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ON THE STABILITY OF THE LAMINAR BOUNDARY LAYER

C. C. Lin
Massachusetts Institute of Technology

1. INTRODUCTION

This paper was originally intended to contain only a discussion of a new treatment of the central mathematical problem arising in the theory of stability of parallel flows. However, at the suggestion of the organizers of the Symposium, there is also included a brief survey of other aspects of the stability of the boundary layer and its transition to turbulence.

As is well known, when the speed of fluid motion is sufficiently high, it in general becomes fluctuating and turbulent even when the external conditions are steady. This is to be expected if one regards a body of fluid as a mechanical system with an infinite number of degrees of freedom. It is only when frictional forces are large, i.e., when the Reynolds number is sufficiently low that most of the possible modes of motion are damped out, resulting in the laminar flow.

The practical engineer is deeply interested in the problem of transition between laminar and turbulent flows because, among other things, turbulent flows are associated with much larger energy losses, and it is often desirable to try to avoid it (although in chemical engineering, it is sometimes desirable to promote turbulent mixing).

Unfortunately, with the present status of our knowledge, it is impossible to predict, with any great degree of certainty, the point of transition between laminar and turbulent flow in a boundary layer (say,) under a given set of flow conditions. One reason for this is the large number of factors that enter into the determination of transition; e.g.,

1. turbulence and noise in the free stream,
2. roughness of the surface,
3. curvature of the surface,
4. pressure distribution along the surface,
5. temperature of the surface.

However, through theoretical and experimental investigations, it has often been found possible to define proper dimensionless parameters for describing the influence of the various factors. Still, it would be very difficult to devise any comprehensive experimental program to include all these factors, and many others, e.g., compressibility, surface suction, etc. It thus appears that a sensible approach is to study the effect of varying the individual factors when other conditions are kept unchanged, and, at the same time, to try to understand the basic mechanism of transition. In this way, we may hope to be better prepared to cope with a given situation.

In Section 5, I shall return to the discussion of the effect of some of the above factors on transition. For the moment, let us consider briefly the inherent instability of the laminar flow and the growth of the small disturbances. Such processes are certainly of primary importance if we wish to understand the basic mechanism of transition.

2. INSTABILITY OF LAMINAR FLOW

It is now known that the instability of the laminar flow with respect to small disturbances does not always play the dominant role in a given transition phenomenon.

Its effect is often obscured by that of the other factors mentioned above. However, since the amplification of small disturbances does provide a definite mechanism of transition in available experiments and since it is an inherent instability for which definite mathematical theories can be developed, it is an important phase of the study of the mechanism of transition. Indiscriminant application of the theory to practical problems should, however, be avoided.

From the theoretical point of view, the study of the stability of simple exact solutions, such as the Couette flow and the plane Poiseuille flow, has merits over the discussion of the boundary layer. (Indeed, Taylor [1] questioned Tollmien's theory [2] of the instability of the laminar boundary layer, because the growth of its thickness in the direction of the flow is neglected with only a mathematical justification.) Fortunately, the two cases of exact solutions mentioned above are indeed typical cases illustrating the two basic types of instability. As is well known, the theory developed for such flows can be—and has been—applied to study the stability of boundary layers.

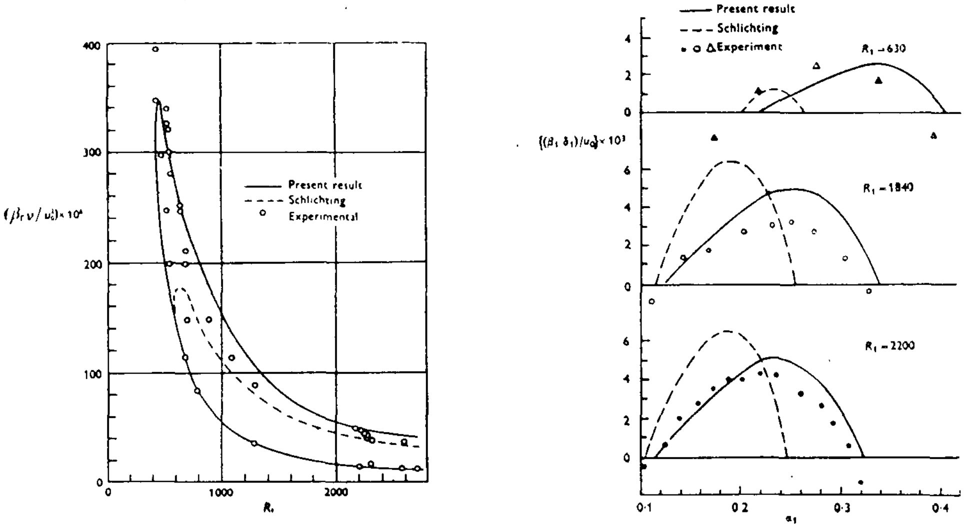
The first successful investigation of the instability of laminar flow was Taylor's work on the motion between rotating cylinders [3]. He obtained experimental confirmation of his theoretical calculations without great difficulty. The cause of instability is also not hard to find. Indeed, when the inner cylinder is rotating and the outer cylinder is at rest, the centrifugal forces tend to produce a secondary flow. The effect of viscosity in this case is then to produce the unstable basic flow, and to provide some damping effect on the secondary motion. On the other hand, when the outer cylinder is rotating and the inner one is at rest, the flow is expected to be stable.

A similar situation my be expected to hold in the case of the boundary layer over a concave wall. Here, the fast moving particles are closer to the center of curvature and the flow is expected to be unstable when the viscosity coefficient is too small to provide sufficient damping. The instability of such flows was first predicted by Görtler, {4] and Liepmann's experiments [5] generally support his conclusions.

The more interesting case is that of parallel flows. Here, one is concerned with wavy disturbances. Lord Rayleigh predicted the instability of the flow when the velocity profile has a point of inflection. There has been ample evidence to support the theory. However, in the case of flow through a channel, where the curvature of the profile is all of the same sign, Heisenberg [6] also predicted instability of the flow caused by the effect of viscosity. The theory was criticized by Noether [7] on mathematical grounds. To some of the mathematical issues, we shall return in Section 4. Tollmien [2] studied the case of the boundary layer and made the theory more convincing. But general acceptance of the theory came only after the calculations of Tollmien and Schlichting [8] were largely verified by the experimental measurements of Schubauer and Skramstad [9]. Figure 1 shows the comparison between theory and experiment. The dashed curves were obtained by Schlichting following Tollmien's approach. The solid curves were obtained by Shen [10] by using a method developed by Lin [11] on the basis of Heisenberg's approach. It is seen that the latter results agree better with experiments. Substantially the same neutral curve was earlier obtained by Tollmien and by Lin.

What is the mechanism of the instability of parallel flows? When there is a point of inflection in the velocity profile, the cause may be traced to a mechanism of vorticity redistribution in a parallel flow. There is a maximum *magnitude* of vorticity at the point of inflection. If we imagine a disturbance giving an exchange of particles on the two sides of the maximum (remembering that such an exchange must be made without change of the vorticity in a *perfect* fluid), we see that a small disturbance can exist without an essential change of the vorticity distribution of the main flow. Thus, there is no "restoring force" to resist the disturbance. On the other hand, with a monotone increase or decrease of vorticity, an exchange of two fluid elements results in an excess and a defect of vorticity; and it is known that such elements will tend to migrate back to the layer where it came from. Thus, there is stability of the basic motion.

Instability of a parallel flow without a point of inflection in the velocity profile



(a) (b)

Figure 1. **From Shen (10), by permission of the Journal of Aeronautical Sciences.**

must therefore be *caused* by viscosity. The viscous forces act as follows. Consider a travelling wave over a solid boundary. It can be shown that the viscous forces at the wall produce a phase shift of the oscillation which in turn produces a Reynolds stress. The Reynolds stress τ is found to be given by the formula [12]

$$\frac{\tau}{\rho \overline{v^2}} = \frac{1}{3\alpha\delta_0}, \tag{1}$$

where

α = wave number
δ_0 = thickness of the Stokes boundary layer
$= \sqrt{\frac{2\nu}{\omega}}$, $\frac{\omega}{2\pi}$ = frequency
$\overline{v^2}$ = mean square of the component of oscillation perpendicular to the wall.

Thus, there is a Reynolds stress acting in the same direction as the shear of the mean motion, if the wave is moving in the direction of the basic flow. At large distances, this stress must be zero. The transition occurs at the critical layer where the wave speed is equal to the flow speed and the effect of viscosity there is again dominant. In a region essentially free from the effect of viscosity, the Reynolds stress remains constant [13]. This Reynolds stress converts energy from the basic flow into the small oscillations. If this energy conversion is sufficiently large to overcome the damping effect of viscosity, the motion will be unstable. Thus, viscosity plays a dual role. It is at the same time the indirect cause of instability and the suppressor of the oscillations. The relative importance of these effects decides whether the flow is stable or not. This is why disturbances of a given frequency are stable both for large and for small Reynolds numbers; because in the one case there is too much damping; and in the other case, there is insufficient excitation.

It has been found possible [12] to pursue this line of reasoning to derive, in a simple manner, quantitative relations between the Reynolds number of the basic flow and other characteristic parameters of the oscillations. These relations could be obtained otherwise only by following very complicated mathematical analysis of the secular determinant of the characteristic value problem.

3. OSCILLATIONS OF FINITE AMPLITUDES

The unstable oscillations predicted by the instability theory are still far from turbulent flow. In fact, experimental observations and theoretical considerations suggest the following five stages of the phenomenon of transition from laminar to turbulent flow:

1. instability of laminar flow
2. amplification of small disturbances
3. amplification of finite disturbances
4. breakdown of finite disturbances
5. formation of turbulence
    - (a) formation of turbulent spots
    - (b) spread of turbulent spots
    - (c) fully developed turbulence.

One of the current issues is whether step 4 is basically a two-dimensional or three-dimensional phenomena. I shall not elaborate on this point, as Mr. Klebanoff will no doubt tell you something more about it in his discussion. I shall now turn to a discussion of the space structure of these oscillations.

The space structure of the oscillations indicates clearly that they are far from turbulent motions, since their scales are far too large. In the direction of the flow, they are of the order of the boundary layer thickness δ; even its scale across the boundary layer is only of the order of $\delta R_\delta^{-1/3}$ close to the boundary. On the other hand, turbulent motions have a vorticity scale of the order of $\delta R_\delta^{-1/2}$ which is still smaller by a factor $R_\delta^{1/6}$. Thus, we must have a further breakdown of the motion into smaller eddies. This is in line with our general concepts of turbulent motion where energy is fed into motions of smaller and smaller scales, and is eventually dissipated into heat. Either the small scale motions are produced as harmonics of the oscillation when its amplitude becomes large, or they are made possible by distortion of the basic flow. In either case, it is necessary to study oscillations of finite amplitudes. By considering the distortion of the basic flow Meksyn and Stuart [14] have made some interesting calculations of such oscillations. More recently, Stuart [15] calculated the development of such motions in the course of time.

The study of higher harmonics, on the other hand, requires a more complete knowledge of the theory of small oscillations. It is known that a perturbation theory based on non-linear oscillations must be based on a complete set of normal modes in the linear theory. In the usual calculations, only *one* of the modes is calculated. To obtain the other modes,* an improved mathematical theory (such as that outlined in Section 4) is desirable.

I shall therefore refrain from discussing in detail the possible mechanisms leading from regular oscillations to the threshold of transition,** except to make the following two observations. First, there is a general concept*** originally suggested by Landau [17] who regards the final state of turbulent motion as reached through the successive instability of the flow system with respect to more and more modes as the basic flow is being distorted by the occurrence of the *lower* modes. Secondly, I wish to point out that a more extensive study of the linear theory, particularly of the amplitude distribution of these oscillations, does show that damped oscillations have a scale $\delta R_\delta^{-1/2}$ even

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* Some calculations of higher modes were recently made by Grohne.

**Cf. Dryden [22] p. 57 for some discussions of this point.

*** Also independently set forth by J. T. Stuart.

away from the boundary. The picture is then possible that the unstable oscillations, when their amplitudes are large, might feed energy into the oscillations basically at the scale of turbulent flow. The amplitude of the latter oscillations also show a change-over from large to small scales like the intermittency phenomenon in fully developed turbulence or the phenomena of turbulent spots in transition.

The above discussions (and other experimental evidence at the Bureau of Standards) suggest the importance of a thorough study of the amplitude distribution of the oscillations, especially that of the damped modes. We therefore turn to the discussion of the improved mathematical theory of such oscillations.

4. IMPROVED MATHEMATICAL THEORY

The small two-dimensional disturbances in a parallel flow are assumed to be given by a perturbation stream function of the wavy type

$$\psi' = \phi(y)e^{i\alpha(x-ct)} \tag{2}$$

such that

$$u' = Re\left(\frac{\partial\psi'}{\partial y}\right), \qquad v' = Re\left(-\frac{\partial\psi'}{\partial x}\right) \tag{3}$$

The linearized Navier-Stokes equations then lead to the Orr-Sommerfeld equation

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R\,[(w-c)(\phi''-\alpha^2\phi) - w''\phi] \tag{4}$$

where $w(y)$ is the basic velocity profile, R is the Reynolds number, and all the variables involved are dimensionless. The conditions at the solid boundary are

$$\phi(0) = \phi'(0) = 0; \tag{5}$$

and those for the outer edge of the boundary layer are

$$\phi, \phi' \rightarrow 0, \qquad \text{as} \quad y \rightarrow \infty. \tag{6}$$

This leads to a characteristic value problem with a secular determinant

$$F(\alpha, R, c) = 0. \tag{7}$$

Instability of the flow obtains when $c_i > 0$.

As is well-known, the solution of the problem is usually based on the fact that αR is large. However, the usual asymptotic solutions used for the calculation of the stability characteristics are valid either in the immediate neighborhood of the critical point where $w = c$, or away from the critical point. Strictly speaking, it is not justifiable to apply both types of solutions at the same time. It is due to a fortunate coincidence of circumstances that the old process can be justified from a numerical point of view in certain important special cases—including the Blasius flow and the plane Poiseuille flow. Actual difficulties arise in other important cases; for example, in the extension of the theory to supersonic boundary layers.*

We are therefore led to seek asymptotic solutions which are uniformly valid in a finite region containing the critical point. Such solutions are desired for at least two other reasons mentioned in Section 3; namely: (1) In trying to study the physical nature of such oscillations, the calculation of the amplitude distribution is essential. Such a calculation cannot be conveniently and accurately made by using the old solutions. (2) In an attempt to study oscillations of finite amplitude, the development of the non-

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* The most illustrative example is the calculation made by Bloom who followed the older theory faithfully and arrived at incorrect results. Solutions adequate for a large number of cases have been found recently by Dunn and Lin (See Ref. 18, p. 72)

linear theory must be based on a well-developed linear theory, including the amplitude distribution.

While the difficulty has been generally recognized by workers in this field, Tollmien [19] was the first to attempt a solution to the problem. His solution is restricted in three ways: (a) only the case of neutral oscillations is treated; (b) only the first approximation is given and there is no obvious algorism to obtain the higher approximations; (c) only the function and its first derivative are considered. Wasow [20] treated the general complex case including damped and amplified oscillations, but his treatment was not free from the restrictions (b) and (c). There is also an obvious difficulty in reconciling his "uniformly valid solution" with his own solution away from the singular point. This difficulty can, however, be easily removed, e.g., by using the transformation [15] used in the method to be described. I shall not go into further discussions of these issues, because Dr. Wasow is here to present a comparison of the various solutions including another one due to Langer. I wish merely to state that the solution to be presented appears to be simple, complete, and most readily applicable.

a. *Older methods*

Let us begin by considering the older methods. For αR large, it is known that formal asymptotic solutions of (4) may be found as follows:

$$\phi = \phi^{(0)}(y) + \frac{1}{\alpha R}\,\phi^{(1)}(y) + \cdots, \tag{8}$$

$$\phi = e^{\pm\sqrt{\alpha R}\,Q(y)}\left[f^{(0)}(y) \pm \frac{1}{\sqrt{\alpha R}}\,f^{(1)}(y) + \cdots\right], \tag{9}$$

where

$$Q(y) = \int_{y_c}^{y} [i(w - c)]^{\frac{1}{2}}\,dy, \qquad f^{(0)}(y) = (w - c)^{-5/4}, \cdots \tag{10}$$

and y_c is the critical point where $w(y) = c$. One of the formal asymptotic solutions of the form (8) (commonly denoted by ϕ_1) is regular, while another (commonly denoted by ϕ_2) exhibits a logarithmic behaviour at the critical point $y = y_c$. The singularity in this asymptotic solution and the solutions of the form (9) shows that they can be valid only when the immediate neighborhood of the critical point y_c is excluded. The multiple-valued nature of the singularity shows further that the region of validity must be restricted to certain sectors in the complex plane in the neighborhood of the point y_c. One has therefore to study very carefully the nature of the solution in that neighborhood. In the older theory, this is accomplished by a change of scale, i.e., by the introduction of the new variable

$$\eta = (y - y_c)/\epsilon, \tag{11}$$

followed by expansion of the solution of (4) in the form

$$\phi(y) = \chi(\eta) = \chi^{(0)}(\eta) + \epsilon\chi^{(1)}(\eta) + \cdots \tag{12}$$

Such expansions are valid for finite values of η. It has been shown, [11] however, that, for large η, the asymptotic expansion of (12) can be formally identified with the solutions (8) and (9) after regrouping of terms.

Two things become clear from such an investigation. First, to obtain $\phi^{(0)}(y)$, we need *all* the terms in (12) showing clearly that a few terms from (12) will not be accurate enough for $y - y_c$ finite. Similarly, a few terms from (8) will not yield accurate results for finite values of η, i.e., for $y - y_c = O(\epsilon)$. Unfortunately, for many applications, (including the case of the boundary layer) one boundary point corresponds to finite or large $y - y_c$ while the other corresponds to finite η. Thus neither (8), (9) nor (12) can be adequately used for the solution of a characteristic value problem. This is a central mathematical difficulty in the stability theory.

The second point is that the asymptotic matching is restricted to certain sectors of the complex plane. This leads to the conclusion (later confirmed by more careful mathematical investigations) that the damped and the self-excited solutions are not complex conjugates in the limiting case of infinite Reynolds number even though the invicid equations appear to lead to that conclusion.

We shall not go into further details of solutions of the type (12) as it will not be used in the following investigations.

b. *Uniformly valid asymptotic solutions of the Orr-Sommerfeld equation.*

To obtain uniformly valid asymptotic solutions of the Orr-Sommerfeld equation, we apply a theory developed for a class of differential equations of that type. In fact, we need only to know the structure of the solution to obtain the desired results. This will now be presented. The general theory will be presented elsewhere.

According to the general theory, asymptotic solutions of the Orr-Sommerfeld equation may be found in the form

$$\phi = K_0 u + K_1 u' + \lambda^{-2}(K_2 u'' + K_3 u'''), \tag{13}$$

In this formula, u is a solution of the equation

$$u^{iv} + \lambda^2(zu'' + \gamma u' + \beta u) = 0 \tag{14}$$

$$\gamma = \lambda^{-2}\gamma^{(1)} + \cdots \tag{14a}$$

$$\beta = \beta^{(0)} + \lambda^{-2}\beta^{(1)} + \cdots$$

where z is related to y by

$$z = \left[\tfrac{3}{2}\int_{y_c}^{y}[i(w - c)]^{\frac{1}{2}}\,dy\right]^{\frac{2}{3}}, \tag{15}$$

and the parameters λ^2 and $\beta^{(0)}$ are given by

$$\lambda^2 = \alpha R, \qquad \beta^{(0)} = -(w_c''/w_c')(iw_c')^{-\frac{1}{3}} \tag{16}$$

In (13) we also have

$$K_i = K_i^{(0)} + \lambda^{-2}K_i^{(1)} + \cdots \qquad i = 1,2,3,4 \tag{17}$$

where $K_i^{(j)}(z)$ are regular in z. We note that to each solution $u(z)$, there is a corresponding solution $\phi(z)$.

According to the general theory, the coefficients $K_i^{(j)}(z)$ are essentially determined except for certain arbitrary constants. Furthermore, the initial approximation $[K_1^{(0)}(z), K_2^{(0)}(z), K_3^{(0)}(z), K_4^{(0)}(z)]$ is uniquely determined up to a common numerical factor. The actual calculation of the functions $K_i^{(j)}(z)$ is, however, to be done by solving certain differential equations and is in general very complicated.

A practical method of obtaining the functions $K_i^{(j)}$ can be devised by merely using the structure of the solution (13). If we calculate the formal asymptotic expansions of ϕ, for finite values of $y - y_c$, by using the known expansions for u, we must obtain asymptotic solutions expressible in terms of the solutions (8) and (9). Since this can be done for *four* linearly independent solutions u, we have essentially *four* sets of relations (for each order in λ^{-2}) for the determinations of K_i. In the above argument, the constants of linear combinations are left undetermined. However, the *regularity* condition for the coefficients $K_i^{(j)}(z)$ is found sufficient for the determination of all the constants to the extent specified in the general theory. Further details of the mathematical theory will be given in the complete publication. Let us now go on to make some remarks on the problem of transition.

5. SOME REMARKS ON TRANSITION

I shall not attempt to give a general survey of the transition problem but wish merely to discuss a few points about the relation between the stability theory and the transition phenomenon.

First, we know that the minimum critical Reynolds number obtained from stability calculations is usually much lower than the transition Reynolds number. Still, there are two useful applications of the stability theory. First, the parameters occurring in the instability theory usually turn out to be the right ones for correlating transition data. For example, the Görtler parameter

$$R_\theta(\theta/r)^{\frac{1}{2}} \tag{18}$$

for the boundary layer over a concave surface. Another example is the form parameter

$$H = \delta^*/\theta \tag{19}$$

which is found to be useful for correlating stability calculations of the boundary layer with suction or pressure gradient or both. A third example is the surface temperature. In all the three cases just cited, the trend of variation of the critical Reynolds number with the parameter in question is the same for stability theory and transition experiment.

Some results obtained by Klebanoff, Schubauer, and Tidstrom [21] give an even more direct correlation between theory and experiment. They found that below the minimum critical Reynolds number of the stability theory, it is practically impossible to induce transition either by roughness or by a spark. The turbulence spot produced by a spark does not spread until it reaches beyond the location defined by the minimum critical Reynolds number. However, if one would consider the situation in a pipe, it is clear that we cannot yet consider such a situation as a general rule for transition from laminar to turbulence. On the other hand, if further experiments can show its validity even only for boundary layers, the practical value of the stability theory would be greatly enhanced.

I wish now to make a few remarks about factors which influence transition but which lie outside the scope of the usual stability theory. Two such important factors are

a. the level of turbulence in the free stream,
b. roughness of the surface.

The influence of the level of turbulence is described by the well-known parameter

$$\frac{u'}{U}\left(\frac{x}{L}\right)^{\frac{1}{5}} \tag{20}$$

given by Taylor [1]. I wish now to turn my attention to the effect of roughness.

Dryden [22] analyzed a series of experiments (especially those of Tani and Hama) where the transitions is induced by a single two-dimensional roughness element and arrived at the correlation parameter

$$k/\delta_k^*, \tag{21}$$

where δ_k^* is the displacement thickness of the boundary layer at the roughness elements of height k. In the case of distributed roughness, Klebanoff, Schubauer and Tidstrom [21] found that such a parameter was not good for correlating experimental data (Fig. 2). They instead proposed the parameter

$$u_k k/\nu, \tag{22}$$

where u_k is the velocity of the flow at the edge of the roughness element. The data still show some scatter, but much less (Fig. 3). Now from purely dimensional considerations, it appears that *both* parameters must be used, since the location of the roughness

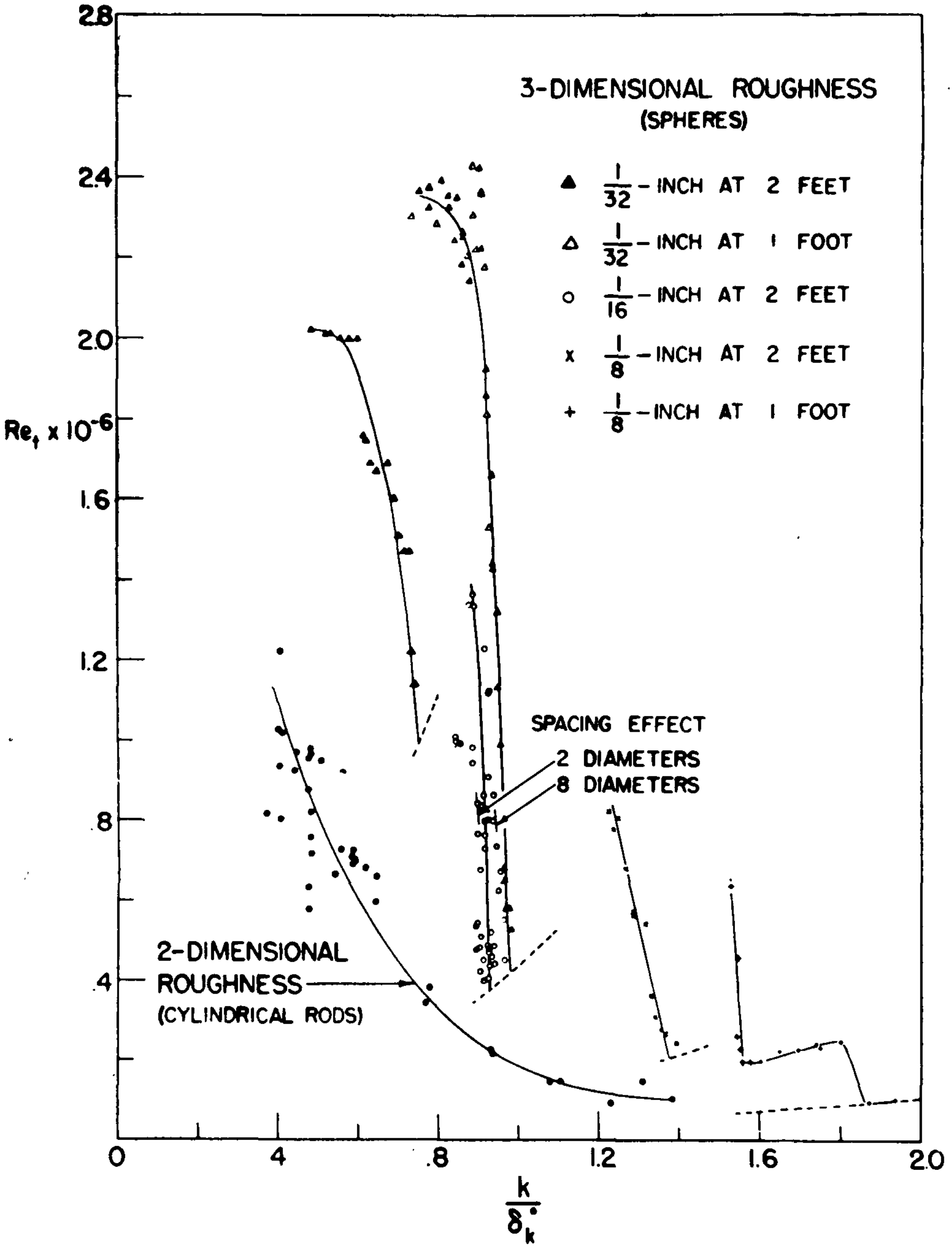

Figure 2. From Klebanoff, Schubauer and Tidstrom (21), by permission of the Journal of Aeronautical Sciences.

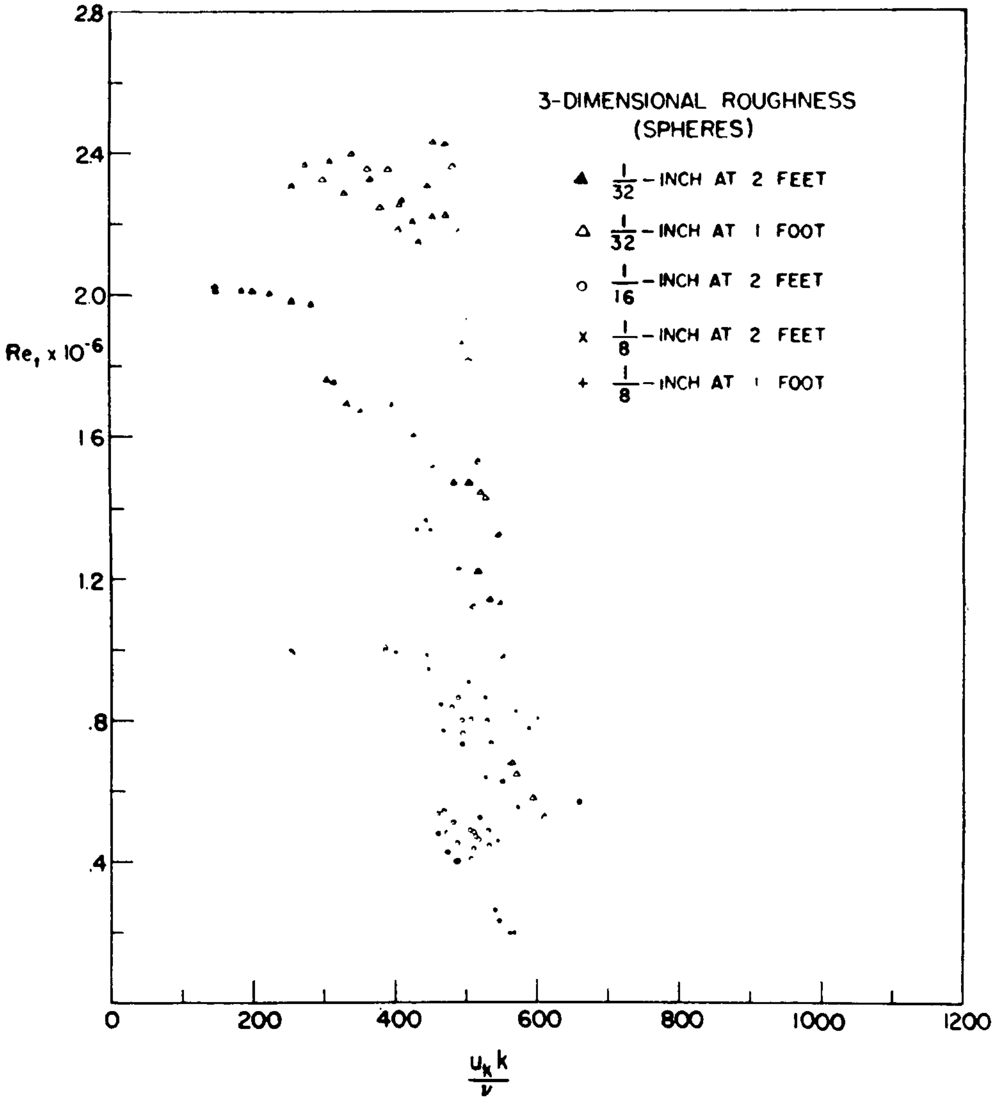

Figure 3. From Klebanoff, Schubauer and Tidstrom (21), by permission of the Journal of Aeronautical Sciences.

element and its height are *two* independent dimensional parameters [cf. Ref. 22]. From them, together with the flow variables, one can indeed construct *both* of the above parameters, which appear to be independent. Thus, if one tries to obtain a one-parameter correlation, there must be an imposed condition (or some other subtle reason) which would reduce a degree of freedom. It therefore seems desirable that experiments should be performed for a wide range of values of *both* parameters in order to clarify the role of roughness.

I wish now to make another suggestion for experiments to decide on the relative significance of two effects on transition. The heating of the solid surface increases the stability of the boundary layer in water by decreasing the viscosity in the immediate neighborhood of the wall, making the profile fuller. However, if a roughness element is present, the roughness parameter $\rho u_k k/\mu$ would be increased indicating early transition. Such experiments would then clarify the relative importance of the two effects on transition: the velocity profile and the roughness elements. When the roughness effect is dominant, the stability theory might actually give the wrong trend for the effect of heating of the surface.

In conclusion, I wish to summarize my discussion as follows. When the inherent instability of the boundary layer is the primary cause of transition, the process can be described in the five stages outlined in Section 3. The first stages may be called the instability of the boundary layer, while the last stages represent transition proper. Thus, although existing results from the instability theory do throw much light on the transition problem, some further effort is needed to determine quantitatively the behavior of the oscillations of finite amplitudes for the purpose of understanding the transition mechanism. On the other hand, even for practical purposes, the presence of so many factors influencing the transition phenomenon makes it almost necessary to study its basic mechanism. I hope that further joint theoretical and experimental efforts on this fascinating subject will lead us to an even better understanding of the phenomenon and will offer us a better basis for its practical control.

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DISCUSSION

W. Wasow

I wish to add some purely mathematical remarks that may help to clarify the relation between Lin's new asymptotic technique and the method I employed in my paper [1] in the Annals of Mathematics, v.58, pp. 223-252 (1953).

Most work on the asymptotic behavior of the solutions of linear differential equations for large values of a parameter λ is based on the idea of comparing the *given differential equation* $L_g[y, \lambda] = 0$ with a *related differential equation* $L_r[y, \lambda] = 0$

whose asymptotic theory is already known and whose solutions are conjectured to have the same asymptotic form as those of the given equation, at least up to terms of some given order $0(\lambda^{-m})$. Since the asymptotic character of the solution of such differential equations is apt to change abruptly from one region of the complex plane to another (so-called "Stokes phenomenon"), it is often difficult to find a related equation whose solutions resemble asymptotically those of the given equation in a domain as large as the physical applications of the theory require. Only very simple types of differential equations can be *directly* analyzed asymptotically. The most common tools for this are integral representations resulting from functional transformation such as the Laplace transformation. Therefore, the asymptotic investigations along the lines described here begin frequently with the study of some *special differential equation* $L_s\,[y, \lambda] = 0$ for which a direct asymptotic theory can be given.

The complete procedure consists then of the following parts.

1. Choice and asymptotic analysis of a suitable special differential equation $L_s\,[y, \lambda] = 0$.
2. Construction of a suitable related differential equation $L_r\,[y, \lambda] = 0$ based on $L_s\,[y, \lambda] = 0$. The related equation should be such that the asymptotic form of its solutions follows readily from that of $L_s\,[y, \lambda] = 0$, e.g., by a transformation of variables.
3. Comparison of $L_r\,[y, \lambda] = 0$ and the given differential equation $L_g\,[y, \lambda] = 0$, i.e., proof that the solutions of the two equations differ by terms that are $O(\lambda^{-m})$, $m \geqslant 0$, in the considered domain of the independent variable z. This part frequently required tedious appraisals.

The given differential equation studied by Lin is of the form

$$L_g\,[y,\lambda] \equiv y^{(4)}+\lambda^2\,[P(z,\lambda)y''+Q\,(z,\lambda)\,y'+R\,(z,\lambda)\,y]=0. \tag{1}$$

This is somewhat—but not essentially—different from the differential equation treated in [1]. The Sommerfeld-Orr equation is a special case of (1). In order to obtain the results needed in the hydrodynamic applications, z should be permitted to range over a domain containing one simple zero of $P(z, \infty)$. Let us assume, e.g., that

$$P(o, \infty) = 0, \qquad \left(\frac{\partial P}{\partial z}\right)_{\substack{z=o\\ \lambda=\infty}} \neq 0. \tag{2}$$

It is the inclusion of this *"turning point"* $z = 0$ into the z-domain that causes most of the difficulties.

The simplest differential equation of the type (1), (2) which still has all its essential features is

$$L_s[y,\lambda] \equiv y^{(4)}+\lambda^2(y''+\alpha y'+\beta y),\ \alpha,\beta \text{ constants.} \tag{3}$$

This is the special equation used by Lin and—with $\alpha = 0$—by myself. Its full asymptotic theory, for $\alpha = 0$, was developed by myself in [1] and in [2], Annals of Math, v. 52, pp. 350-361 (1950), by means of complex Laplace transformation.

In a sufficiently small region *not containing the turning point* asymptotic expansions for the solutions of (1), and hence of (3), have been known for a long time (cf. [3], Annals of Math, v. 53, pp. 852-871 (1948)). Both Lin and myself use these expansions for the construction of a related equation. First, a preliminary transformation of the independent variable has to be performed. In order to explain the meaning of this transformation without entering into technicalities we refer to the fact, proved, e.g. in [3], that the previously mentioned abrupt change of asymptotic character takes place upon the crossing of certain curves ("transition lines"), which, in the case of the equation (3), form three rays issuing at equal angles from $z = 0$. By properly transforming the independent variable the transition lines of (1) can be made to coincide with those of (3). Let us assume that (1) is already written in terms of this new variable. Now let $u_j\,(z,\lambda), j = 1, 2, 3, 4$ be a suitable fundamental system of solutions of (3) of known asymptotic form. Then I determine, in [1], four functions $v_j\,(z,\lambda)$ of the form

$$v_j = k_j(z)u_j, \qquad j = 1, \cdot \cdot \cdot , 4, \tag{4}$$

where the $k_j(z)$ are chosen so that the $v_j = v_j(z, \lambda)$ have, in some region away from $z = 0$, the same asymptotically leading term as the solution of (1). The functions $k_j(z)$ may fail to be regular at $z = 0$, which makes the subsequent arguments more involved. The related equation used in [1] is then the easily constructed fourth order linear differential equation that has these v_j, $j = 1, \cdot\cdot\cdot, 4$ as a fundamental system.

By contrast, Lin computes a different set of four functions, which we shall also call v_j, by setting

$$v_j = C_1(z,\lambda)u_j + C_2(z,\lambda)u_j' + C_3(z,\lambda)u_j'' + C_4(z,\lambda)u_j''', \qquad j = 1, \cdot \cdot \cdot , 4 \tag{5}$$

and requiring that in a region away from $z = 0$ these v_j have the same asymptotic form as a fundamental system of (1). Since the $C_j\,(z,\lambda)$ are permitted to depend on λ, Lin has much more freedom in his construction so that he can construct functions v_j that agree with the solutions of (1) up to an arbitrary but finite order $0\,(\lambda^{-m})$. His coefficients are of the form

$$C_\nu(z,\lambda) = \sum_{n=1}^{m} r_{\nu n}(z)\lambda^{-n}, \tag{6}$$

and it turns out that the $r_{\nu n}\,(z)$ can be chosen so as to be regular at $z = 0$. These are important improvements.

The work of part 3 of the general procedure outlined above, in which the related and the given equations have to be compared as to their asymptotic form in a domain *including* the turning point $z = 0$, has not yet been fully carried out for Lin's method. A well developed technique for this task exists, but its application to Lin's differential equations may still involve considerable labor.

T. B. Benjamin

I believe it may be appropriate here to refer to another problem of hydrodynamic stability, one which has received only little attention compared with the stability problems for the boundary layer and pipe flow, which have been in the forefront of theoretical hydrodynamics for about a half a century now.

The case I would like to mention is the flow of a viscous liquid in a thin film, bounded on one side by a plane wall, and on the other by a free surface. Flow of this type is a fairly common experience.

For instance, rain water may sometimes be observed running in a thin sheet on the outside of a window. Again, one may often observe motor oil, or paint, or mayonnaise, flowing in a thin sheet down the sides of some solid object immersed for sampling purposes.

The general mechanics, and in particular, the stability of such flows, is an important question in chemical engineering, for instance, particularly where the film is in contact with a gas stream.

Some work on this problem has been done recently in Cambridge, both in the Chemical Engineering Department and in the hydraulics laboratory, and I would like to outline what the problem was, briefly, and what progress has been made on the theoretical side at least, since this seems the aspect most appropriate to this session of the Symposium.

It is a well-established fact that when a film of water runs down a vertical plate, the flow is laminar and uniform until the Reynolds number, based on film thickness, is raised to a value of the order of unity. Waves with a well defined periodicity then appear, their velocity downwards being about three times the mean flow velocity, and it is in fact, quite straight forward to measure all features of the transition. It should be noted that this transition is far removed from the development of turbulence, unlike that in the more familiar cases of instability of laminar flow. The film cannot be described as fully turbulent until the Reynolds number is increased to values in the hundreds, at least.

The influence of surface tension must be recognized in any theory which is to be of practical use for water or other common liquids. Surface tension is the only restoring force when the plate is vertical, although, of course, gravity exerts a restoring force when the plate is inclined.

During his talk on Monday, Professor Lighthill remarked that the existing theory of roll waves, as for instance developed by R. F. Dressler, is applicable to flow in thin films. Since this theory has been developed by Dressler for a generalized law of frictional resistance, one may use his results simply by substituting the appropriate resistance of a laminar film. In this case, of course, the resistance is directly proportional to flow speed, and inversely to the film thickness.

In fact, a lengthy treatment having much in common with this approach, was published a number of years ago by the Russian scientist Kapitza. However, a more exact treatment of this problem is possible. The problem may be formulated as a characteristic-value problem in the familiar manner of stability theory for the boundary layer. The only differences are that special boundary conditions are needed to account for the free surface, and that, in the range of interest, the Reynolds number is small. Thus one may resort to asymptotic expansions in ascending powers of Reynolds number, instead of descending powers as in the usual case.

Dr. Yih has already published a description of this method of approach, but restricted his treatment to the case of zero surface tension, and to terms of only the first power of αR. Actually the Reynolds number is about two or three in the range of greatest interest, but alpha is kept fairly small due to the result of surface tension. αR is considerably less than one, but couldn't be regarded as a small quantity in the usual sense.

In our work at Cambridge, we have considered expansions up to the third power of αR, which has required a power series expansion of the stream function in terms of the coordinate perpendicular to the flow as far as the 15th power. It remains to prove the convergence of this approximation, but the physical aspects are sufficiently reassuring.

The result of a good deal of heavy algebra is a series of curves of neutral stability, with R and alpha as coordinates in the familiar fashion of stability theory, and with Weber number, representing surface tension, and the slope of the plane as parameters.

A good agreement with experiment has been found, and we have also made some progress with the more difficult problem where the film is stressed by a turbulent gas stream confined with an additional fixed boundary.

In conclusion, I would very much like to comment the film problem for further study by workers in this field. It has a number of important practical aspects. I hope, for instance, that it will attract the great talents of Professor Lin.

P. S. Klebanoff

The comments I have to make consist of some of the recent results obtained from a continuing experimental investigation of boundary layer transition which is being conducted at the National Bureau of Standards under the sponsorship of the National Advisory Committee for Aeronautics. They bear on Dr. Lin's paper in that they deal with the region of finite amplitude with which we are all concerned. One objective of the investigation is to observe by hot-wire techniques the growth and evolution of a wave from its source to transition. It was possible to carry out this objective by using the vibrating ribbon technique used so successfully by Schubauer and Skramstad in their experimental confirmation of the Tollmien-Schlicting stability

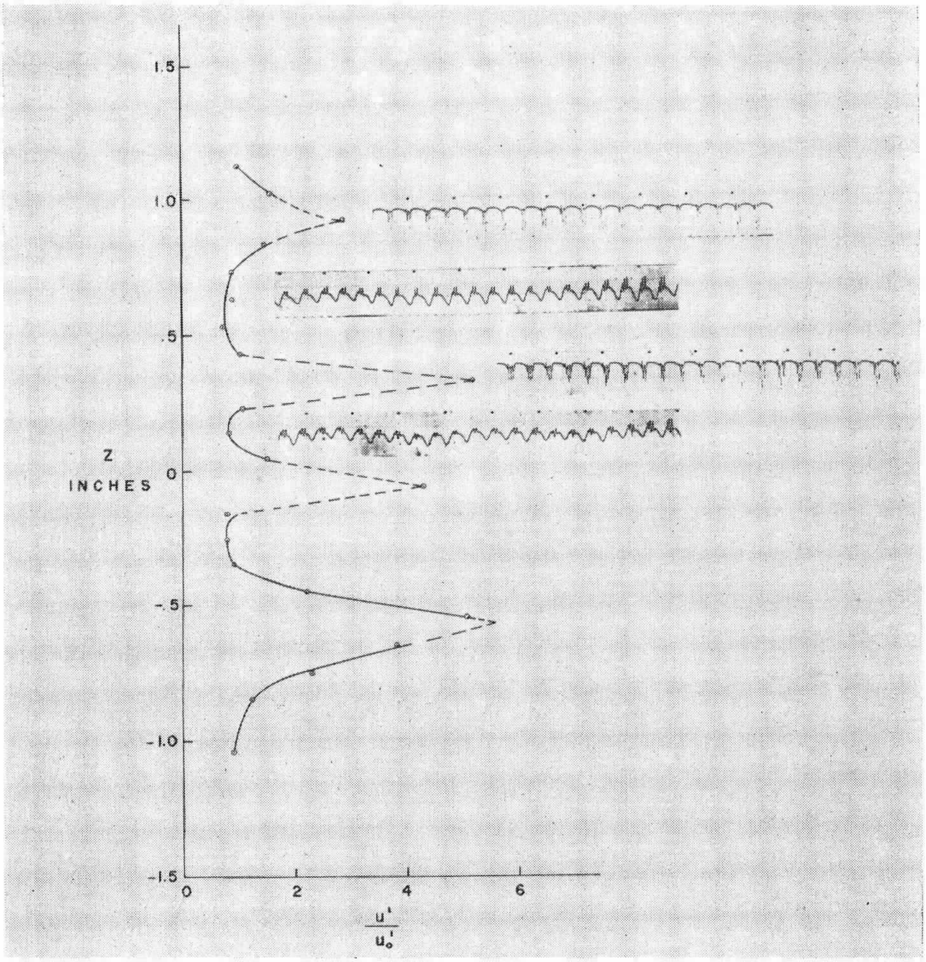

Figure 1.

theory. The study is being conducted on a flat-plate boundary layer with zero pressure gradient and wind speed of 50 feet per second.

Figure 1 represents the intensity of the longitudinal fluctuation and shows its distribution in the z-direction, which is the direction parallel to the plate and normal to the flow. The measurements were made one foot downstream from the vibrating ribbon, at which point transition began, and at a distance from the surface equal to half the boundary layer thickness. The intensity is plotted in terms of the root-mean-square value and is referred to the intensity u_0', which is the value at a fixed point 1.2 inches above the center line of the plate, and was on the order of 1 percent. Although the vibrating ribbon excites a wave which is 2-dimensional and uniform along its crest for a distance of about one foot there has developed a strong 3-dimensionality as shown by the large variation of intensity. This variation had its beginning well upstream from transition where the wave is still quite pure. The breaking pattern associated with transition is also shown. At the peak the breaking pattern is different from that in the hollow. The breaking pattern at the peak consists of bursts of high frequency fluctuations occuring each cycle on the low-velocity side of the primary wave. In the hollow the fluctuations are of much smaller intensity and occur on the high velocity side of the wave. The frequency of these fluctuations is about six times that of the primary wave which had a frequency of 145 cycles per second. Frequencies this high would indicate that these high intensity fluctuations are probably not a consequence of non-linear effects but are more in the nature of a secondary instability. However, more study is called for in this connection. Preliminary measurements have indicated that these high frequency fluctuations at the peak and in the hollow occur at the same time, indicating that the smaller intensity fluctuation may be just an excitation caused by the very intense pattern at the peak, and that the initial breakdown of laminar flow has a very short extent in the z-direction which might be described loosely as point-like. These localized regions are spaced along the wave front giving the impression of streets along which a succession of bursts were occuring.

In Figure 2 there is shown the breakdown of laminar flow from the beginning of transition to where it becomes fully turbulent. It was too inconvenient to move the hot wire through the transition region; instead the oscillograph traces shown in the figure were obtained by moving the transition region over the hot-wire by continuously increasing the ribbon amplitude. The hot-wire was at a position corresponding to a peak in the intensity distribution shown in Figure 1. At first we see the rather pure wave somewhat distorted with a flat-top appearance on the high velocity side, then the appearance of high-frequency fluctuations which consume more and more of the primary wave, and then the onset of fully turbulent flow. This process, i.e. from the beginning of transition to where we have fully developed turbulent flow, takes place within a distance equivalent to 50 boundary layer thicknesses.

An attempt was made using the china clay technique to determine whether the 3-dimensionality associated with the intensity of the wave in the region where the wave was still quite pure could possibly be due to Görtler-type vortices. The drying pattern on the china-clay surface which was initially sprayed with oil of wintergreen is shown in Figure 3. The dark line at the left of the figure is the vibrating ribbon. The limitations of the method prevented any definite conclusions as to the existence of Görtler-type vortices in the region before transition. As far as could be determined the street-

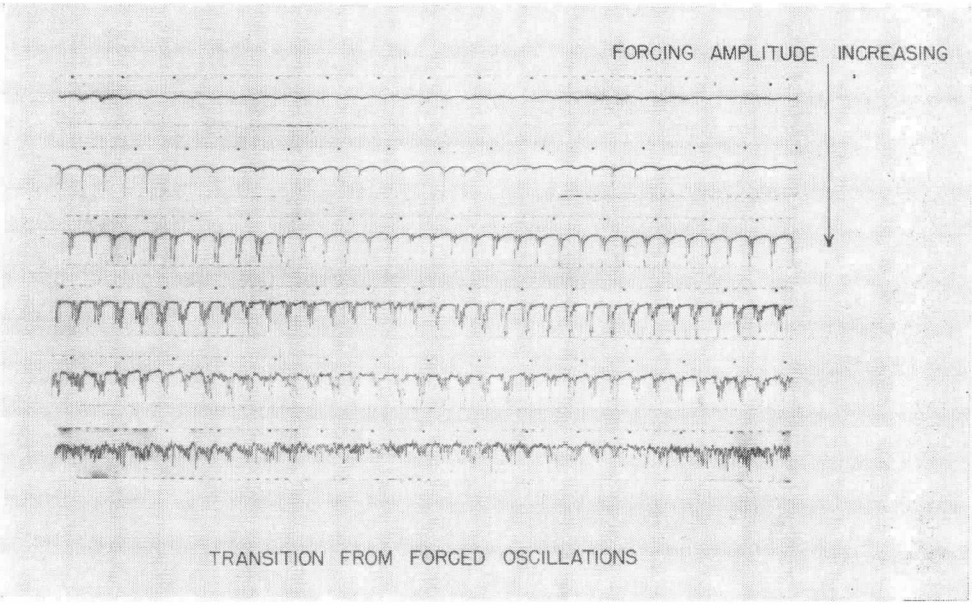

Figure 2.

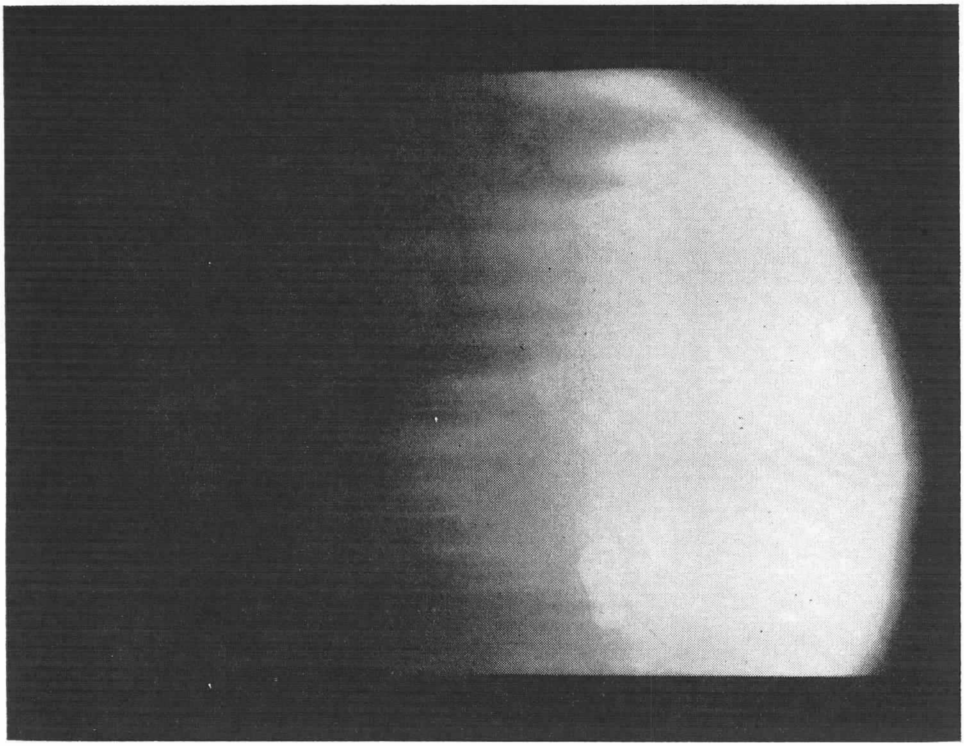

Figure 3.

like array manifested by the transition process as shown began only after the laminar flow was disrupted and turbulent bursts were present.

An important aspect especially in the region of finite amplitude is the amplification of the wave. Figure 4 shows preliminary measurements of the amplification of the wave from the ribbon to transition at a frequency of 160 c.p.s. for different input levels, and was measured along a line corresponding to a peak in the intensity distribution. The intensity u_0' is the level at x_0 which is a reference position two inches downstream from the ribbon. The distance from the surface at which the measurements were made was at about 0.2 the thickness of the boundary layer, which is where the maximum in the intensity distribution across the boundary layer occurs in the Tollmien-Schlicting theory. At input levels of 0.2 and 0.3 percent transition did occur past the neutral point, but here it occurred in an intermittent fashion with the wave sometimes damping and at other times increasing and causing transition. As the input level was raised still further, transition occurred steadily and at positions successively farther upstream. The data shown ends at about the beginning of the transition region. The departure from the small disturbance rate is an exponential-like growth. The departure for the four highest input levels takes place at an intensity level of about 1.5 percent. As pointed out by Dr. Lin on a visit to our laboratory the other day, a level of this magnitude would correspond to a fluctuating shear stress which is on the order of one-tenth the viscous shear and would be a reasonable assumption for the beginning of non-linear effects. However, it should be mentioned that there is no discernible effect on the mean-velocity distribution. In fact there is no significant distortion of the mean-velocity profile until the actual breakdown of laminar flow occurs. At any rate it is felt that the significance of non-linearity needs further investigation and that the effect of curvature as proposed by Görtler should not be overlooked.

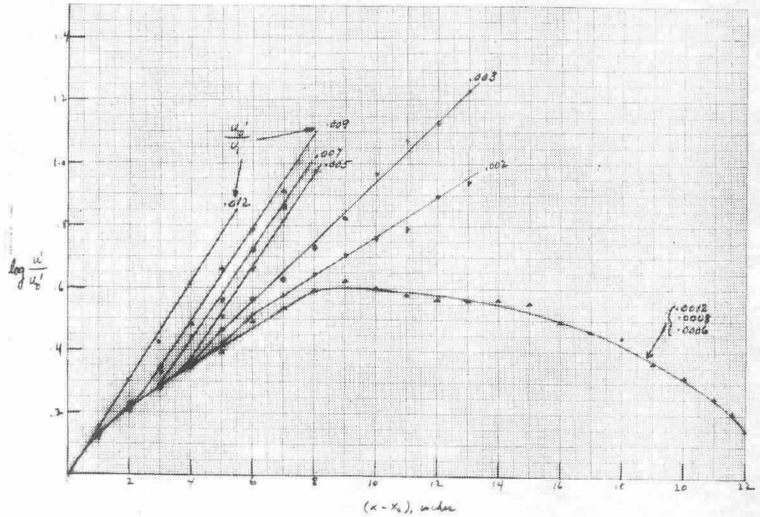

Figure 4.

This briefly summarizes some of the experimental data. The 3-dimensionality that I have been talking about was eventually traced to small span-wise variations in the mean flow in the boundary layer, which when coupled with the circumstance that the frequency of the primary wave was near branch II of the neutral curve, gave rise to local variations in the amplification rate. In fact by choosing a frequency near branch I of the neutral curve the peaks and hollows could be reversed. The 3-dimensional character is of practical significance inasmuch as the frequency selected for amplification in natural transition lies near branch II and would be especially susceptible to irregularities in the mean flow. Also a random input would give rise to distorted waves. Although the results obtained to date indicate the presence of 3-dimensionality prior to transition the question as to whether it is an inherent part of the transition process has not as yet been resolved. In this connection it should be mentioned that Hama at the University of Maryland, conducted an investigation of transition on a flat plate in water, using dye and a trip wire and concluded that distortion in the wave always occurs prior to transition. The difficulty has been in getting a flow free of irregularities. Span-wise variations in mean velocity on the order of a few percent appear to be significant. The results obtained do not exclude the possibility of break-down of a 2-dimensional wave, nor exclude the usefulness of a 2-dimensional finite amplitude theory.

C. C. Lin

I wish to thank Dr. Wasow for this able comparison between his work and our present approach. It may be well to emphasize that the connection coefficients $C_\nu(z, \lambda)$ are independent of the particular solutions u_j while $k_j(z)$ depends on the index j. The appropriateness and desirability of using the form

$$v = \sum_{\nu=0}^{3} C_\nu(z,\lambda) u^{(\nu)} \tag{29}$$

become especially obvious, when the original equation is regarded as a system of four differential equations of the first order.

The third step mentioned by Dr. Wasow is essentially an existence proof. Whereas this is usually omitted in applied mathematical research—since the solution constructed carries every conviction of being a uniformly valid solution—Mr. Rabenstein and I are carrying out this proof in view of the fundamental nature of the problem.

Note added in proof (Nov. 7, 1957). The proof referred to in the above discussions has been completed in the meantime by Rabenstein and myself. A paper is in preparation to give a comprehensive description of the theory.