2 + \tau^2)^{1/2} \tag{33}$$

to be small of the first order. From this assumption follows the usual theory of waves of small amplitude and slope.

There is, however, a serious limitation to the use of the linearized approximation. A wave will break at the crest when the fluid velocity there exceeds the velocity of the wave. The critical case is when the fluid velocity at the crest is equal to the velocity of the wave, that is to say $q = ce^{\tau} = 0$, so that $\tau = -\infty$.

It follows that no approximation based upon the assumption that τ is small can throw any light on the case of breaking.

A way to avoid this difficulty has been proposed by T. V. Davies [19] namely in Levi-Civita's boundary condition to replace $\sin \theta$ by $\sin 3\theta$.

This substitution replaces one non-linear boundary condition by another. It still preserves the essential feature but allows τ to be large.

The boundary condition then becomes

$$\frac{\partial\theta}{\partial\psi} = -\frac{g}{3c^3} e^{-3\tau} \sin 3\theta, \qquad \psi = 0 \tag{34}$$

which leads to

$$e^{-3i\omega} = 1 - 3A \exp\left(\frac{2\pi i w}{c\lambda}\right) \tag{35}$$

where $w = \varphi + i\psi$ is the complex potential and $c^2 = g\lambda/2\pi$, where λ is the wave-length.

The condition for breaking at the crest is $u - iv = 0$ when $w = 0$ which since $u - iv = ce^{-i\omega}$, leads to $3A = 1$.

In the neighborhood of the crest the wave then forms a wedge of angle 120°.

Moreover if we write $A = 2\pi a/\lambda$ where a is small we recover the ordinary linearized theory.

Thus the method discovered by Davies yields an approximation which applies over the whole range from waves of small amplitude to those on the point of breaking at the crest.

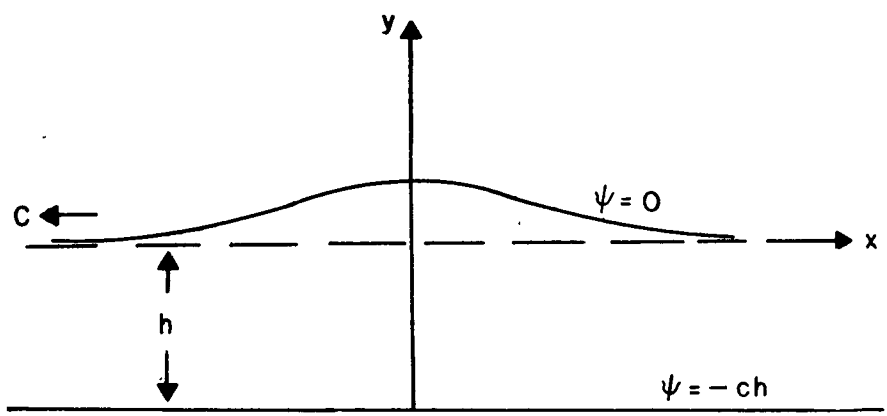

FIG 9

The method has also been applied by my assistant B. A. Packham [20] at the Royal Naval College to obtain a unique solution in closed form of the problem of the solitary wave, Fig. 9. The solution is so simple that I must give it

$$e^{-3i\omega} = 1 - \sin^2 kch \operatorname{sech}^2 \tfrac{1}{2}k(w - ich) \tag{36}$$

$$\frac{c^2}{gh} = \frac{\tanh kch}{kch}, \qquad 0 \leq kch \leq \tfrac{1}{3}\pi \tag{37}$$

In this result $k = 0$ corresponds to rest and $kch = \pi/3$ corresponds to breaking at the crest.

In connection with waves I should like to mention a striking phenomenon to which Dr. Gawn called my attention. He also demonstrated it to me at Haslar.

One method of making waves in a ship tank is by means of the periodic heaving motion of a cylinder. The result in general is to generate straight crested longitudinal waves.

If, however, the frequency of the oscillations of the cylinder exceeds a certain well defined critical value, transverse waves are also set up and a choppy sea results. You would hardly credit this without seeing it. I believe that no theoretical explanation of this phenomenon has so far been found. The suggestion that it is connected with the presence of the vertical walls of the tank is apparently without foundation. Here then is a matter to which research workers might usefully direct their attention.

What I have been describing is a method of approximation, very much superior to linearization but still an approximation.

It is therefore with gratification that we can turn to another method of approach to the free surface problem which reduces the attack on the two-dimensional case to the solution of a linear partial differential equation of the second order of parabolic type.

This method was originated by F. John [21] and reposes on the fact that on a surface of constant pressure the acceleration of the liquid relative to the acceleration due to gravity is a vector whose direction is normal to the surface of constant pressure. This statement is a simple consequence [22] of the equation of motion in the form

$$\boldsymbol{a} - \boldsymbol{g} = -\frac{1}{\rho}\nabla p. \tag{38}$$

If then the surface has the equation

$$z = f(\alpha, \theta), \qquad z = x + iy, \tag{39}$$

where α is a real Lagrangian parameter, we have

$$\frac{\partial^2 f}{\partial t^2} + ig = ir(\alpha, t)\frac{\partial f}{\partial \alpha}, \tag{40}$$

where $r(\alpha, t)$ is an arbitrary function. This is the equation just mentioned. Every two-dimensional continuous free surface must satisfy this equation, whether the motion is rotational or irrotational, steady or unsteady.

When the motion is irrotational it is not difficult to deduce the equation satisfied by the velocity potential.

Further John has shown that the form $y = k(x, t)$ of the free surface can be prescribed. For a particle, x, y are functions of t and the constant pressure condition then implies that x as a function of t satisfied a non-linear ordinary differential equation of the second order.

In the case of steady motion the problem can be simplified still further to depend on the solution of an ordinary second order linear equation of the form

$$f''(\beta) + ig = iS(\beta)f'(\beta), \tag{41}$$

where $S(\beta)$ is an arbitrary function.

It follows from these considerations that two-dimensional free surface problems can be reduced to the study of a limited class of differential equations. Nevertheless progress will necessarily depend on divining the proper form of the arbitrary function involved.

On the other hand we have here a means of generating an unlimited number of free surfaces by assigning the arbitrary function. In particular putting $S(\beta) = \omega$, a constant, leads to a trochoidal free surface, from which we can proceed to a trochoidal progressive wave.

Unfortunately this wave has to be associated with a moving ocean bed since the singularities of the progressive wave are no longer fixed.

But let us take heart. A function devoid of singularities is also devoid of interest: it is a constant.

A different representation of steady irrotational flow is due to H. Lewy [23]. By proper choice of axes and units the surface condition can be expressed in the form

$$\frac{dz}{dw}\left(\frac{dz}{dw} - 2i\frac{dy}{dw}\right) + \frac{1}{2y} = 0, \qquad \psi = 0, \qquad w = \varphi + i\psi. \tag{42}$$

Let us regard this as an equation in the complex domain involving a complex function $y = -\eta(w)$, real when $\psi = 0$.

The equation has the solution

$$z = -\, i\eta(w) + \int\left[\frac{1}{2\eta} - \left(\frac{d\eta}{dw}\right)^2\right]^{1/2} dw, \tag{43}$$

and provided the integral is real on some segment of $\psi = 0$, we have

$$y = -\, \eta(w) \qquad \text{on} \qquad \psi = 0, \tag{44}$$

and the point z describes a free streamline.

Lewy has also proved that the flow is analytic on the free surface in the steady case and has thus established that the formula gives the most general steady irrotational motion with a free surface.

M. J. Vitousek [24] has studied in detail flows obtained by attributing certain forms to $\eta(w)$ and in particular waves of trochoidal and cycloidal profile.

You will no doubt expect to hear something concerning the motion of a body through a fluid, typically the problem of the submarine well below the surface. The classical case governed by Kirchhoff's equations, when the vessel moves with velocity $u = u(t)$ and angular velocity ω in liquid otherwise at rest gives [25] for the force $\boldsymbol{R}_0$ and the moment $\boldsymbol{M}_0$

$$\boldsymbol{R}_0 = -\frac{\partial \boldsymbol{K}}{\partial t} - \omega_\wedge \boldsymbol{K}, \tag{45}$$

$$\boldsymbol{M}_0 = -\frac{\partial \boldsymbol{L}}{\partial t} - \omega_\wedge \boldsymbol{L} - \boldsymbol{u}_\wedge \boldsymbol{K}, \tag{46}$$

where

$$\boldsymbol{K} = \rho \int_S \varphi d\boldsymbol{S}, \qquad \boldsymbol{L} = -\,\rho \int_S \varphi \boldsymbol{r}_\wedge d\boldsymbol{S}. \tag{47}$$

The integrals are taken over the surface S of the body. M. D. Haskind [26] has this year extended these formulae to include the case where the liquid has velocity $\boldsymbol{v} = \boldsymbol{v}(t)$ at infinity and acceleration $\boldsymbol{f} = \frac{d\boldsymbol{v}}{dt}$. If Φ is the velocity potential, he defines $\varphi = \Phi - \boldsymbol{v} \cdot \boldsymbol{r}$ and the corresponding force and moment are

$$\boldsymbol{R} = \boldsymbol{R}_0 + M'\boldsymbol{f} \tag{48}$$

$$\boldsymbol{M} = \boldsymbol{M}_0 + \boldsymbol{v}_\wedge \boldsymbol{K} + M'\boldsymbol{r}_{c\wedge}\boldsymbol{f}, \tag{49}$$

where $\boldsymbol{R}_0$ and $\boldsymbol{M}_0$ are calculated by the formulae above, M' is the mass of liquid displaced and $\boldsymbol{r}_c$ is the position vector of the centroid of the volume displaced.

The motion of a submarine when gravity is taken into account has been discussed this year by P. V. Harlamov [27]. The problem he studies is that of a body totally immersed with buoyancy equal to the weight of the body so that in general the weight and buoyancy form a couple. A thorough going discussion of directions of permanent translation is made. The results are unsuited to reproduction here but I mention one as a specimen. If $\boldsymbol{r}$ is the position vector of the centroid of the displaced water relative to the centre of gravity of the body, permanent translation is impossible if the body is immersed so that the plane containing the vector $\boldsymbol{r}$ and one of the principal axes of the virtual mass ellipsoid is vertical but the vector $\boldsymbol{r}$ is not vertical.

Hydrodynamics, as appears sufficiently from the little I have been able to say presents many problems whose solution is to be desired. To keep in touch with the literature which is published in so many parts of the world is an increasingly difficult task. There is one obvious means by which knowledge could be diffused more quickly, namely by a greater readiness of authors to distribute their off-prints to those who are interested in the subject. Not every separatum will interest the recipient immediately, but nevertheless it makes him aware of what is happening. Were it not for Mathematical Reviews and Applied Mechanics Reviews much important work which is published in various journals and in many languages from Russian to Roumanian would be quite unknown. Moreover this work has often to endure a long delay before appearing in print and so to that extent is old before it is born.

I have quoted to you three pieces of work printed this year. Let me bring you bang up to date by quoting a result [28] which has not yet been published.

The equation of motion and equation of continuity of a viscous fluid, moving in a conservative field of force when incompressible, and under no external force when compressible, are satisfied identically by

$$\rho = \nabla^2\chi, \qquad \rho\boldsymbol{q} = -\nabla\frac{\partial\chi}{\partial t} \tag{50}$$

$$\Phi = \rho(\boldsymbol{q};\boldsymbol{q}) + \left(V - \frac{\partial^2\chi}{\partial t^2}\right)I + \nabla_\wedge\Psi_\wedge\nabla \tag{51}$$

Here Φ is the stress-tensor, $V = \rho\Omega$, where Ω is the potential of the field, χ is an arbitrary function and Ψ is an arbitrary symmetric two-tensor.

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