A THEORY OF THE STABILITY OF LAMINAR FLOW ALONG COMPLIANT PLATES

F. W. Boggs and N. Tokita
Research Center, U.S. Rubber Company,
Wayne, New Jersey

1. INTRODUCTION

A great deal of theoretical work has been done in recent years on viscous flow in the neighborhood of a rigid wall. This has made it possible to explain the initiation of turbulence through the development of flow instability. Laminar flow has been shown by the theory to be stable under conditions when it is actually observed, and it has been shown to be theoretically unstable under conditions when turbulent flow is known to prevail. These results developed by Tollmien, Schlichting, Lin, and many others [1-7] have been verified by Schubauer [8] at the National Bureau of Standards. Today the controversy which existed for many years over the nature of turbulent flow and its relationship to laminar flow seems to be settled.

The recent work of Kramer [9-11] showing the effect of a flexible boundary wall on fluid flow has emphasized the need to extend the boundary-layer theory to include this interesting and important case. This present paper undertakes the development of a theory which will predict the stability conditions of the boundary layer in contact with a flexible wall under the same general assumptions regarding the flow that were made by previous authors for the rigid wall and to the same degree of approximation. The result should, therefore, have the same kind of validity.

In this treatment the properties of the flexible wall are expressed in terms of its acoustical compliances.

One of the significant conclusions which follows from this work is that the conditions of stabilizing a flow are fairly critical and that all coatings which are flexible do not necessarily have a favorable effect on the flow conditions.

In subsequent papers we will attempt to extend this theory to some of the special cases of practical importance, but here we will confine ourselves to some very general conclusions.

The theories of boundary-layer stabilization are based on the supposition that the Navier-Stokes equations hold at all times and that transition from laminar to turbulent flow results when the laminar flow is unstable for some arbitrarily small perturbation. Stable laminar flow exists only if the flow is stable for every possible infinitesimal perturbation. It assumes but does not really prove that when a laminar flow fulfills these conditions, it will be the dominant flow pattern and that the drag coefficient calculated from such a laminar flow is the one applicable.

The question of the lower limit of turbulent flow is not considered by the Schlichting theory nor do we consider it here. Certainly, before the picture of flexible walls can be complete, a consideration of their influence on the fully turbulent boundary layer should be made.

The theory of boundary-layer stability starts by solving the Navier-Stokes equations and the equation of continuity for a steady state. An arbitrary perturbation developable in a Fourier series is added to this solution, and the conditions are studied under which this arbitrary perturbation will either increase or decrease as time proceeds. If no possible perturbation can increase, then the flow is assumed stable. Instability will exist if any perturbation exists capable of increasing in magnitude without external excitation. Between these stable and unstable conditions there exists a boundary at which a perturbation neither increases nor decreases in magnitude. This is known as neutral stability, and it separates the stable from the unstable conditions. The values of the parameters of the system which lie on the boundary separating the regions of stable and unstable flow are known as conditions of neutral stability. The plotting of these conditions makes it possible to separate the regions of stable and unstable flow, and on this basis to predict when instability will occur. The principal aim of this paper is to express the changes in the curves of neutral stability which are brought about by the presence of a flexible wall, the properties of this wall being expressed in terms of its compliances. These compliances may not only be calculated from the nature of the walls, but they may also be measured.

2. PERTURBATION OF TWO DIMENSIONAL PARALLEL FLOW

If we have a flow exclusively in the x direction which, however, depends on y, then we may examine the effect of a small perturbation on this flow. Suppose that $\overline{V}_x$ satisfies the Navier-Stokes equation and the equation of continuity and that it vanishes along rectilinear boundaries extending in the x direction. It has been shown that if the perturbation has the form

$$v_x = \int_{-\infty}^{+\infty} \frac{\partial\Phi(\bar{\alpha}, y)}{\partial y} e^{i(\bar{\alpha}x - \beta t)} \, d\bar{\alpha} \tag{2.1}$$

$$v_y = -\int_{-\infty}^{+\infty} i\bar{\alpha}\Phi(\bar{\alpha}, y) \, e^{i(\bar{\alpha}x - \beta t)} \, d\bar{\alpha} \tag{2.2}$$

then the function $\Phi(\bar{\alpha},y)$ satisfies the Orr-Summerfeld equation

$$\left(\overline{V}_x(y) - \frac{\beta}{\bar{\alpha}}\right)\left(\frac{d^2\Phi}{dy^2} - \bar{\alpha}^2\Phi\right) - \frac{d^2\overline{V}_x}{dy^2}\Phi = -\frac{i\nu}{\bar{\alpha}}\left(\frac{d^4\Phi}{dy^4} - 2\bar{\alpha}^2\frac{d^2\Phi}{dy^2} + \bar{\alpha}^4\Phi\right) . \tag{2.3}$$

It is usual to express the solution of Eq. (2.3) in terms of dimensionless variables appropriate to the system. Let us choose a characteristic length which may be the distance between the edges in Couette and plane Poisseuille flows or the thickness of a boundary layer in a boundary-layer problem. Let us express the velocity of flow in terms of the maximum velocity $\bar{U}_m$. We will take $Y = y/\delta$ as an independent variable and introduce the following parameters:

$$R = \frac{\delta \bar{U}_m}{\nu}, \qquad \alpha = \bar{\alpha}\,\delta, \qquad C = \frac{\beta}{\bar{\alpha}\bar{U}_m}, \qquad g = \frac{\bar{V}_x}{\bar{U}_m}. \tag{2.4}$$

Equation (2.3) then becomes

$$(g - c)\,(\Phi'' - \alpha^2\Phi) - g''\Phi = -\frac{i}{\alpha R}\,(\Phi'''' - 2\alpha^2\Phi'' + \alpha^4\Phi). \tag{2.5}$$

The solution of Eq. (2.5) has been extensively discussed in the literature. It has been found that it may be expressed in terms of solutions of the left-hand side of Eq. (2.5) set equal to zero (proper attention being paid to the behavior at the branch point) and in terms of solutions of

$$\eta \frac{d^2\psi}{d\eta^2} + i\,\frac{d^4\psi}{d\eta^4} = 0 \tag{2.6}$$

where $\eta = (a_1\alpha R)^{1/3}\,(Y - Y_K)$, Y_K is the value of y for which $g(y) = C$ and a_1 is the derivative of g at the point $y = Y_K$. All previous solutions to these equations have assumed that v_x and v_y vanished on all boundaries and at infinity if the flow was not confined. In the case of the boundary layer with which we will primarily be concerned, the conditions are $v_x = v_y = 0$ for $y = 0$ and $y = \infty$. This requires four boundary conditions. If we choose an appropriate solution ϕ of the left-hand side of Eq. (2.5) which vanishes with its first derivative for large values of y and a similar solution of ψ of Eq. (2.6), then it can be shown that for the familiar solution of the stable equation, we obtain

$$\begin{vmatrix} \phi & \psi \\ \phi' & (a_1\alpha R)^{1/3}\psi' \end{vmatrix} = 0. \tag{2.7}$$

This determinant is a complex function of the three variables α, R, and C. If for all real values of α and R the imaginary part of C is negative, then the flow will be stable. If we choose C real, we can plot any one of these variables (α, R, and C) as a function of one of the others for the condition that satisfies Eq. (2.7). The curves so obtained are known as curves of neutral stability and establish the boundary of stable flow conditions. Diagrams of this type permit one to predict not only whether the flow will be stable but the wavelengths which occur in the unstable regions of the diagrams. Our objective in this paper is to examine how these curves of neutral stability will be influenced by the compliant surfaces. This paper, however, contains only the basic principles which will need to be developed somewhat more fully.

3. BOUNDARY CONDITIONS FOR COMPLIANT SURFACES

The velocity of flow of a viscous liquid must be identical with the velocity of the wall at every point of contact. Suppose that $\vec{r}$ is the position of a point on the undeformed wall and that the deformation caused by forces in the flow is designated by the vector $\vec{\xi}(r)$. The rate of deformation of the wall at the point $\vec{r} + \vec{\xi}(r)$ will be given by $\dot{\vec{\xi}}(r)$. This must be identical with the flow velocity at the position $\vec{r} + \vec{\xi}$. Hence, we will have

$$\dot{\vec{\xi}}(r) = \vec{v}(r+\xi) = \vec{v}(r) + \nabla\vec{v} \cdot \vec{\xi} + \cdots . \tag{3.1}$$

Since we are considering only small perturbations, we can neglect the effect of $\vec{\xi}$ on $\vec{v}$ and identify the rate of deformation of the surface with the velocity of flow at every point along the boundary. This will lead to the approximate relationship

$$\dot{\vec{\xi}}(r) = \vec{v}(r). \tag{3.2}$$

If the motion of the wall is due exclusively to the forces in the fluid boundary layer, we must express the surface deformation in terms of these forces. To completely establish the boundary conditions, we must, therefore, express the surface forces in terms of the flow. When this has been done Eq. (3.2) will become a differential expression in the stream function which must be satisfied on the boundary.

If the response of the surface to external forces is linear, we can always express both the forces and the corresponding response of the surface in terms of progressive waves, just as we did in the case of perturbation of the flow velocity. If ξ_x and ξ_y are the amplitude of a wave of wave number $\bar{\alpha}$ and frequency β traveling along the surface and if $P(\bar{\alpha}, \beta)$ and $T(\bar{\alpha}, \beta)$ are the corresponding expressions for the amplitude of the wave of pressure and tangential force, we can define a set of compliances such that the following equations are satisfied:

$$\begin{aligned} \dot{\vec{\xi}}_x &= Y_{11}T + Y_{12}P \\ \dot{\vec{\xi}}_y &= Y_{21}T + Y_{22}P. \end{aligned} \tag{3.3}$$

The constants Y_{ij} are dependent on the parameters $\bar{\alpha}$ and β and are determined by the specific nature of the wall. At a later date we hope to present detailed treatments in which these constants are calculated from the structure of the coating. In this paper we will confine ourselves to their general properties.

To complete the calculation of the boundary conditions, we must express the surface forces T and P in terms of the stream function. From the perturbation of the steady-state solution of the Navier-Stokes equation, we obtain

$$\frac{\partial v_x}{\partial t} + \bar{V}_x \frac{\partial v_x}{\partial x} + v_y \frac{d\bar{V}_x}{dy} + \frac{1}{\rho} \frac{\partial P}{\partial x} = \nu\nabla^2 v_x . \tag{3.4}$$

We assume that $\bar{V}_x$ vanishes on the boundary and that its derivative at this point is given by a constant depending on the nature of the boundary layer profile. In the case of Blasius profile this is equal to $\gamma\bar{U}_m/\delta$ where γ is a dimensionless constant and the same for all boundary layers of this type. If we express v_x and v_y as was done in Eqs. (2.1) and (2.2) and if we use the dimensionless variable defined in the previous section as the independent variable and if we use the same dimensionless parameters, it is readily shown that the pressure assumes the form

$$P = \frac{\bar{U}_m\rho}{\delta}\left[\frac{1}{i\alpha R}\Phi'''(y) + \left(C + \frac{i\alpha}{R}\right)\Phi'(y) + \gamma\Phi(y)\right]. \tag{3.5}$$

The tangential force will be given in turn by

$$T = \frac{\nu\rho}{\delta^2}\Phi'' = \frac{\bar{U}_m\rho}{\delta}\frac{\Phi''}{R}. \tag{3.6}$$

If we substitute Eqs. (3.5) and (3.6) into Eq. (3.3) which expresses the deformation of the surface in terms of the surface forces, we will obtain a relationship between the velocity on the boundary and its derivatives along the wall. Every term in this new expression will depend upon the stream function or on its derivatives and on the parameters of the system. These two equations, therefore, replace as boundary conditions the vanishing of the components of velocity on the surface of the body. The conditions at infinity will not be changed by the flexible wall and will, therefore, remain $v_x = v_y = 0$ as in the classical equation.

It was pointed out previously that the stream function Φ could be expressed as a linear sum of two functions ϕ and ψ. This will still be true since the differential equation is unchanged by the flexible wall. Consequently, these two functions obey the boundary conditions at infinity both for the flexible and for the inflexible wall and can be used in linear combination to develop the solution of the equation of the flow in the presence of a flexible wall. If we substitute into Eq. (3.3) a linear sum of these two functions, multiplying by arbitrary coefficients, we will obtain a set of two equations in the two arbitrary coefficients which much be satisfied. The condition of existence of solutions to these equations is the vanishing of a determinant which is related to the determinant obtained for the rigid wall. The set of linear equations which must be satisfied are given in

$$\begin{aligned}
&A\left\{\phi'(y) - \bar{U}_m\rho\, Y_{12}\left[\frac{\Phi'''}{i\alpha R} + \left(C + \frac{i\alpha}{R}\right)\phi' + \gamma\phi\right] - \frac{\bar{U}_m\rho}{R}Y_{11}\phi''\right\}\\
&+ B\left\{(a_1\alpha R)^{1/3}\psi'(y) - \bar{U}_m\rho Y_{12}\left[\frac{a_1}{i}\psi''' + \left(C + \frac{i\alpha}{R}\right)(a_1\alpha R)^{1/3}\psi' + \gamma\psi\right]\right.\\
&\qquad\left. - \frac{\bar{U}_m\rho}{R}(a_1\alpha R)^{1/3}Y_{11}\psi''\right\} = 0\\
&A\left\{-i\alpha\phi - \bar{U}_m\rho\, Y_{22}\left[\frac{\phi'''}{i\alpha R} + \left(C + \frac{i\alpha}{R}\right)\phi' + \gamma\phi\right] - \frac{\bar{U}_m\rho}{R}Y_{21}\phi''\right\}\\
&+ B\left\{-i\alpha\psi - \bar{U}_m\rho\, Y_{22}\left[\frac{a_1}{i}\psi''' + \left(C + \frac{i\alpha}{R}\right)(a_1\alpha R)^{1/3}\psi' + \gamma\psi\right]\right.\\
&\qquad\left. - \frac{\bar{U}_m\rho}{R}(a_1\alpha R)^{1/3}Y_{21}\psi''\right\} = 0.
\end{aligned} \tag{3.7}$$

Retaining only the terms in $(a_1 \alpha R)^{1/3}$ and those independent of this quantity, in other words neglecting the negative powers of αR, we will obtain the set of equations

$$
\begin{aligned}
&A\,\left[\phi' - \bar{U}_m \rho\, Y_{12}(C\phi' + \gamma\phi)\right] \\
&+ B\left\{(a_1\alpha R)^{1/3}\psi' - \bar{U}_m \rho Y_{12}\left[-ia_1\psi''' + C(a_1\alpha R)^{1/3}\psi' + \gamma\psi\right]\right\} = 0 \\
&A\,\left[-i\alpha\phi - \bar{U}_m\rho\, Y_{22}(C\phi' + \gamma\phi)\right] \\
&+ B\left\{-i\alpha\psi - \bar{U}_m\rho\, Y_{22}\left[-ia_1\psi''' + C(a_1\alpha R)^{1/3}\psi' + \gamma\psi\right]\right\} = 0.
\end{aligned}
\tag{3.8}
$$

We make the substitution

$$
\begin{aligned}
e_1 &= \bar{U}_m \rho\, Y_{22} \\
e_2 &= \bar{U}_m \rho\, Y_{12}
\end{aligned}
\tag{3.9}
$$

and rearrange terms, and we obtain

$$
\begin{aligned}
&A\,\left[(1 - e_2 C)\phi' - e_2\gamma\phi\right] \\
&+ B\,\left[(a_1\alpha R)^{1/3}(1 - e_2 C)\,\psi' - e_2\gamma\psi + ie_2 a_1\psi'''\right] = 0 \\
&A\,\left[-e_1 C\phi' - (i\alpha + e_1\gamma)\phi\right] \\
&+ B\,\left[-e_1 C(a_1\alpha R)^{1/3}\psi' - (i\alpha + e_2\gamma)\psi + ie_2 a_1\psi'''\right] = 0.
\end{aligned}
\tag{3.10}
$$

The condition of existence of solutions in this set of equations is the setting equal to zero of the determinant of the coefficients of A and B. Again we will note that when e_1 and e_2 are equal to zero this condition will reduce to the one obtained for a rigid plate. This is as it should be since e_1 and e_2 vanishing means zero compliance; the plate cannot be deformed.

It may be readily seen that the matrix of the coefficients of A and B in Eq. (3.10) is given by

$$
\begin{pmatrix} 1 - e_2 C & -e_2\gamma \\ -e_1 C & -(i\alpha + e_1\gamma) \end{pmatrix}
\begin{pmatrix} \phi' & (a_1\alpha R)^{1/3}\,\psi' \\ \phi & \psi \end{pmatrix}
+ \begin{pmatrix} 0 & ie_2 a_1\psi''' \\ 0 & ie_1 a_1\psi''' \end{pmatrix} .
\tag{3.11}
$$

To obtain the eigenvalues of the differential equation, we must set the determinant of this matrix equal to zero.

Since the second matrix in Eq. (3.11) is singular, we cannot divide through by it, but if the prefactor on the first term is not singular, we may multiply through by its inverse so that Eq. (3.11) is transformed to

$$\begin{pmatrix} \phi' & (a_1\alpha R)^{1/3}\,\psi' \\ & \\ \phi & \psi \end{pmatrix} + \frac{1}{\Delta}\begin{pmatrix} -(i\alpha + e_1\gamma) & e_2\gamma \\ & \\ e_1 C & (1 - e_2 C) \end{pmatrix}\begin{pmatrix} 0 & ie_2 a_1\psi''' \\ & \\ 0 & ie_1 a_1\psi''' \end{pmatrix}. \tag{3.12}$$

This step requires examination. If the matrix has no inverse, if the operation cannot be carried out, then we have a special problem. In general, the boundary layer will be unstable if the zeros of the determinant (3.13) have positive real parts:

$$\Delta = \begin{vmatrix} 1 - e_2 C & -e_2\gamma \\ & \\ -e_1 C & -(i\alpha + e_1\gamma) \end{vmatrix} = -i\alpha + i\alpha e_2 C - e_1\gamma$$

$$= -i\alpha + i\alpha C\bar{U}_m\rho Y_{12} - \gamma\bar{U}_m\rho Y_{22}. \tag{3.13}$$

Since we assume the compliances are passive, this will generally not be the case, so that we will be able actually to divide through by the matrix and obtain (3.12). Our stability conditions are then given in

$$\begin{vmatrix} \phi' & (a_1\alpha R)^{1/3}\,\psi' \\ & \\ \phi & \psi \end{vmatrix} + \begin{vmatrix} \phi' & \alpha e_2 \\ & \\ \phi & ie_1 \end{vmatrix} \frac{a_1\psi'''}{\Delta} = 0\,. \tag{3.14}$$

As can be seen the first term is the expression which must be set equal to zero in the case of the rigid boundary. The second term contains the added conditions imposed by the flexibility. Developing this determinant, we finally obtain the following equation for the conditions of the boundary:

$$\frac{\psi}{Y_\varkappa(a_1\alpha R)^{1/3}\,\psi'} - \frac{\phi}{Y_\varkappa\phi'} + \left(\frac{ie_1}{\Delta} - \frac{\alpha e_2\,\phi}{\Delta\phi'}\right)\frac{a_1\psi'''}{Y_\varkappa(a_1\alpha R)^{1/3}\,\psi'} = 0. \tag{3.15}$$

We will note that this boundary condition is expressed in terms of three functions depending upon the hydrodynamic conditions and on the two compliances. Two of the three functions which are involved are identical with the ones which appeared in the solution for the rigid body as developed by previous authors. We may rewrite in the form

$$F(\zeta) - G(\alpha, C) = -\left[ie_1 - \alpha e_2 Y_\varkappa G(\alpha, C)\right] a_1 H(\zeta)/\Delta$$

$$F(\zeta) = \frac{\psi(\zeta)}{\zeta\psi'(\zeta)}\,, \qquad G(\alpha, C) = \frac{\phi}{Y_\varkappa\phi'} \tag{3.16}$$

(Cont.)

$$
\begin{aligned}
H(\zeta) &= \frac{\psi'''(\zeta)}{\zeta\psi'(\zeta)} \\
\zeta &= -(a_1\alpha R)^{1/3} Y_\varkappa \\
\frac{F(\zeta) - G(\alpha,C)}{a_1 H(\zeta)} &= -\frac{Y_{22} + i\alpha Y_\varkappa G(\alpha,C)\, Y_{12}}{i\gamma Y_{22} + \alpha C Y_{12} - \dfrac{\alpha}{\bar{U}_m \rho}} \, .
\end{aligned}
\tag{3.16}
$$

4. SOME GENERAL PROPERTIES OF A SURFACE COMPLIANCE

Rayleigh was the first to show that solid bodies are subject to waves propagating over their surface. Such waves will exist in the case of a turbulent boundary layer in contact with a flexible surface and will play an important part in the nature of the compliance. Basically, there are two factors to be considered; the deformation which occurs at the point where the pressure is applied and the way this deformation propagates along the surface. We will consider the effect of both of these reactions of the surface. Let us suppose that we have a pressure of the type mentioned previously, and for the sake of simplicity, let us suppose that it varies periodically with the time so that it may be represented by the product of a complex exponential in the time and a function of x. In this case we may represent the pressure by an expression similar to

$$
e^{-i\beta t} P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P(\bar{\alpha})\, e^{i\bar{\alpha}x}\, d\bar{\alpha}\, e^{-i\beta t} . \tag{4.1}
$$

This will cause periodic displacements of the surface velocity in the x and y direction which may be represented by

$$
\begin{aligned}
v_y &= -i\beta e^{-i\beta t} \xi_y(x) = \int_{-\infty}^{+\infty} Y_{22}(\bar{\alpha},\beta)\, P(\bar{\alpha})\, e^{i\bar{\alpha}x}\, d\bar{\alpha}\, e^{-i\beta t} \\
v_x &= -i\beta e^{-i\beta t} \xi_x(x) = \int_{-\infty}^{+\infty} Y_{21}(\bar{\alpha},\beta)\, P(\bar{\alpha})\, e^{i\bar{\alpha}x}\, d\bar{\alpha}\, e^{-i\beta t} .
\end{aligned}
\tag{4.2}
$$

In Eq. (4.2) the Y's are the surface compliances for a wave having a wave number $\bar{\alpha}$. If we use the Fourier inversion theorem for Eq. (4.1) and Eq. (4.2), we will obtain

$$
\begin{aligned}
P(\bar{\alpha}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P(x)\, e^{-i\bar{\alpha}x}\, dx \\
Y_{22}(\bar{\alpha},\beta)\, P(\bar{\alpha}) &= -\frac{i\beta}{2\pi} \int_{-\infty}^{+\infty} \xi_y(x)\, e^{-i\bar{\alpha}x}\, dx
\end{aligned}
\tag{4.3}
$$

$$Y_{21}(\bar{\alpha},\beta)\ P(\bar{\alpha}) = -\frac{i\beta}{2\pi}\int_{-\infty}^{+\infty} \xi_x(x)\ e^{-i\bar{\alpha}x}\ dx. \tag{4.4}$$

Combining these equations we obtain for the surface compliances

$$Y_{22}(\bar{\alpha},\beta) = \frac{\dfrac{-i\beta}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} \xi_y(x)\ e^{-i\bar{\alpha}x}\ dx}{\displaystyle\int_{-\infty}^{+\infty} P(x)\ e^{-i\bar{\alpha}x}\ dx} \tag{4.5}$$

$$Y_{21}(\bar{\alpha},\beta) = \frac{\dfrac{-i\beta}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} \xi_x(x)\ e^{-i\bar{\alpha}x}\ dx}{\displaystyle\int_{-\infty}^{+\infty} P(x)\ e^{-i\bar{\alpha}x}\ dx}. \tag{4.6}$$

So far these are purely formal relationships which, for example, would allow us to determine the surface compliance experimentally from the reaction of the surface to a pressure. This is useful in that equipment can be designed and has been designed to use this relation for the determination of the surface compliance of arbitrary surfaces. However, we must put it in more manageable shape for the determination of experimental quantities, and we will show how it can be used to relate the surface compliance to the propagation constant of Rayleigh waves at a given frequency. Let us suppose that the pressure is applied over a very small length which is allowed then to go to zero. Experimentally, this could be achieved by the use of a very narrow knife edge as a driving unit. We will have for the pressure

$$\begin{aligned} P(x) &= \frac{F}{2\varepsilon} \qquad \text{when} \quad -\varepsilon < x < \varepsilon \\ P(x) &= 0 \qquad \text{when} \quad x < -\varepsilon \quad \text{or} \quad x > \varepsilon. \end{aligned} \tag{4.7}$$

Using these in Eq. (4.4) to obtain the Fourier integral of the pressure, we have

$$P(\bar{\alpha},\beta) = \frac{1}{\sqrt{2\pi}} F\ \frac{\sin\bar{\alpha}\varepsilon}{\bar{\alpha}\varepsilon}. \tag{4.8}$$

Taking the limit as $\varepsilon \to 0$ we finally obtain

$$P(\bar{\alpha},\beta) = \frac{1}{\sqrt{2\pi}} F \lim_{\varepsilon\to 0} \frac{\sin\bar{\alpha}\varepsilon}{\bar{\alpha}\varepsilon} = \frac{1}{\sqrt{2\pi}} F. \tag{4.9}$$

The compliances of the surface will be given respectively by

$$Y_{22} = -\frac{i\beta}{\sqrt{2\pi}\,F}\int_{-\infty}^{+\infty} \xi_y(x)\; e^{-i\bar{\alpha}x}\; dx \tag{4.10}$$

$$Y_{21} = -\frac{i\beta}{\sqrt{2\pi}\,F}\int_{-\infty}^{+\infty} \xi_x(x)\; e^{-i\bar{\alpha}x}\; dx. \tag{4.11}$$

The Fourier transforms in this case are carried out from minus infinity to plus infinity in the complex plane. It is convenient to use a real notation, in which case the integrals will go from zero to infinity. It is worth noting, however, that the normal displacement will be an even function of x, whereas the tangential displacement will be an odd function of x. As a consequence only the cosine transformation from zero to infinity will be important in the first instance, and the sine transform will be important in the second instance.

This gives us for the surface compliance

$$Y_{22}(\bar{\alpha},\beta) = -\frac{2i\beta}{\sqrt{2\pi}\,F}\int_{0}^{\infty} \xi_y(x)\;\cos\,\bar{\alpha}x\; dx \tag{4.12}$$

$$Y_{21}(\bar{\alpha},\beta) = -\frac{2i\beta}{\sqrt{2\pi}\,F}\int_{0}^{\infty} \xi_x(x)\;\sin\,\bar{\alpha}x\; dx. \tag{4.13}$$

Let us now suppose that the vertical vibration imparted by the knife edge gives rise to propagating waves having the propagating constant Γ. The compliances which apply to the normal and tangential components will be respectively given by

$$Y_{22}(\bar{\alpha},\beta) = -\frac{2i\beta\; f_o(\beta)}{\sqrt{2\pi}\,F}\int_{0}^{\infty} e^{i\Gamma x}\;\cos\,\bar{\alpha}x\; dx \tag{4.14}$$

$$Y_{21}(\bar{\alpha},\beta) = -\frac{2i\beta\; g_o(\beta)}{\sqrt{2\pi}\,F}\int_{0}^{\infty} e^{i\Gamma x}\;\sin\,\bar{\alpha}x\; dx \tag{4.15}$$

where $f_o(\beta)$ and $g_o(\beta)$ are functions of the frequency.

Carrying out the integral we obtain

$$Y_{22}(\bar{\alpha},\beta) = \frac{2\beta\ f_o(\beta)}{\sqrt{2\pi}\ \ \bar{\alpha}F} \frac{\left(\frac{\Gamma}{\bar{\alpha}}\right)}{\left(\frac{\Gamma}{\bar{\alpha}}\right)^2 - 1} \tag{4.16}$$

$$Y_{21}(\bar{\alpha},\beta) = \frac{2\beta\ g_o(\beta)}{\sqrt{2\pi}\ \ \bar{\alpha}F} \frac{1}{\left(\frac{\Gamma}{\bar{\alpha}}\right)^2 - 1}\ . \tag{4.17}$$

The quantity $\Gamma/\bar{\alpha}$ is the ratio of the wavelength of the disturbance in the boundary layer to the wavelength in the coating. One of the points of interest will be to consider the effect of a large difference between these two wavelengths and also what the behavior will be when these two wavelengths are close together.

If we calculate the corresponding dimensionless compliances, we obtain

$$\begin{aligned}
e_1 = \rho\bar{U}_m Y_{22} &= \frac{2}{\sqrt{2\pi}} \frac{\beta}{\bar{\alpha}} \frac{f_o(\beta)}{F} \left[\frac{\left(\frac{\Gamma}{\bar{\alpha}}\right)}{\left(\frac{\Gamma}{\bar{\alpha}}\right)^2 - 1}\right] \rho\bar{U}_m \\
&= \frac{2}{\sqrt{2\pi}} C \frac{f_o(\beta)}{F} \rho\bar{U}_m^2 \left[\frac{\left(\frac{\Gamma}{\bar{\alpha}}\right)}{\left(\frac{\Gamma}{\bar{\alpha}}\right)^2 - 1}\right] = \frac{2}{\sqrt{2\pi}} C \frac{f_o(\beta)}{F} \rho\bar{U}_m^2 \left[\frac{\left(\frac{C}{v}\right)}{\left(\frac{C}{v}\right)^2 - 1}\right] \\
&\simeq \frac{2}{\sqrt{2\pi}} \frac{\beta}{\Gamma} \frac{f_o(\beta)}{F} \rho\bar{U}_m^2 = \frac{2}{\sqrt{2\pi}} v \frac{f_o(\beta)}{F} \rho\bar{U}_m^2, \qquad \text{for } \frac{\Gamma}{\bar{\alpha}} >> 1 \\
&\simeq -\frac{2}{\sqrt{2\pi}} \frac{\beta\Gamma}{\bar{\alpha}^2} \frac{f_o(\beta)}{F} \rho\bar{U}_m^2 = -\frac{2}{\sqrt{2\pi}} \frac{C^2}{v} \frac{f_o(\beta)}{F} \rho\bar{U}_m^2, \quad \text{for } \frac{\Gamma}{\bar{\alpha}} << 1
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
e_2 = \rho\bar{U}_m Y_{21} &= \frac{2}{\sqrt{2\pi}} \frac{\beta}{\bar{\alpha}} \frac{g_o(\beta)}{F} \left[\frac{1}{\left(\frac{\Gamma}{\bar{\alpha}}\right)^2 - 1}\right] \rho\bar{U}_m \\
&= \frac{2}{\sqrt{2\pi}} C \frac{g_o(\beta)}{F} \left[\frac{1}{\left(\frac{\Gamma}{\bar{\alpha}}\right)^2 - 1}\right] \rho\bar{U}_m^2 = \frac{2}{\sqrt{2\pi}} C \frac{g_o(\beta)}{F} \rho\bar{U}_m^2 \left[\frac{1}{\left(\frac{C}{v}\right)^2 - 1}\right] \\
&\simeq \frac{2}{\sqrt{2\pi}} \frac{\beta\bar{\alpha}}{\Gamma^2} \frac{g_o(\beta)}{F} \rho\bar{U}_m^2 = \frac{2}{\sqrt{2\pi}} \frac{v^2}{C} \frac{g_o(\beta)}{F} \rho\bar{U}_m^2, \quad \text{for } \frac{\Gamma}{\bar{\alpha}} >> 1 \\
&\simeq -\frac{2}{\sqrt{2\pi}} \frac{\beta}{\bar{\alpha}} \frac{g_o(\beta)}{F} \rho\bar{U}_m^2 = -\frac{2}{\sqrt{2\pi}} C \frac{g_o(\beta)}{F} \rho\bar{U}_m^2, \quad \text{for } \frac{\Gamma}{\bar{\alpha}} << 1.
\end{aligned} \tag{4.19}$$

It is apparent that e_1 and e_2 both pass through the origin when β is equal to zero. It follows that they will both pass through the origin when C vanishes. Furthermore, when $\bar{\alpha}$ is small e_2 will be negligible and e_1 will be a function of β only. These two characteristics will have some important practical consequences which we will consider in the next section of this paper. It can also be seen by substituting that similar conditions will be obtained when the propagation constant of waves over the surface is proportional to the velocity of flow.

5. CURVE OF NEUTRAL STABILITY FOR A FLEXIBLE WALL

If Eq. (3.16) is rewritten in terms of e_1 and e_2, we obtain

$$\frac{F(\zeta) - G(\alpha,C)}{a_1 H(\zeta)} = -\frac{e_1 + i\alpha e_2 Y_\varkappa G(\alpha,C)}{C\alpha e_2 + i\gamma e_1 - \alpha} = \frac{-\frac{e_1}{\alpha} - iY_\varkappa G(\alpha,C)\ e_2}{Ce_2 + i\gamma\frac{e_1}{\alpha} - 1} . \tag{5.1}$$

The left-hand side depends on the Reynolds number only through its dependence on ζ, whereas the right-hand side depends on the Reynolds number through the frequency β and wave number $\bar{\alpha}$, which depend on both the velocity and the Reynolds number. The curve of neutral stability can be obtained by identifying the real and imaginary parts of both sides of Eq. (5.1) and eliminating either α or C. For each value of $\bar{U}_m$ there will be a different curve of neutral stability and a corresponding critical Reynolds number which, if it exists, will depend on $\bar{U}_m$. The procedure outlined above represents a formidable amount of work which, even with the aid of calculators, would be forbidding. Can we find a procedure for obtaining bounds for the Reynolds number which will simplify the general discussion?

For a given velocity the right hand-side of Eq. (5.1) will be a function of two variables only, α and β. Furthermore, as we saw in Section 4 all the curves will pass through the origin and be tangential to each other when α and β are small.

If we hold α constant we can plot the right-hand side of Eq. (5.1) as a function of C treating R as a parameter. For a given value of α the right-hand side of Eq. (5.1) will be a single curve passing through the origin while the left-hand side will give a family of curves, one for each Reynolds number. The intersection of the curve corresponding to the right-hand side of (5.1) with each one of the family of curves for the left-hand side will correspond to a value of α and of R on the curve of neutral stability. If the family of curves representing the left has an envelope through the elimination of R, the value of the frequency corresponding to the intersection of this envelops with the right plotted for α and $\bar{U}_m$ constant will be an extreme value for C. The corresponding value of the frequency for each value of α will form a boundary separating areas of stability and instability. The value of the frequency along this boundary will be a function of the velocity and of α, and its value will be given by an equation of the form

$$\beta(\bar{U}_m, \alpha) = \frac{\bar{U}_m^2 \alpha C}{\nu\sqrt{R}} . \tag{5.2}$$

In addition, we must specify that we are on the envelope. This will give a relationship between α and C. These two relations together will, therefore, lead to a relationship between the Reynolds number and the velocity, from which a family of curves of critical Reynolds numbers can be constructed.

A delineation of the family formed of the envelopes obtained by the elimination of the Reynolds numbers through the construction of the envelope requires a more detailed analysis of the way the right-hand side of Eq. (5.1) is formed. If we consider the function given to the left of Eq. (5.1), it is not difficult formally to construct the envelope.

The analytic expressions for the functions F, G, and H are inconvenient. We have used an approximate expression for the denominator and have constructed the envelope graphically. The value of the function in the numerator for a given set of values of α and C is obtained by simple graphical subtraction. Holding α constant and varying C, we obtain a family of curves with the general appearance given in Fig. 1. This family of curves has an envelope which is formed by the tangent to the common apices of the curves. There will be one such envelope for each value of α. This family of envelopes is given in Fig. 2, which was constructed from existing data. Each member of this family of envelopes separates the plane into two domains. Any point in the lower domain will have two members of the family of which it is an envelope passing through it, a point on the envelope will have only one, and finally a point above the envelope will have none. The locus of the points of intersection between the family of envelopes on the one hand and the compliances on the other will form the boundary between areas of stability and instability. Points corresponding to the compliance which are above the envelope cannot lead to unstable solutions.

In the discussion that follows we will consider the special case when e_2 vanishes. For small values of β and $\bar{\alpha}$, e.g., moderate speeds and large Reynolds numbers, this will be the most important case.

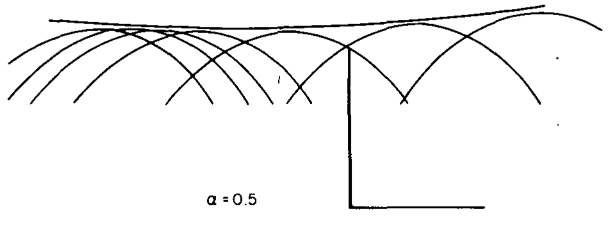

Fig. 1. Family of curves from Eq. (5.1) for constant α and a set of values of C

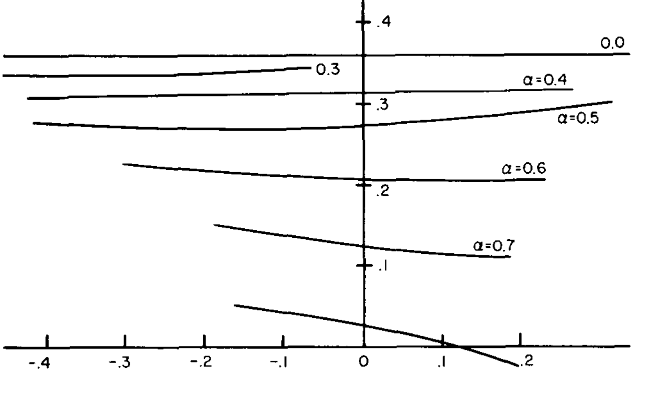

Fig. 2. Envelopes obtained as shown in Fig. 1

We can solve Eq. (5.1) for $e_1/i\alpha$, giving us

$$\frac{F - G}{a_1 H} = \frac{-\dfrac{e_1}{\alpha}}{i\gamma \dfrac{e_1}{\alpha} - 1}, \qquad \frac{\gamma e_1}{i\alpha} = \frac{-(F-G)}{(F-G) - iH},$$

$$\frac{\gamma e_1}{i\alpha} = \frac{-(F-G)}{(F-G) - 1}, \quad \text{for large } \zeta \tag{5.3}$$

$$\frac{F-G}{(F-G) - 1} = Y \quad \text{or} \quad \frac{1}{Y} = 1 - \frac{1}{F-G}; \quad \text{here } Y = i\gamma \frac{e_1}{\alpha}.$$

The family of curves given in Fig. 2 can be transformed graphically by a succession of projective transformations to give the family of curves given in Fig. 3. This is the appropriate form to use for Eq. (5.3). It should be noted that there is one member of this family corresponding to C equals 0, which is a boundary for all members of the family. The value of C for which the family of curves passes through the origin corresponds to a minimum value of C compatible with complete stability for a rigid flat plate. The study of this envelope is in fact a convenient way of visualizing the stability problem for a rigid wall as well as for a nonrigid one and leads to essentially the same results in the former case as the procedure employed by Schlichting.

Let us now proceed to a consideration of the general effect that e_1 will have on the stability of the flow. e_1 will be represented by a family of curves of the two variables

$$\beta = \frac{\bar{U}_m^2 \alpha C}{\nu \sqrt{R}}$$

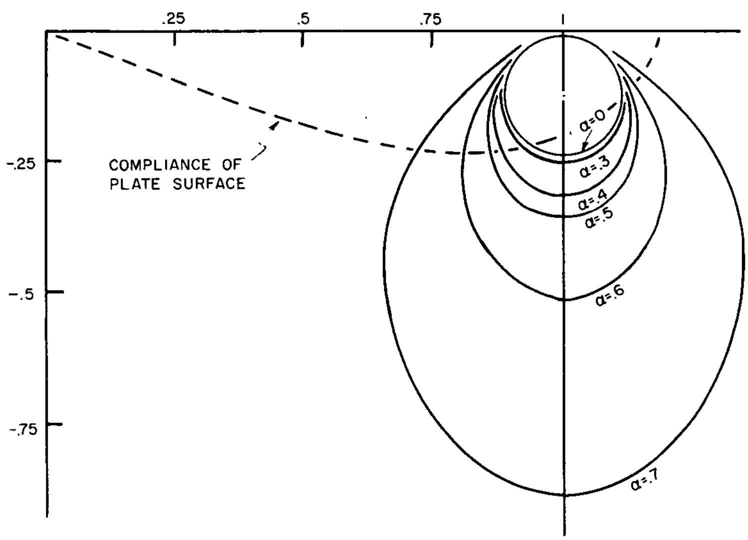

Fig. 3. Family of contours in the compliance, obtained by transformation from Fig. 2

and

$$\bar{\alpha} = \frac{\bar{U}_m \alpha}{\nu\sqrt{R}} ,$$

which we know must be mutually tangent and pass through the origin for small values of both variables. If now for a fixed value of α we describe one of the families of curves, its intersection or intersections with the member of the envelope for a corresponding value of α will give points on the boundary line separating stable conditions from unstable conditions. In general, there will be two such intersections and usually the stable conditions will be those sections between the points of intersection. If the members of the family of curves of the compliance do not intersect the corresponding member of the family of envelopes, then either the system will always be stable or will always be unstable. Since the envelopes corresponding to unstable conditions for a rigid wall always enclose the origin and since the compliances must pass through the origin, it follows that there will be at all times at least one intersection with the compliance unless the compliance is so small as to be completely enclosed by the envelope. Since, however, in this family of curves the portion closest to the origin represents unstable conditions, it follows also that in the case where there is no intersection between the compliance and the envelope that the boundary layer will always be unstable. Consequently, in the case of interest there will always be at least one intersection corresponding to low frequencies between the members of the envelope and the family of compliances. These curves considered as a function of R, β, and the velocity of flow will form a family from which the extreme values of the Reynolds number can be deduced.

SUMMARY

We have given a very brief discussion of the principles on which an analysis of the effect of a flexible surface on fluid flow is based. It shows that through the introduction of the concept of surface compliance, the conditions of stability of laminar flow may be analyzed. A more detailed presentation would allow us to give conditions which will be fulfilled for a stable flow. In general, we show that the flow will be stable when certain contours in the complex plane exclude the origin. The presence of the flexible wall replaces the origin by a curve and the points of instability are the intersection of this curve with the contours.

APPENDIX

Reference 4 gives the relation

$$F(\zeta) = \frac{\psi(\zeta)}{\zeta\psi'(\zeta)} = \frac{\displaystyle\int_{+\infty}^{-\zeta}\int_{+\infty}^{\xi} \rho^{1/2} H'_{1/3}\left[\frac{2}{3}(i\rho)^{3/2}\right] d\rho\, d\xi}{\displaystyle -\int_{+\infty}^{-\zeta} \rho^{1/2} H'_{1/3}\left[\frac{2}{3}(i\rho)^{3/2}\right] d\rho} . \tag{A1}$$

For large values of ζ it is shown that

$$F(\zeta) = \zeta^{-3/2} e^{i\pi/4} \tag{A2}$$

We wish to calculate $\psi'''/\zeta\psi'$ in terms of F and its derivatives. Let

$$h = \frac{\psi'}{\psi} = \frac{1}{\zeta F(\zeta)} . \tag{A3}$$

In terms of h we obtain

$$\frac{\psi'''}{\zeta\psi'} = \frac{1}{\zeta}\left(h^2 + 3h' + \frac{h''}{h}\right) . \tag{A4}$$

For large values of ζ we obtain by combination of (A2) and (A4)

$$\frac{\psi'''}{\zeta\psi'} = \frac{1}{\zeta} \zeta e^{-i\pi/2} + \frac{3e^{-i\pi/4}}{2\zeta^{1/2}} - \frac{1}{4\zeta^2} \simeq e^{-i\pi/2} . \tag{A5}$$

ACKNOWLEDGMENTS

The authors take pleasure in thanking Dr. Max O. Kramer and Professor P. J. W. Debye for many helpful suggestions made during the course of this work.

REFERENCES

[1] Tollmein, W., "Göttinger Monographie uber Grenzchichten," Pt B 3, 1956

[2] Schlichting, H., "Zur Entstchung des Turbulentz bei der Plallenstromung," Nachr. Ges. Wiss. Göttinger, Math. Phys. Klasse 182, 1933

[3] Schlichting, H., "Boundary Layer Theory," New York:McGraw-Hill, 1955

[4] Lin, C.C., "On the Stability of Two-Dimensional Parallel Flow," Quarterly of Appl. Math., Vol. III, pp. 117, 218, and 277

[5] Lin, C.C., "The Theory of Hydrodynamic Stability," Cambridge:The University Press, 1955

[6] Squire, H.B., "On the Stability of Three Dimensional Disturbances of Viscous Fluid Between Parallel Walls," Proc. Roy. Soc. A-142 (1933)

[7] Schlichting, H., and Ulrich, A., "Zur Berechnung des Umschlungs Laminar-Turbulent," Jahrbuch d. dt Luftfahrtforshung 18, 1942

[8] Schubauer, G.B., and Shramstad, H.K., "Laminar Boundary Layer Oscillations and Stability of Laminar Flow," National Bureau of Standards Research Paper 1772

[9] Max O. Kramer, Communication Jour. Aero-Space Sciences, June 1957

[10] Max O. Kramer, Communication Jour. Aero-Space Sciences, May 1959

[11] Max O. Kramer, "Boundary Layer Stabilization by Distributed Damping," Jour. Amer. Soc. of Naval Engineers 72:25 (1960)

DISCUSSION

T. G. Lang (U.S. Naval Ordnance Test Station)

I would like to mention that I have observed Dr. Kramer's experiments on models utilizing compliant surfaces in Long Beach Harbor in California and consider his test apparatus to be well designed and his reported results to be accurate. When the theory of distributed damping using compliant surfaces was first proposed by Dr. Kramer, it was mentioned that sea mammals such as porpoises and whales utilize this phenomena to reduce their drag. Reports by many observers of fish and sea mammals indicate high performance. I would like to describe a study which was recently conducted by the U.S. Naval Ordnance Test Station (NOTS) on the performance characteristics of a live porpoise. These results which I shall present are still preliminary since additional analysis is planned.

The performance tests were conducted by a group of NOTS personnel in the towing tank at Convair, San Diego. This tank is 315 feet long, 12 feet wide, and 6-1/2 feet deep. It was filled to a depth of 4-1/2 feet on June 3, 4, and 5 and to 6 feet on June 15 with sea water. This water was continuously filtered except during tests, and chemicals were added to prevent growth of plankton and bacteria. The tests were composed of two types. One type was a peak-effort run down the tank, and the other was a motionless glide through a series of large underwater hoops. The porpoise was tested both in its natural condition and with a ring, whose thickness varies from 1/16 to 1 inch, placed around its head section. The purpose of the thinnest ring was to induce turbulence on the body. The thicker rings were used to significantly increase the drag of the porpoise by a fixed amount and thereby aid in determining its horsepower output when the top speed with each ring is known.

Figure D1 shows a porpoise with such a ring placed around its head section. This is a porpoise similar to the one used in these tests, except it was trained to support itself in the position shown in the figure for several seconds. Figure D2 shows the dimensions of the porpoise (Pacific white-sided dolphin) which was tested in this program. It is 6.7 feet long, has a maximum diameter of 1.2 feet, and weighs 200 pounds. The distance versus time data were primarily measured by overhead cameras. Two cameras were mounted at the beginning of the runs behind underwater windows, but much of their data was lost due to camera malfunction.

Figure D3 shows that portion of the horsepower which was required on the peak effort runs to produce the recorded acceleration. It is noted that horsepowers up to 1.8 were

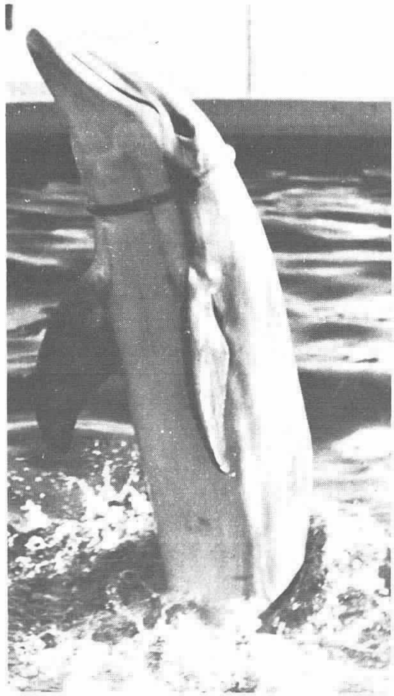

Fig. D1. Placement of ring on the porpoise head section

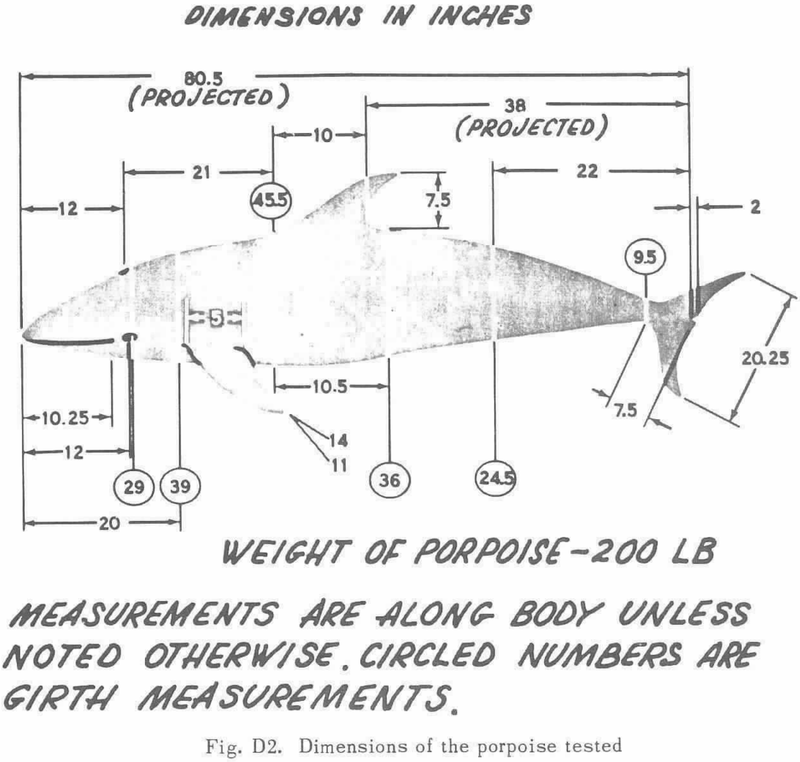

Fig. D2. Dimensions of the porpoise tested

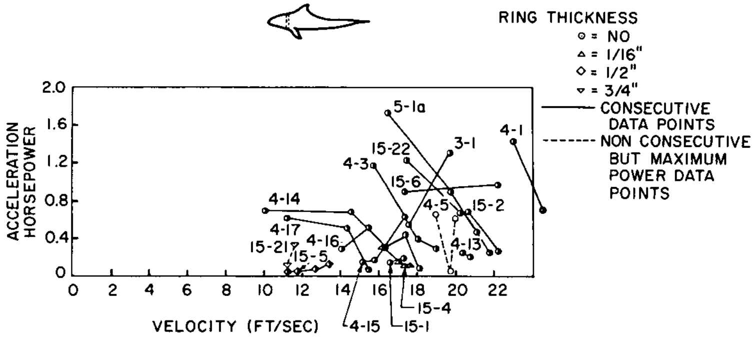

Fig. D3. Acceleration horsepower versus velocity

measured, and this does not include the horsepower expended in overcoming drag. This value of power is in agreement with that which humans can exert for the same time period. The lack of acceleration data at low speeds is due to the previously mentioned camera malfunction.

Figure D4 shows the results of the glide runs wherein the drag was calculated from the deceleration rate as the porpoise glided through underwater hoops. The data are plotted as drag area versus glide velocity, wherein the effects of virtual mass are included. This drag area is the drag divided by $1/2\ \rho V^2$. The scatter of points for any one configuration is believed due to movement of the porpoise while gliding. For numerous reasons, the maxima of these points are considered to approach the correct drag value of a motionless gliding porpoise. Figure D5 is a plot of this same data against ring thickness. The solid lines are the estimated drag area at various glide speeds. The dotted line is a curve faired through the maximum drag area data points and is believed to represent the minimum value of the experimental drag area of the porpoise. The wave drag has not been subtracted from this data, but it has been calculated to be small at the higher glide speeds.

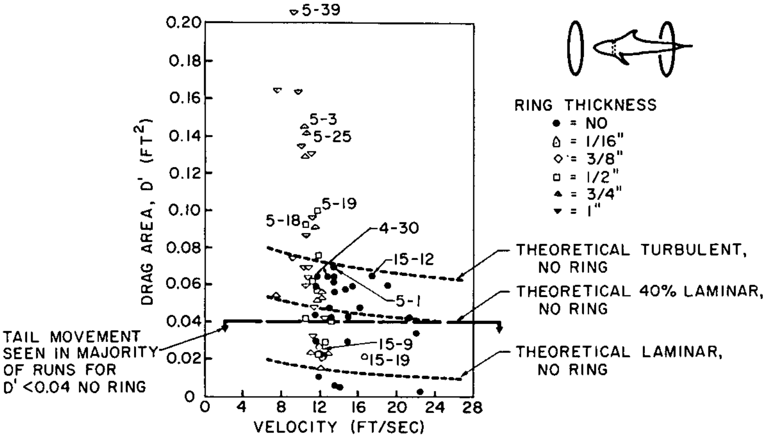

Fig. D4. Drag area versus velocity

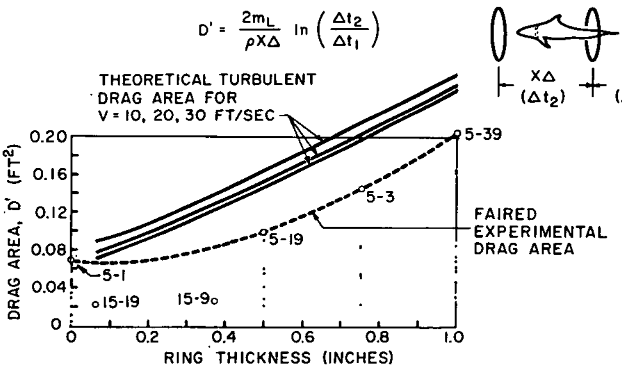

Fig. D5. Drag area versus ring thickness

Figure D6 shows the maximum recorded speed versus ring thickness. With no ring, the maximum speed was 25 ft/sec, which is only 15 knots. It is noted that the water depth has no effect on top speed, since it was 4-1/2 feet for the runs on June 3, 4, and 5, and 6 feet for the June 15 runs. Figure D7 shows the drag horsepower versus ring thickness. The drag horsepower was not measured but was calculated using the dotted-line experimental drag area of Fig. D5 and the top speed from Fig. D6. Of primary interest is the fact that the 2.0-horsepower maximum, recorded for the no-ring condition, agrees well with the recorded maximum acceleration horsepower. These results tend to indicate that the effective drag while swimming is essentially the same as that while gliding, and that the boundary layer in each case is effectively turbulent. It is noted, however, that the horsepower for the runs with rings is only 0.8 rather than 2.0. This fact would tend to indicate either that the rings had a large effect on the boundary layer while swimming, that they caused the porpoise to exert less effort due to irritation or some other cause, or else that the experimental ring drag values are low. The time during which the purpoise traveled at top speed was several seconds longer when a ring was carried than when it was not, so the expended horsepower with rings would have been expected to be around 1.3 to 1.6 rather than 2.0.

Figure D8 shows the porpoise body movement during a typical cycle while accelerating. The nose stations are aligned and the body position is sketched from different film frames

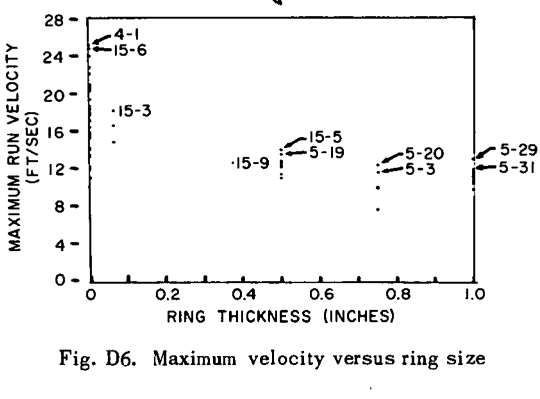

Fig. D6. Maximum velocity versus ring size

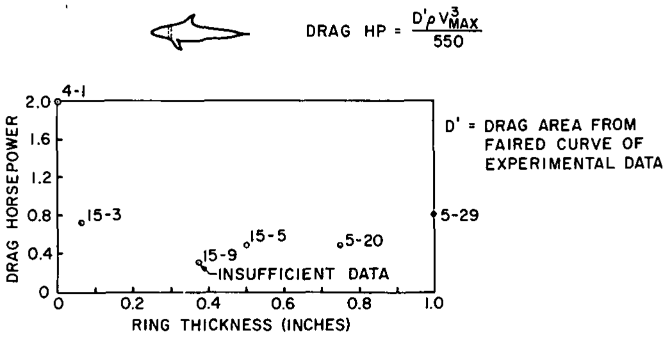

Fig. D7. Maximum drag horsepower versus ring thickness

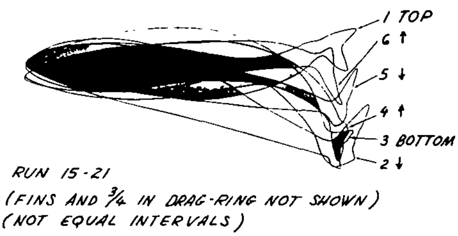

Fig. D8. Porpoise movements during acceleration, nose sections aligned

to show body and tail movement. The fins have been removed from the sketches to improve clarity.

In conclusion, it can be stated that no unusual physical or hydrodynamic phenomena were apparent in this porpoise study. These tests, however, have not proved that unusual characteristics do not exist. It is possible that: (a) turbulence in the tank water or the added chemicals affected the ability of the porpoise to control its boundary layer, (b) the porpoise did not exert maximum effort, or (c) that unknown factors affected the results due to the unnatural environment. A report is in process which contains more detailed information on the study.

M. Landahl (Massachusetts Institute of Technology)

I want to comment on an error in the boundary condition, but first I want to point out that the present author here is not the only one who happened to fall in that trap because it is very easy to do it. The reason for it is as follows: When you have a wall, of course, you know that the steady boundary layer is such that whatever waves you may have superimposed on top of this layer, all components of velocity must vanish at the wall. But when you have a flexible wall you have to consider that you must satisfy the boundary condition on the wall itself and not on the mean position and this gives an extra term due to the finer slope of the profile which accounts for this extra term. This problem is really quite intriguing in a way; when you straighten things like this out it turns out that this problem mathematically is very simple and, as a matter of fact, is simpler than the classical problem. The reason for this I will point out in a minute, I will first mention that other investigations in this area have been published and are on the way to being published. There is a forthcoming paper in the Journal of Fluid Mechanics by Dr. Bannerman of Cambridge and I think that his paper is by far the most comprehensive and has very interesting physical discussion on it. I said that the mathematical problem is really quite simple, however, the difficulties are in the physical interpretation of the results which are, I must say, extremely difficult. First of all, the reason why the mathematical problem is so simple is that you can state this problem as a direct problem instead of as an indirect problem. In this particular case you can, instead of asking, what is the stability for a particular layer, ask what kind of wall admittance should one choose to maintain neutrally stable oscillations for waves of a particular wavelength and wave number. This is quite straightforward and as a matter of fact if you straighten out all these initial difficulties the results come out to be extremely simple. The results show that in order to take care of the instability completely you have to supply energy into the boundary layer and consequently a passive wall layer will never do the trick. However, it turns out that a pure compliance of the surface, a pure spring compliance, that is, a very flexible wall, will push the minimum Reynolds number up quite a bit. You can push it up, as Dr. Boggs pointed out, as far as possible, but there is one particular thing here I must say I don't quite understand completely yet. It looks as if you say to the boundary layer, I take 5 percent of the energy out of you, and then the boundary layer says, all right, I give you 10 percent. The more energy you take out of it the more energy the boundary layer supplies, and this is a very peculiar thing, which I think has to be understood more completely before you can really design these layers with good knowledge.

F. S. Burt (Admiralty Research Laboratory)

At ARL we also discussed this problem of the use of damping in the skin to stabilize the laminar boundary layer and came to a similar conclusion, that in the normal instability case one might have to put energy into the layer rather than take it out. The only other thing I would like to ask Mr. Boggs is if he can give us some figures for the critical Reynolds number, both maximum and minimum that he was referring to in his figure, preferably in terms of length to the transition point, which is the more normal Reynolds number used for transition.

F. W. Boggs

To the question of the energy, I would like to point out one thing. We have found, as a practical matter, that when the mechanical loss in the rubber that we use in the coating becomes large, the coating becomes substantially ineffective, maybe even detrimental. As to how energy might be transferred I wonder whether it wouldn't be through the propagation; our feeling is, and my impression is also, from an examination of our data that it is only when you have propagation that you will have stabilization. If you have propagation of the Schlichting wave which differs from the propagation of the wave in the coating (let's suppose, just to make things simple, that there is a factor of 2), then if they are in phase at one spot, half a wavelength ahead they will be out of phase. So whereas in one case you would be transferring the energy into the boundary, a wavelength or half a wavelength away the energy transfer would be taking place in the other direction. I would think that this is the type of mechanism that might explain it. As to Mr. Lang's comments, we have discussed this and we have seen reports on porpoises that travel at very high speeds, but I don't know, every time you talk to somebody he gives you something higher. The Scientific American, where we all get our information now in the United States, had an article on this subject and they quoted reliably something of the order of 25 knots, if I recall correctly. Two horsepower is about what the biologists will tell us the porpoise should deliver. I am quite interested that this is the figure that is gotten because this is just about right on the basis of the body weight and the amount of muscle and what we know about the muscle efficiency. The only question is, does the porpoise go 25 knots? The 2-horsepower figure cannot explain the 25 knots, even though it can explain the 15, so I think this is the question, are the speeds of the porpoises that we have just heard proverbial fish stories or are they the truth? I think that Mr. Burt's question perhaps I have answered already, except for the Reynolds number. We have rough calculations. The limits, I am sorry, I don't know in length; I haven't made the calculation, but they lie somewhere (and these calculations are uncorrected by Mr. Landahl's remarks so they are subject to some doubt) within the area of maybe between 10^7 and 10^9, about 45 miles an hour (excuse my nonnautical language) for the coatings which have given us the best results. This is fairly consistent with our experimental data. The calculations are rough and could easily be off by a factor of two and they might be off by a lot more, but we do not get 10^{20} or anything like that. These are modest figures.

* * *