UNSTEADY SUPERCAVITATING FLOWS

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INTRODUCTION

The problem of unsteady flows with wake formation of with separated free streamlines has recently stimulated some research interest. To fix ideas, consider the flow past a plate with its broad side facing the stream and oscillating about a fixed point as shown in Fig. 1. The flow separates from the plate at points A and A', and we can adopt the hypothesis that the free streamlines AB and A'B' bound a region, called the wake or the cavity, of constant pressure p_c which may be different from the pressure at infinity, p_∞. In the absence of the unsteady motion, the streamlines of constant pressure in this incompressible irrotational flow imply streamlines of constant velocity. However, the velocity on the free streamlines in an unsteady flow is no longer constant since the pressure depends also on the time-rate of change of the local velocity field, and furthermore, the unsteady motion of the body causes vorticity of varying strength to be shed from the body and carried downstream along the free boundaries. When the plate inclination decreases to a small angle with the undisturbed flow, the wake may disappear, the two vortex sheets then coalesce into a single sheet, and the flow becomes that considered in the unsteady airfoil theory. Thus the unsteady flow with wake formation differs from the airfoil theory essentially by the presence of two vortex sheets instead of one, and the problem is thereby considerably complicated.

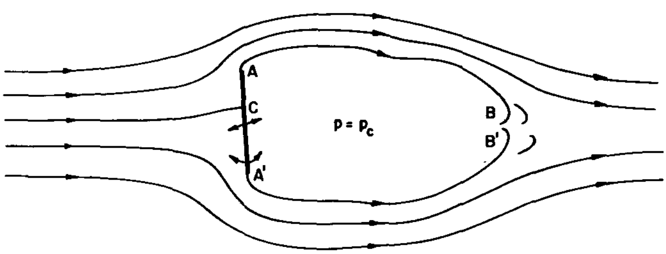

Fig. 1 - Flow past a plate

Aside from these conspicuous points, other new features may arise in the problem of the unsteady motion with wake formation; they may be basically different from the corresponding features characterizing the steady cavity flows and hence are pertinent to the general formulation of the problem. One of these features is the possibility for the volume of a finite or infinite wake to vary with time. For cavity flows of an incompressible liquid surrounding a vapor cavity, it is obvious that, when the volume of the vapor cavity changes, the conservation of mass and conservation of volume of the entire flow become incompatible because of the difference in the liquid density and the vapor density. Consequently, any variation of the cavity volume must come from a source distribution whose net strength depends on the time rate of change of the cavity volume. A direct consequence of this source distribution in a two-dimensional flow of infinite extent is that it generates a pressure field which is logarithmically singular at infinite distances. Thus it is clear that the change in the cavity volume depends on the flow conditions imposed at infinity. More precisely, the cavity volume will change if the flow at infinity is of a source type, and it will not change if otherwise. Physically this can be seen as follows. Consider an obstacle moving through an incompressible liquid contained in a rigid container, large enough to make the wall effects negligible. After the cavity trailing behind the body is established, e.g., by removing the liquid through openings of the container and by allowing the obstacle to move fast enough, the container is then assumed to be perfectly rigid and free from leakage. Then, as the velocity of the obstacle changes in time, the cavity, whether still attached to or detached from the obstacle depending on the later state of the motion, certainly cannot change its volume. On the other hand, if the liquid has a free surface above the obstacle, then the cavity volume may change with the velocity of the motion. Therefore, in order to make the problem completely specified and the solution uniquely determinate, the condition of the velocity and pressure at infinity must also be given as functions of the time. For simplicity, we shall confine ourselves in this work to the case in which the flow at infinity is source-free, and hence the cavity volume will remain constant. However, it should be remarked here that the generalization of this argument to the case of separated flow in air would be open to question; for in this case of one-phase flow, the fluid inside the wake may actually consist of fluid particles originally outside due to the turbulent mixing in the unsteady motion. In any case, if the fluid particles outside a cavity essentially remain outside, then one may assume that the wake or cavity does not change in volume. This is perhaps a reasonable assumption when the unsteady part of the motion is only a small perturbation of a basic steady flow.

Another question about the unsteady wake flow is whether the wake pressure may vary with time. For a liquid flow with a vapor-gas cavity, the thermal process of changing phase on the free boundary and the process of any gas diffusion across the boundary may be regarded to be almost instantaneous with respect to the unsteady motion. Furthermore, the inertia of the vapor or gas inside the cavity has negligible effect on the outer flow because of the large difference in density of the two phases. Hence, for this case at least, the assumption of constant pressure in the cavity is as realistic as for the steady motion. On the other hand, the same assumption, when applied to a separated flow of a single-phase medium, such as air, seems somewhat questionable, because the unsteady motion of the body imparts acceleration to the fluid both outside and inside the cavity, and they respond in turn with the same inertia. However, the assumption of constant pressure in the wake may still be retained as the first approximation. The effect of gravity, which arises in the two-phase flows, is neglected here as usual.

In the following we shall first describe the particular case when the time and space variables can be separated. Next, the hydrodynamic motion shortly after an unsteady motion is introduced will be considered. The discussion will then be concentrated on the case in which the unsteady part of the motion is only a small perturbation of the basic steady flow. Two methods of approach will be described. For a blunt body with a wide wake, the small perturbation is made on the velocity potential; whereas, when the body-wake system is very thin, the acceleration potential is employed. In the former case, the acceleration potential of course also exists, but it is not divergence free. Consequently the value of using it is greatly reduced.

FINITE CAVITIES WITH CONSTANT SHAPE

In 1949 Von Kármán (1) treated an accelerated flow normal to a flat plate which is held fixed in an inertial frame such that with given acceleration of the flow, the flow will separate from the plate to form a closed wake of constant shape behind the plate (Fig. 2). This incompressible, irrotational flow has a velocity potential of the form

$$\Phi(x,y,t) = U(t)\,\varphi(x,y) \tag{1}$$

where x and y are the Cartesian coordinates in the flow plane, fixed with respect to the plate and with the x-axis parallel to the undisturbed uniform stream, and t is the time. The pressure is given by

$$\frac{p}{\rho} + \left(\frac{dU}{dt}\right)\varphi + \frac{1}{2}U^2(\mathrm{grad}\,\varphi)^2 = C(t). \tag{2}$$

Here, φ is a harmonic function of x and y, and satisfies the following boundary conditions:

(i) at infinity $\varphi_x = 1, \quad \varphi_y = 0$

(ii) on the plate $\varphi_x = 0$ (3)

(iii) on the free boundary of the wake, $p = p_c = \text{const.}$

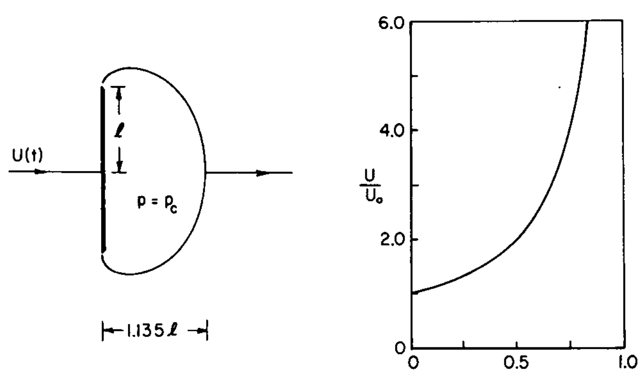

Fig. 2 - Accelerated flow past a fixed plate forming a closed wake of constant shape

The only boundary condition that involves t is (iii). Hence φ will not depend on t, as assumed, when and only when the two time factors in Eq. (2) are proportional to each other; that is,

$$k = a/U^2 = a_0/U_0^2 \tag{4}$$

where k is a real constant, $a(t) = dU/dt$, $a_0 = a(0)$, and the initial velocity of the flow at infinity $U_0 = U(0)$ is assumed to be positive but otherwise arbitrary. Then condition (iii) becomes

$$(\text{grad } \varphi)^2 + 2k\varphi = 0 \qquad \text{on the free boundary.} \tag{5}$$

The arbitrary constant k is determined such that the problem of φ with conditions (3i), (3ii), and (5) possesses a solution. Now Eq. (4) has the general integral

$$U(t) = U_0 \left(1 - \frac{a_0 t}{U_0}\right)^{-1} \tag{6}$$

which characterizes the flow velocity at infinity. Von Kármán obtained the solution of φ for a particular case in which the two branches of the free boundary of the wake meet and close the cavity at a stagnation point, and showed that a closed wake of constant shape is possible only when the "Froude number"

$$U^2/a\ell = U_0^2/a_0\ell = \left[\left(\frac{\sqrt{2}+1}{2}\right)^{1/2} + \frac{1}{2} \text{ arc cos } (\sqrt{2} - 1)\right]^{-1} = 0.599 \tag{7}$$

where ℓ is the half-width of the plate. It can be seen that when the flow at infinity obeys (6) and (7), the velocity on the free boundary decreases from 3.11 U at the separation point to zero at the rear. This is the only case for which a stagnation point can occur on a free boundary of constant pressure.

Writing $w = dF/dz$, with $z = x + iy$ and F the complex potential, we see that F/w, and hence z as determined from $\int dF/w$, is evidently independent of t by virtue of (1). It then follows that the wake shape is constant in time. In fact the rear end of the wake is at a distance 1.135 ℓ from the plate (see Fig. 2). Furthermore, the pressure coefficient defined by

$$C_p = (p - p_c)\ \frac{1}{2}\rho U^2$$

is also independent of t by (2) and (4), and so is the acceleration coefficient

$$C_a = \frac{D}{Dt}(\text{grad } \Phi)\ \frac{1}{2}\rho U^2.$$

This flow is thus also characterized by having C_p and C_a constant in time. It therefore follows that the drag coefficient has a fixed value which can be verified to be

$$C_D = \text{Drag}/\rho U^2 \ell = 10.48\,. \tag{8}$$

The magnitude $m' =$ drag/acceleration may be called the cavity-induced mass; then by (7) and (8),

$$m' = \rho U^2 \ell\, C_D/a = 6.26\, \rho \ell^2. \tag{9a}$$

If the flat plate had undergone an acceleration a normal to the flow without wake formation (a postulated Dirichlet flow), then the induced mass would be

$$m_o' = \pi \rho \ell^2\,. \tag{9b}$$

Thus,

$$m'/m_o' = 1.995. \tag{9c}$$

As pointed out by Tan (2), the law of motion (Eq. 6) for preserving the wake pattern (with Eq. 7 as a special case) is general and unique. The law is general in the sense that it applies to any body shape with well-defined separation points, provided there exists a solution for φ. The law is unique because no other fluid motion will produce such a preserved pattern of the wake flow. For the case of the flat plate, the class of flows characterized by (Eq. 6) has been investigated by Gilbarg (3). It was shown that only under (Eq. 7) the flow (which is that given by Von Kármán) has a stagnation point in the rear, whereas for other values of acceleration obtained from the solution of φ, the cavity shapes all have cusps at the rear. Another type of the flows in this case was discovered to contain a doubly covered cavity subregion (3); these flows were considered as physically unrealistic. The case of polygonal obstacles was also treated by Gilbarg (3).

It should be remarked that the wake flow generated by a flat plate accelerating broadwise through the fluid which is otherwise at rest in an inertial frame is a problem different from the present one since the pressure equation in this case assumes a different form.

SUDDEN ACCELERATION OF CAVITY FLOWS

It is of importance to note an essential difference between the unsteady flows with and without free boundaries. In the determination of the velocity field of the unsteady potential flows without a free boundary, the time appears only as a parameter since no knowledge about the time rate of change of the physical quantities is required in the problem. Only when the pressure field is calculated does the effect of unsteady motion explicitly appear (through the term $\partial\Phi/\partial t$ in the Bernoulli equation). On the other hand, when the wake is present, the flow will depend on its previous time history, because the explicit role of the time enters the problem through the boundary conditions on the free surface. However, when a basic steady wake flow is given a sudden acceleration, there will be no past history of any time-varying disturbances at the moment of the application of this sudden change. Therefore, the problem of finding the flow characteristics at the initial instant of the unsteady cavity flow is expected to be not any more complicated than the general unsteady flow without a cavity. For this reason we shall first consider the initial stage of the reaction of a cavity flow to the sudden acceleration of a rigid boundary in contact with it.

Let $\varphi(\vec{x})$ be the velocity potential of an established steady cavity flow past a stationary rigid body. Suppose now the body is given a sudden acceleration a at time $t = 0$ in the direction opposite to the undisturbed flow, say. Then the velocity potential for small $t > 0$ will assume the form (4)

$$\Phi(\vec{x},t) = \varphi(\vec{x}) + at\,A(\vec{x}) + O(t^2)\,. \tag{10}$$

The quantity A will be referred to as the "initial acceleration potential" for unit acceleration. Since Φ of the resulting flow and φ of the basic steady flow are both harmonic functions, it follows that

$$\nabla^2 A = 0\,. \tag{11}$$

From the momentum equation we see at once that the pressure will assume the form

$$P(\vec{x},t) = p(\vec{x}) + p_1(\vec{x}) + O(t), \qquad p_1 = -\rho a\,A(\vec{x}), \tag{12}$$

where $p(\vec{x})$ described the pressure field of the original steady flow, and p_1 is the pressure generated by the sudden acceleration.

From the boundary condition that the relative normal velocity on the rigid body must vanish, $\partial\Phi/\partial n = -at\,n_1$, where $\vec{n}$ is the unit normal vector on the body and n_1 its component along the direction of undisturbed flow. Since the displacement of the rigid body is $O(t^2)$, this boundary condition can be applied on its undisturbed position. Hence, at $t = 0^+$, we have from (10),

$$\partial A/\partial n = -n_1 \qquad \text{on the rigid boundary.} \tag{13a}$$

By hypothesis, the pressure in the cavity remains constant. Hence, at $t = 0^+$, we have from (12)

$$A = 0 \qquad \text{on the free boundary.} \tag{13b}$$

Furthermore, since this sudden acceleration of the body cannot affect the flow at infinity, we require that

$$A \to 0 \ \text{ as } |\vec{x}| \to \infty \quad \text{in the flow.} \tag{13c}$$

The acceleration potential A is then uniquely determined by (11) and conditions (13a)-(13c). It may be noted that A depends on the basic steady flow through condition (13b), in which the location of the free boundary must be specified.

Sudden Acceleration of a Helmholtz Flow

A simple interesting case is the sudden acceleration of a flat plate in the Helmholtz flow, which has been treated recently by Gurevich (5). The steady Helmholtz flow past a plate set normal to the stream is given by the solution

$$z = -\frac{4\ell}{4+\pi}\left[\frac{2w}{(1-w^2)^2} + \frac{w}{1-w^2} + \tanh^{-1} w\right] \tag{14}$$

in which the origin of $z = x + iy$ is chosen at the plate center, ℓ is the half-width of the plate, the region of the complex velocity $w = u - iv$ is $u \geq 0$ and $|w| \leq 1$, and $w = 1$ at $z = \infty$. The parametric representation of the free boundary is obtained by letting $w = \exp(-i\theta)$, since we have there $|w| = 1$. The boundary condition of A on the plate is now

$$\partial A/\partial x = -1 \quad \text{on} \quad x = 0, \qquad |y| < \ell. \tag{15}$$

The solution of (11) satisfying conditions (15), (13b), and (13c), with the free boundary determined from (14), is readily obtained. It is conveniently expressed as

$$\frac{dG}{dw} = \frac{8i\ell}{\pi(4+\pi)}\,\frac{(1+w^2)^2}{(1-w^2)^3}\left[\log\frac{1+iw}{1-iw} - \frac{i\pi}{1+w^2} + (\pi+2)\,\frac{iw(1-w^2)}{(1+w^2)^2}\right]$$

where $G = A + iB$, B is the conjugate harmonic function of A, and the real part of gives the solution of the problem.

The additional drag due to the sudden acceleration of the plate can be shown to be

$$D_1 = \int_{-\ell}^{\ell} p_1\, dy = -\rho a \int_{-\ell}^{\ell} A\, dy = 1.6896\,\rho a \ell^2 . \tag{16}$$

In this case the initial value of the cavity-induced mass, defined by D_1/a, is

$$m' = D_1/a = 1.6396\,\rho \ell^2 \quad \text{at} \quad t = 0^+ . \tag{17}$$

The ratio of m' to the induced mass m_o' of (9b) is therefore

$$m'/m_o' = 1.6896/\pi = 0.5377 \; (\doteq 7/13) . \tag{18}$$

The result that this ratio is less than unity may be explained in that the cavitated side of the plate, being located in the region of a constant pressure, has no capacity of imparting kinetic energy to the exterior flow.

It may be recalled that the induced mass of an unsteady flow without a cavity or other free boundaries has always the same value whether the flow in the past has been steady or not. However, this is certainly not the case in the presence of free boundary for the reason previously given. Consequently, the present analysis is applicable only to the initial instant when the original steady cavity flow is subjected to a sudden change. Even though its application is limited, this theory nevertheless gives us some general idea about the reaction of a cavity flow to a sudden change of state.

Basic Steady Flows with a Finite Cavity

Since the steady flow will be the basic reference on which the small unsteady perturbations are to be superimposed, the use of the simplest possible representation of the basic flow will obviously facilitate the analysis of the problem. There are several mathematical potential flow models proposed to give a good representation of the steady cavity flows, such as the Riabouchinsky image model, the reentrant jet model, and a third one which will be called here the dissipation wake model, as investigated independently by Joukowsky, Roshko, and Eppler. The physical significance underlying these models and the accuracy of their results in estimating the flow quantities in comparison with the experimental observations have been discussed elsewhere (e.g., see Refs. 4 and 6 and the references cited therein).

For a simple example of the dissipation wake model, consider the case of a flat plate set normal to an otherwise uniform flow, with the physically realistic situation of cavity pressure p_c less than p_∞. The flow in the physical space is represented by this model as shown in Fig. 3. According to this model, the flow separates from the plate at points A and A′, and becomes parallel to the undisturbed flow at D and D′, the

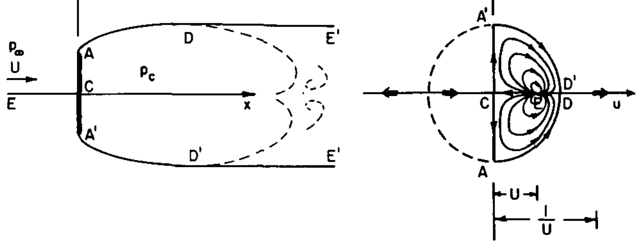

Fig. 3 - Example of flow according to the dissipation wake model

location of which can be determined from the theory. The wake pressure in between AD and A'D' takes the assigned value p_c, and the flow further downstream of D and D' is supposed to be dissipated in such a way that the pressure increases gradually from p_c back to p_∞ in a strip parallel to the flow at infinity. Outside the approximated wake the flow is assumed to be everywhere irrotational. Thus, from the complex potential

$$f(z) = \varphi(x,y) + i\psi(x,y) \tag{19}$$

the complex velocity w can be derived as

$$w(z) = df/dz = u - iv = qe^{-i\theta} \tag{20}$$

where q and θ are the magnitude and direction of the velocity. The pressure is given by

$$p + \frac{1}{2}\rho q^2 = p_\infty + \frac{1}{2}\rho U^2 . \tag{21}$$

By hypothesis, $p = p_c$ on AD and A'D'; hence we have there $q = q_c = 1$ by normalization. Then, in terms of p_c and $q_c = 1$,

$$U = q_c(1 + \kappa)^{-1/2} = (1 + \kappa)^{-1/2}, \qquad \kappa = (p_\infty - p_c)\Big/\frac{1}{2}\rho U^2 \tag{22}$$

where κ is called the cavitation number, an important parameter of the flow. Since $p_c < p_\infty$, then $\kappa > 0$ and $U < 1$. Also by hypothesis, along DE' and D'E', $\theta = 0$, while q decreases from unity back to the free-stream value U.

The boundary condition of $f(z)$ is that $\psi = 0$ on ECADE' and on ECA'D'E'. In view of the qualitative shape of the streamlines $\psi = \text{const.}$ near the point $w = U$ in the w-plane, it is obvious that f must have there a singularity of the form $(w - U)^{-1}$. Hence the boundary condition on ψ can be satisfied exactly by using the method of images, first reflecting $(w - U)^{-1}$ into the unit circle and then into the line $u = 0$, to give

$$f = B\left(\frac{1}{w - U} - \frac{1}{w + U} - \frac{U^{-2}}{w - U^{-1}} + \frac{U^{-2}}{w + U^{-1}}\right) \tag{23}$$

where B is a real constant. Hence

$$z = \int_0^w \frac{df}{dw}\frac{dw}{w} = \frac{2B}{U}\left[\frac{(U^2 - U^{-2})w}{(w^2 - U^2)(w^2 - U^{-2})} - \frac{1}{U}\tanh^{-1}\frac{w}{U} + U\tanh^{-1}(Uw)\right]. \tag{24}$$

If ℓ is the half-width of the plate, so that $z = i\ell$ at $w = -i$, then B is determined by the relation

$$\ell = \frac{2B}{U}\left\{\frac{1 - U^2}{1 + U^2} + \frac{1}{U}\left[\frac{\pi}{2} - (1 + U^2)\tan^{-1}U\right]\right\}. \tag{25}$$

The drag coefficient of this flow is readily obtained from the above solution

$$C_{D_0} = \frac{\text{Drag}}{\rho U^2 \ell} = \frac{\pi}{2}\left\{\frac{U^3}{1 + U^2} + \frac{U^2}{1 - U^2}\left[\frac{\pi}{2} - (1 + U^2)\tan^{-1}U\right]\right\}^{-1}. \tag{26}$$

For small values of κ, (26) may be expanded to yield

$$C_{D_o} \cong \frac{2\pi}{\pi + 4}\left[1 + \kappa + \frac{\kappa^2}{6(4 + \pi)}\right] \tag{26'}$$

which provides a good approximation of (26) within 2 percent error for $\kappa \leqq 1$. The above result (26) agrees with the result of Riabouchinsky's model or of the reentrant jet model within 0.5 percent for $\kappa \leqq 1$ (or $0.707 \leqq U/q_c \leqq 1$). In some other known cases of inclined or curved (or polygonal) obstacles, the result yielded by using this model has been found also to be in good agreement with the experiments (6, 7).

Equations (24) and (25) are Roshko's solution (8), which here has been rederived conveniently by the image method. For the general case of the steady cavity flow past an inclined obstacle with small curvature, it is possible to generalize this method with linearization of the curvature. The same problem solved by using Riabouchinsky's model or the reentrant jet model would be considerably more complicated. Because of the simplicity of the mathematical details generally involved in the dissipation model relative to the other models, this model is preferred here for representing the basic flow.

The Effect of Cavity Size on Accelerated Cavity Flows

Consider now the problem of the sudden acceleration of a flat plate in a cavity flow characterized by the cavitation number κ as defined by (22). By using the dissipation model, the solution of the basic steady flow is given by Eqs. (23)-(25). When the plate is accelerated with acceleration a, opposite to the undisturbed flow, the initial acceleration potential A again obeys the boundary conditions (13b), (13c), and (15). By the conformal transformation

$$f = \frac{2B}{U}\frac{1 - U^2}{1 + U^2}\zeta^2, \qquad \zeta = \xi + i\eta \tag{27}$$

the entire flow region is mapped onto the upper half ζ-plane, with the plate lying on $\eta = 0$, $|\xi| \leqq 1$ and the cavity boundary at $\eta = 0$, $|\xi| \geqq 1$. Again, with $G = A + iB$, condition (13b) may be written as

$$\mathrm{Re}\,\frac{dG}{d\zeta} = \frac{\partial A}{\partial \xi} = 0 \quad \text{on} \quad \eta = 0, \quad |\xi| > 1\,. \tag{28}$$

On the plate, by making use of condition (15), we have

$$\mathrm{Im}\,\frac{dG}{d\zeta} = \mathrm{Im}\,\frac{dG}{dz}\frac{dz}{d\zeta} = \frac{\partial y}{\partial \xi}\frac{\partial A}{\partial x} = -\frac{\partial y}{\partial \xi} \quad \text{on} \quad \eta = 0, \quad |\xi| < 1 \tag{29a}$$

where the value of $\partial y/\partial\xi$ on the plate can be calculated from (23) and (27), to give

$$\frac{\partial y}{\partial \xi} = \frac{2B}{U}\left(\frac{1}{U} - U\right)\left[\sqrt{1 - \xi^2} + \sqrt{1 - \left(\frac{1 - U^2}{1 + U^2}\right)^2 \xi^2}\right] \quad \text{for} \quad |\xi| < 1\,. \tag{29b}$$

The solution of this mixed boundary value problem, prescribed by (28) and (29), is

$$\frac{dG}{d\zeta} = \frac{1}{\pi\sqrt{1 - \zeta^2}}\int_{-1}^{1}\left(-\frac{\partial y}{\partial \xi_1}\right)\frac{\sqrt{1 - \xi_1^2}}{\xi_1 - \zeta}\, d \geqq 1 \tag{30}$$

with $\partial y/\partial\xi$ given by (29b).

The additional drag due to the sudden acceleration of the cavitated plate can be calculated from

$$D_1 = \int_{-\ell}^{\ell} p_1 \, dy = -\rho a \int_{-1}^{1} A \frac{\partial y}{\partial \xi} d\xi = \rho a \int_{-1}^{1} y(\xi) \frac{\partial A}{\partial \xi} d\xi .$$

Substituting (29b) and (30) into the above expression and evaluating the integrals for small values of ϵ, one obtains

$$D_1 = 1.6896 \rho a \ell^2 \, [1 + 0.269 \epsilon^2 + 0.075 \epsilon^4 + 0.049 \epsilon^6 + O(\epsilon^8)] \tag{31}$$

where

$$\epsilon = \frac{1 - U^2}{1 + U^2} = \frac{\kappa}{2 + \kappa} .$$

In the practical range of interest, the cavitation number κ is generally not large, hence ϵ is always appreciably less than unity. Consequently (31) may be used to cover a wide range of κ. For $\kappa << 1$ in particular, (31) may be approximated by

$$D_1 = 1.6896 \rho a \ell^2 \, [1 + 0.067 \kappa^2 (1 - \kappa) + O(\kappa^4)] . \tag{32}$$

As $\kappa \to 0$, this result reduces to the previous solution, Eq. (16).

It is of interest to note that the effect of the cavitation number on the part of the drag due to sudden acceleration, denoted by D_1, starts with the term of $O(\kappa^2)$; hence D_1 increases at a much slower rate than the steady part of the drag, D_o (see Eq. 26'), with increasing κ.

PERTURBATION OF UNSTEADY PLANE FLOWS PAST BLUNT BODIES WITH FINITE CAVITIES

Since the general case of the unsteady wake flow having a large time-varying part is virtually unsolved, we shall first consider the special case of the small unsteady perturbations superimposed on the basic steady flow past a blunt body. It is convenient to formulate this latter problem by dividing the flow into two parts, namely, (a) the basic steady flow, the problem of which is in general nonlinear, and (b) the small unsteady perturbations, the problem of determining them to be linearized. This method of approach has already been used in several recent works on this general subject. A scheme of small perturbations of the basic steady flows has been applied by Ablow and Hayes (9) to treat two unsteady free surface flows: the hollow vortex and the problem of the Borda mouthpiece. The problem of unsteady flow past curved obstacles with an infinite wake has been treated by Woods (10-12), applying small perturbations to the basic Helmholtz-Kirchhoff flow. Here this method will be generalized to the case of finite cavities.

Suppose for $t \geqq 0$ the rigid body in the cavity flow is given an unsteady motion of small amplitude. The small perturbation assumption then implies that for $t \geqq 0$ the complex potential, complex velocity, and the pressure of the resulting flow may be expressed as

$$\begin{aligned} F(z,t) &= f_0(z) + f_1(z,t), \\ W(z,t) &= dF/dz = w_0(z) + w_1(z,t) = Q\, e^{-i\Theta} \\ P(x,y,t) &= p_0(x,y) + p_1(x,y,t) \end{aligned} \tag{33}$$

in which $f_0(z)$, $w_0(z)$, and $p_0(x,y)$ represent the basic steady flow, which, as was discussed in the section "Basic Steady Flows with a Finite Cavity," are assumed to be known here. The time-dependent perturbations f_1, w_1, and p_1 are assumed to have moduli much smaller than those of the basic flow, except possibly at some isolated points. Consequently, relations among the flow quantities are assumed to hold only to the first order in the small perturbations. Here, the space variable z, following the Eulerian description, is not perturbed. It is further noted that $f_1(z,t)$ and $w_1(z,t)$ are both analytic functions of z for every t and continuously differentiable in t if the motion of the rigid body has a continuous time derivative. The pressure equation is now

$$\frac{P}{\rho} + \frac{\partial \varphi_1}{\partial t} + \frac{1}{2} Q^2 = C \tag{34}$$

where φ_1 is the real part of f_1, $Q = |W|$, and C may be taken as a constant.

To apply the boundary conditions, it is convenient to introduce the new variables:

$$\Omega_1 \equiv \tau_1 + i\theta_1 = \log \frac{w_0}{W} = \log \frac{g_0}{Q} + i(\Theta - \theta_0). \tag{35a}$$

Hence, to the first-order terms,

$$\tau_1 = -\log\left(1 + \frac{Q - q_0}{q_0}\right) = -\frac{Q - q_0}{q_0}, \qquad \theta_1 = \Theta - \theta_0 \tag{35b}$$

so that τ_1 represents the percentage change in the velocity magnitude, and θ_1 denotes the change in the flow angle, both with respect to the basic flow (see Fig. 4). Thus, Ω_1 is an analytic function of z, and hence of any analytic function of z, and Ω_1 is a small quantity of the first order. The equation for the pressure disturbance, from (21) and (34), is

$$\frac{p_1}{\rho} + \frac{\partial \varphi_1}{\partial t} - q_0^2 \tau = 0 . \tag{36}$$

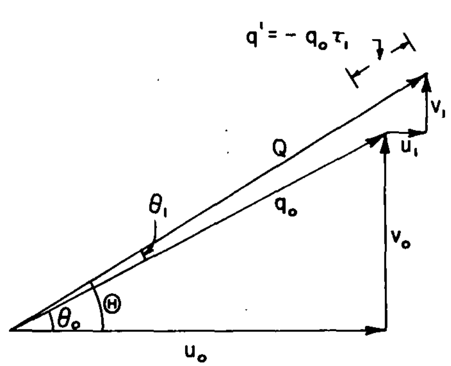

Fig. 4 - Unsteady flow expressed as a perturbation on the basic steady flow

The boundary conditions of the perturbed flow may be stated as follows:

1. In general, the motion of the rigid boundary consists of a small translation and a small rotation. Let (x_0, y_0) be a point on the rigid boundary in the basic flow; then its perturbed position may be expressed as

$$\begin{aligned} X &= x_0 + \epsilon(t) - \alpha(t) y_0 , \\ Y &= y_0 + \delta(t) + \alpha(t) x_0 \end{aligned} \tag{37}$$

where ϵ, δ, and α are arbitrary functions of t with small magnitudes. From this expression, the perturbation velocities q_n and q_s of the flow normal and tangential to the rigid surface are readily obtained. Since the motion is infinitesimal, the value of (q_n, q_s) at (X, Y) may be approximated by their value at (x_0, y_0). Doing so, we have, to the first order in q_n and q_s,

$$\theta_1 = \tan^{-1}\frac{q_n}{q_0 + q_s} = \frac{q_n}{q_0} \quad \text{on the unperturbed rigid surface} \tag{38}$$

where q_n is the component of the velocity $(\dot{\epsilon} - v_0\alpha - y_0\dot{\alpha},\ \dot{\delta} + u_0\alpha + x_0\dot{\alpha})$ normal to the rigid surface. Hence θ_1 is known over the rigid boundary.

2. Over the free boundary, the pressure everywhere in the unsteady cavity wake is assumed to retain its value in the basic flow. Hence, to the first order of approximation,

$$p_1 = 0, \qquad \text{over the unperturbed free boundary.} \tag{39}$$

Hence (33) becomes

$$\frac{\partial \varphi_1}{\partial t} - q_0^2\tau = 0 \quad \text{over the free boundary.}$$

Let s be the arc length along a streamline; then from

$$Q = \frac{\partial \Phi}{\partial s} = \frac{\partial}{\partial s}(\varphi_0 + \varphi_1) = q_0 + \frac{\partial \varphi_1}{\partial s}, \qquad \text{or} \qquad \frac{\partial \varphi_1}{\partial \varphi_0} = -\frac{Q - q_0}{q_0} = -\tau_1$$

we can write the above equation as

$$\frac{\partial}{\partial \varphi_0}(q_0^2\tau_1) + \frac{\partial \tau_1}{\partial t} = 0 \quad \text{on the unperturbed free boundary.} \tag{40}$$

This equation may also be written as

$$\left(\frac{\partial}{\partial t} + q_0\frac{\partial}{\partial s}\right)(q_0^2\tau_1) = 0$$

which expresses the fact that, for any disturbance in τ_1 due to the unsteady motion of the rigid boundary, the quantity $q_0^2\tau_1$ is convected downstream along the free boundary with the basic flow, unchanged in magnitude. In particular, it can be asserted at once that for any $t > 0$, $\tau_1 = 0$ over the portion of the free boundary for $s \geqq s_e$, where s_e is given by

$$t = \int_0^{s_e} ds/q_o .$$

Since q_o is a known function of s, the above condition reduces the problem to that of determining the value of τ_1 at the points of separation. The physical significance of $q_o^2\tau_1$ can be shown to be the vorticity variation due to the unsteady motion, and the above condition is actually the statement of the conservation of vorticity. It can also be seen that the disturbance in τ_1 will generate a surface wave along the free boundary; this surface wave is sustained by the velocity field of the flow in much the same way as the wavy water jet emitted from an oscillating garden hose.

3. At $z = \infty$, we further require that the perturbation does not change the flow direction and pressure at infinity; that is,

$$\text{(a)} \quad \theta_1 \to 0 \quad \text{as} \quad |z| \to \infty \tag{41a}$$

$$\text{(b)} \quad p_1 \to 0 \quad \text{as} \quad |z| \to \infty . \tag{41b}$$

Furthermore, by Kelvin's circulation theorem, the unsteady motion cannot, for any finite t, result in a net vortex at infinity; that is,

$$\text{(c)} \quad \mathrm{Re} \oint_{\Gamma} w_1(z,t)\, dz = 0 \qquad \text{for} \quad t < \infty \tag{42}$$

where Γ is a closed contour with $|z| >> Ut$. It is not difficult to show that these conditions (a)-(c) imply that the perturbation flow at infinity has no net source or vortex and $\Omega_1 = \tau_1 + i\theta_1$ may at most be

$$\Omega_1 = O(z^{-3/2}) \qquad \text{as} \quad |z| \to \infty . \tag{43}$$

The problem of determining $\Omega_1 = \tau_1 + i\theta_1$ with the above boundary conditions of a mixed type (i.e., with θ_1 given over a part of the boundary and τ_1 governed by (40) over the remaining part) can in principle be solved, sometimes conveniently with application of the Laplace transform. Such a detail, however, will be omitted here. Because many of the functions that appeared in the numerical calculation of the general problems are not yet tabulated, a few relatively simple cases have been carried out. Woods (10) treated an interesting example - the problem of a flat plate in the basic Helmholtz flow undergoing an impulsive motion and a harmonic motion. The result shown in Fig. 5 corresponds to the case when the velocity of the plate is increased impulsively from U to (21/20)U for the time period $\tau = 2$ and is then reduced back to U.

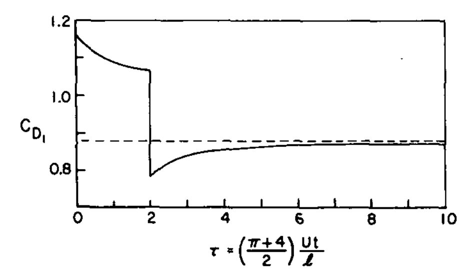

Fig. 5 - Result of impulsively increasing the velocity of a flat plate in the basic Helmholtz flow from U to (21/20)U for the time period $\tau = 2$

UNSTEADY CAVITY FLOWS PAST THIN BODIES: ACCELERATION POTENTIAL

For the case of unsteady cavity flows past slender or thin bodies the method of using the acceleration potential is readily applicable, provided the body-cavity system remains slender in the flow motion. Thus the rigid body under consideration will be a thin body of small thickness and camber, held at a small angle to the flow and with the finite cavity extending downstream of the body. The unsteady motion may consist of a translation and a rotation, both of small amplitude. It is further assumed that (a) in the absence of the unsteady motion, a steady cavity of finite size has already been established, with cavitation number $\kappa > 0$ and hence U and q_c related by (22), and that (b) the unsteady motion is again only a small perturbation of the basic flow. Because the body-cavity system is very thin, it is reasonable to expect that the flow is perturbed by the system only slightly from the uniform state, and the problem may therefore be linearized with respect to the uniform flow, taking the steady and unsteady part as a whole. Based on this approach, a few cases of interest have been treated by Parkin (13) and Wu (14).

The General Formulation

It is of interest to note that, unlike the thin airfoil theory, there are in this problem two characteristic velocities instead of one: U, the characteristic velocity near infinity, and q_c, the other characteristic velocity in the basic steady flow near the cavity. Since the flow quantities near the body and cavity are of direct interest, it is convenient to take q_c as the reference for the entire velocity field.

The flow outside the body-cavity system is governed by

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\, \vec{q} = -\frac{1}{\rho} \nabla P. \tag{44}$$

When the motion is steady, this equation can be integrated to give (21), and hence the relation (22) between U and q_c follows. For the unsteady flows in general, (44) implies that the acceleration vector has a scalar potential $\mu(x, y, t)$ such that

$$\vec{a} \equiv \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\, \vec{q} = q_c^2 \text{ grad } \mu \tag{45}$$

where

$$\mu = -(P - p_c)/\rho q_c^2 . \tag{46}$$

In the above, the constant cavity pressure p_c and the constant velocity q_c on the cavity of the basic flow are chosen for convenience. For definiteness, we take the Cartesian coordinates (x, y) in the flow plane, with the planar body-cavity system lying over a certain region of the x-axis, and write the total velocity as

$$\vec{q}/q_c = (1 + u, v) \tag{47}$$

so that (u, v) represents the nondimensional perturbation velocity.

Now we shall introduce the small perturbation assumption that the flow is only perturbed slightly from a uniform state $(q_c, 0)$ so that $|u|, |v| << 1$ almost everywhere in the flow and all quantities of second and higher order in (u, v) may be neglected. Then (45) is linearized to yield

$$\frac{\partial \mu}{\partial x} = \frac{1}{q_c} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}, \qquad \frac{\partial \mu}{\partial y} = \frac{1}{q_c} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}. \tag{48}$$

Hence from the continuity equation $\operatorname{div} \vec{q} = 0$, it follows that

$$\nabla^2 \mu = 0 . \tag{49}$$

The above formulation can of course be extended to three-dimensional cavity flows past slender bodies as well. It may also be noted that while the acceleration potential μ defined by (45) always exists (even in the flows past blunt bodies), μ satisfies the Laplace equation (49), to the first-order terms, only in the planar or slender flows.*

For two-dimensional problems, (49) implies the existence of a conjugate harmonic function $\nu(x, y, t)$ such that

$$\frac{\partial \mu}{\partial x} = \frac{\partial \nu}{\partial y}, \qquad \frac{\partial \mu}{\partial y} = -\frac{\partial \nu}{\partial x}. \tag{50}$$

*For the case of cavity flows past blunt bodies as treated in the previous section, writing $\vec{q} = \vec{q}_o(x, y) + \vec{q}_1(x, y, t)$ and assuming $|\vec{q}_1| << |\vec{q}_o|$, one readily observes that $\nabla^2 \mu = 4(u_{0x} u_{1x} + u_{0y} u_{1y})$ which is in general not zero; hence the merit of using μ in that case is greatly reduced.

Hence, the complex acceleration potential $G = \mu + i\nu$ must be an analytic function of $z = x + iy$ at any time instant:

$$G(z,t) = \mu(x,y,t) + i\nu(x,y,t). \tag{51}$$

This potential G is related to the complex velocity $w = u - iv$ by

$$\frac{dG}{dz} = \left(\frac{1}{q_c}\ \frac{\partial}{\partial t} + \frac{d}{dz}\right) w(z,t). \tag{52}$$

It may further be noted that although $w(z,t)$ is still an analytic function of z for all t, it is in general not regular everywhere inside the flow, because vortex sheets may be shed continuously from the body due to its unsteady motion, whereas $G(z,t)$, with its real part corresponding to the pressure, will certainly be regular there. It is for this reason that the analysis is simpler with the direct use of $G(z,t)$.

The boundary conditions of this problem are as follows:

1. At upstream infinity, where the disturbances are required to die out, $P = p_\infty$ and $v = 0$. Hence from (46), (22), and (48),

$$\mu = -\frac{\kappa}{2(1+\kappa)}, \qquad \nu = 0, \qquad \text{at} \quad z = \infty. \tag{53}$$

2. The cavity pressure is assumed to remain constant, $P = p_c$. Hence by (46), we have on the cavity boundary

$$\mu = 0 \tag{54}$$

which, after linearization, may be applied on the portion of the x-axis that is the projection of the cavity boundary from above or from below. This condition is valid as long as the intermediate portion of the cavity boundary does not reattach to the solid surface. For simplicity, the body length is normalized to unity so that it occupies the region $0 < x < 1$ and the cavity extends from certain known separation points to $x = \ell(t)$, $\ell > 1$ or $\ell < 1$. The value $\ell(t)$ is determined as a part of the problem for given cavitation number κ, or vice versa.

3. The motion of the wetted rigid boundary may be described in general by

$$y = h(x,t) \tag{55}$$

with $|h| << 1$ for all t, and the motion is limited to such a type that the wetted portion of the boundary always remains being wetted. Then the y-component of the flow at the rigid surface, in the linearized nondimensional form, is

$$v(x,0,t) = \frac{1}{q_c}\ \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}, \qquad 0 < x < 1. \tag{56}$$

Hence from (48) and (52), on $0 < x < 1$,

$$\frac{\partial}{\partial x}\,\nu(x,0,t) = -\left(\frac{1}{q_c}\ \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v(x,0,t) = -\left(\frac{1}{q_c}\ \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 h(x,t). \tag{57}$$

Here, again, both the above conditions may be applied on the projection of the wetted rigid surface on the x-axis. The integral of v in terms of ν may be obtained from (57) as

$$v(x,0,t) = - \int_{-\infty}^{x} H\left(\xi, 0, t + \frac{\xi - x}{q_c}\right) d\xi, \qquad H(x,y,t) = \frac{\partial}{\partial x} \nu(x,y,t) \tag{58}$$

provided, of course, the integral converges. In this integral the condition that $v = 0$ at $x = -\infty$ is used. It may be emphasized that the solution satisfying the boundary condition on acceleration (57) will also satisfy the boundary condition on velocity (56), provided (58) satisfies (56) also.

4. The condition that the cavity boundary and the wetted rigid surface must form at every instant a closed contour enclosing a constant cavity volume may be expressed by

$$\oint_C v(x, 0\pm, t)\, dx = \mathrm{Im} \oint_C w(z,t)\, dz = 0, \qquad \text{for all } t \tag{59}$$

where C is a contour enclosing the boundary of the body and the cavity.

5. The Kutta condition for the lifting problem, fully cavitated or not, is that

$$G(z,t) \text{ is regular at the trailing edge for all } t. \tag{60}$$

This completes the formulation of the problem. With the cavitation number κ given and the motion of the wetted boundary prescribed, the cavity length $\ell(t,\kappa)$ and the complex acceleration potential $G(z,t,\kappa)$ satisfying these conditions can, at least in principle, be found.

Hydrodynamic Force and Moment

Since $G(z,t,\kappa)$ is an analytic function of z for all t, regular everywhere outside the body-cavity system, the expansion of G near $z = \infty$ must assume the form

$$G(z,t,\kappa) = a_0 + ib_0 + \frac{a_1 + ib_1}{z} + \frac{a_2 + ib_2}{z^2} + \cdots \tag{61}$$

in which the coefficients a_n, b_n may depend on t and κ. From (53) it follows that

$$a_0 = - \frac{\kappa}{2(1+\kappa)}, \qquad b_0 = 0 . \tag{62}$$

These two conditions and condition (59) in general determine the cavity length $\ell(t,\kappa)$ since they contain no other unknowns except ℓ and κ.

It is of interest to analyze the singularities of the solution $G(z,t,\kappa)$. From the boundary conditions imposed it is not difficult to see that for $\kappa > 0$ (and hence $\ell(t,\kappa) < \infty$) $G(z,t,\kappa)$ may admit singularities only at the leading edge of the rigid body and at the rear end of the cavity. Near the cavity end $z = \ell$, we take a path ϵ_ℓ encircling the point $z = \ell$ from the lower to the upper side of the cavity boundary; then $\arg(z - \ell)$ increases by 2π along ϵ_ℓ. Since on the lower surface, $\mu = 0$ and $\nu < 0$ (it can be seen from (48) and (50) that near $z = \ell$, $\nu_x \cong -v_x$ and v is positive) and on the upper surface, $\mu = 0$ and $\nu > 0$, the image point in the G-plane of the point z on ϵ_ℓ describes in the G-plane a path along which $\arg G$ decreases by π, provided along this path $\mu \nmid 0$ (so that $P \nmid p_c$). Then it follows from the theorem of conformal transformation that

$$G(z,t,\kappa) = K(t)(z-\ell)^{-\frac{1}{2}}[1 + O(z-\ell)] \quad \text{near} \quad z = \ell(t) \tag{63}$$

where K is a real coefficient which may depend on t. One may interpret this result by saying that the detailed structure of the wake behind the cavity, often represented by an image of the rigid body or a reentrant jet in the nonlinear problems, is now collapsed to a point singularity in planar cavity flows. This singular behavior of the solution also facilitates the calculation of the drag force.

If a sharp leading edge in a lifting flow is wetted on both sides, then it is known in the airfoil theory that

$$G(z,t,\kappa) = C_1(t)\, z^{-\frac{1}{2}}[1 + O(z)] \qquad \text{near} \quad z = 0. \tag{64}$$

However, if the leading edge is cavitated on one side, then an argument similar to the above leads to the conclusion that

$$G(z,t,\kappa) = C_2(t)\, z^{-\frac{1}{4}}[1 + O(z)] \qquad \text{near} \quad z = 0. \tag{65}$$

The leading edge singularity is thus weaker than in the former case.

From the above solution it is readily found that the lift coefficient is given by (14)

$$C_L = \frac{\text{Lift}}{\left(\frac{1}{2}\rho U^2\right)(\text{chord})} = 4\pi(1+\kappa)\, b_1(t,\kappa) \tag{66}$$

where b_1 is defined in (61). Similarly, the moment coefficient about the leading edge, positive in the nose-up sense, is

$$C_M = \frac{\text{Moment}}{\left(\frac{1}{2}\rho U^2\right)(\text{chord})^2} = -4\pi(1+\kappa)\, b_2(t,\kappa). \tag{67}$$

The calculation of the drag coefficient is more complicated. However, it can be shown (14) that for $\ell(t) > 1$,

$$C_D = 2\pi(1+\kappa)K^2 - (v+\nu)_{x=1}\, C_L + \frac{2(1+\kappa)}{q_c} \oint_B \left[v_t \int_0^x \mu(\xi,y,t)\, d\xi - \mu h_t\right] dx \tag{68}$$

where K is defined in (63) and B denotes the contour enclosing the rigid surface only. For $\ell < 1$, the first term on the right side should be halved.

Oscillating Hydrofoil at $\kappa = 0$

An interesting example of the theory, treated by Parkin (13), is that of a flat plate held at a small angle of attack α_0, fully cavitated with an infinite cavity ($\kappa = 0$), and performing a small plunging oscillation normal to the undisturbed flow. The hydrofoil motion is given by

$$y = h_0\, c\, e^{j\omega t} \qquad 0 < x < c \tag{69}$$

where c is the chord of the plate, h_0 is the nondimensional amplitude, $j = \sqrt{-1}$ is the imaginary unit for the time motion, and $\omega/2\pi$ is the frequency. The solution obtained by Parkin gives

$$C_L = \frac{\pi}{2}\left\{\alpha_0 + \left[\frac{9}{16}k^2 - jkW(k)\right] h_0 e^{j\omega t}\right\} \tag{70}$$

where $k = \omega c/U$ is the "reduced frequency" based on the full chord, and $W(k)$ is a special function which is plotted in Fig. 6. The term $\pi\alpha_0/2$ gives the steady part of the lift as experienced by an inclined lamina in the Helmholtz flow. The second term with $9k^2/16$, being independent of the forward speed when the lift coefficient is converted to the lift force, represents the contribution of the apparent mass to the lift. The last term accounts for the influence of the cavity upon the lift. On the other hand, the unsteady lift on a flat plate airfoil without separation is

$$C_L = 2\pi\left\{\alpha_0 + \left[\frac{1}{4}k^2 - jk\,C\left(\frac{k}{2}\right)\right] h_0 e^{j\omega t}\right\} \tag{71}$$

where $C(k/2)$ is the Theodorsen circulation function. A comparison of these results shows that the apparent mass of an inclined flat plate with an infinite cavity is only 9/16 the apparent mass of the plate in fully wetted flow. Furthermore, as plotted in Fig. 6, $W(k)$ is quite different from $C(k/2)$. This is probably due to the wake structure and the presence of two vortex sheets instead of one single sheet as in the airfoil problem.

Rewrite (70) as

$$C_L - \frac{\pi}{2}\alpha_0 = 2(\ell_1 + j\ell_2)\, h_0\, e^{j\omega t} \tag{72}$$

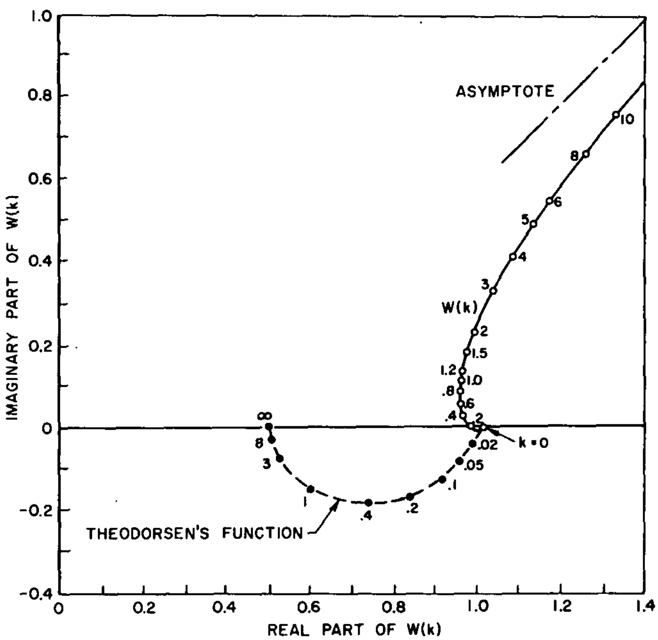

Fig. 6 - Special function in Eq. (70)

with

$$\ell_1 = \frac{\pi}{4}\left[\frac{9}{16} k^2 + k \text{ Im } W(k)\right], \qquad \ell_2 = -\frac{\pi}{4} k \text{ Re } W(k).$$

The quantity ℓ_1 is usually called the "stiffness derivative," and ℓ_2 the "damping derivative." In flutter problems it is the sign of ℓ_2 in a certain degree of freedom which determines whether the motion in that degree of freedom is damped or not by the hydrodynamic forces. The damping derivative ℓ_2 is plotted in Fig. 7, with its value in the fully wetted flow shown for comparison. The same problem with more general locations of the upper separation point has been treated by Woods (11) by using the method described in the section "Perturbation of Unsteady Plane Flows Past Blunt Bodies with Finite Cavities;" his result for the present case agrees exactly with this theory. A comparison with the case of the fully wetted flow shows that there is a loss of damping for plunging oscillation in the cavity flow. In fact, as

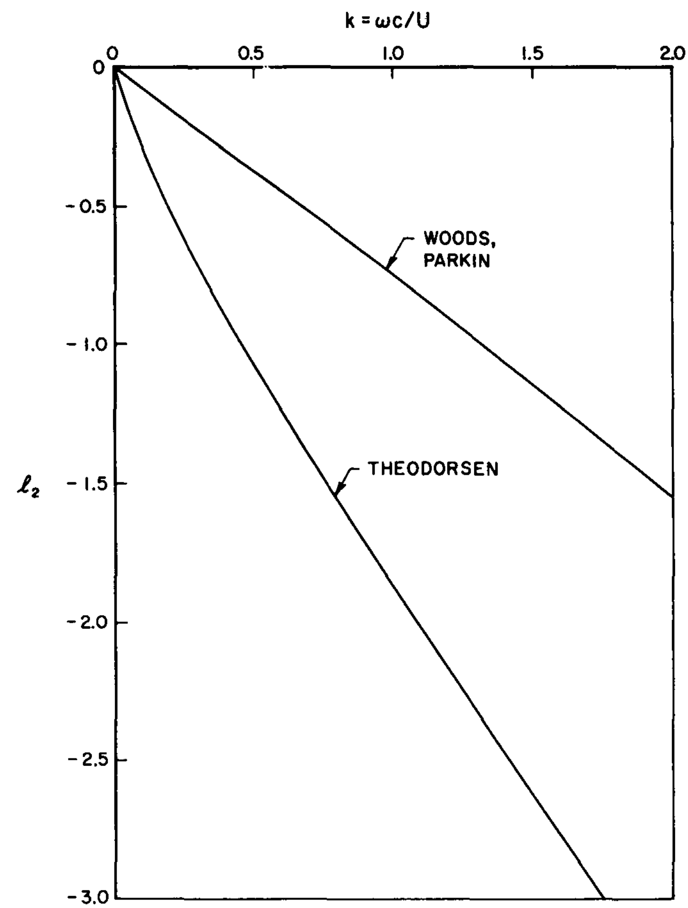

Fig. 7 - Plot (top curve) of the damping derivative ℓ_2 of Eq. (72), compared with the corresponding quantity for flow without separation

shown by Woods, when the flow is allowed to separate from different locations on the upper surface of the plate, in almost all frequency ranges there is a loss of damping effect for both plunging and pitching oscillations, and this loss increases with increasing extent of flow separation. It was also predicted by Woods that a fully stalled airfoil pivoted at its midchord would tend to flutter in the frequency range $0.4 < k < 1.6$ in our notation.

Supercavitating Slender Wedge in Plunging Oscillation

Another example is the problem of a supercavitating slender wedge of half vertex angle γ performing a vertical plunging oscillation. The instantaneous position of the wedge surface is given by

$$h(x,t) = \pm\gamma x + h_0 e^{j\omega t} \qquad \text{on} \quad y = \pm 0, \qquad 0 < x < 1. \tag{73}$$

It has been found (14) that the lift and moment experienced by the cavitated oscillating wedge have the same expression as those experienced by an oscillating airfoil, except that the reduced frequency for $\kappa > 0$ is $k = \omega c/q_c$ instead of $\omega c/U$ as for the fully wetted flow. The drag exerted on the cavitated wedge due to the cavity formation, however, is now

$$C_D = \frac{8\gamma^2}{\pi}(1+\kappa)\frac{\ell}{\ell-1} + 2\pi(1+\kappa)\,h_0^2 k^2\, C^2\left(\frac{k}{2}\right) e^{2j\omega t} \tag{74}$$

where $k = \omega c/q_c$, $C(k/2)$ is the Theodorsen function, and ℓ is cavity length given by the relation

$$\frac{\kappa}{2(1+\kappa)} = \frac{\gamma}{\pi}\left(\frac{2\sqrt{\ell}}{\ell-1} + \log\frac{\sqrt{\ell}+1}{\sqrt{\ell}-1}\right). \tag{75}$$

The first term in (74) gives the drag coefficient of the basic cavity flow over the wedge, while the last term expresses the contribution of the unsteady motion.

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DISCUSSION

B. R. Parkin (California Institute of Technology)

I would like to put a question to Professor Wu concerning his first topic, that is, his elucidation of the theory of Von Kármán. As Professor Wu has mentioned, the coordinate system is chosen in such a way that, being fixed to the body, it is not an inertial frame of reference, and thus it introduces a pressure gradient at infinite, noticed throughout the flow. I would appreciate Professor Wu's indicating the bearing of this pressure gradient at infinity on the values of the coefficients which he quoted from Von Kármán's work. Would these coefficients and the cavity shape be any different from those which would obtain if the liquid at infinity had been at rest?

P. Kaplan (Stevens Institute of Technology)

My comment is concerned with the application of Wu's results, and results to follow, in the general field of naval hydrodynamics. In particular, the requirements of supercavitating devices, which are being emphasized, are such that they have thin sections, combined with great sensitivity of the hydrodynamic forces to the camber. This makes the problem of hydroelasticity appear important for high-speed operation. The forces on the oscillating supercavitating foil are also essential for the study of flutter, and I am happy that Dr. Wu has mentioned the applicability of some of his results to this problem.

T. Y. Wu

In reply to Dr. Parkin's question, I wish to say that the wake flow generated by a flat plate accelerating broadwise through the fluid, which is otherwise at rest in an inertial frame, is a problem different from that treated by Kármán. In this case one may take a coordinate system fixed with respect to the fluid at infinity so that at infinity the velocity is zero and the pressure is equal to p_∞, a constant. Then the plate will be moving with acceleration a. If the shape of the free boundary may still be constant in time, then when the condition of constant pressure is applied on the moving free boundary, one has a different expression for the pressure equation.

Next, I wish to give a uniqueness proof with respect to the leading-edge singularity, $z^{-3/4}$ or $z^{-1/4}$. The first time I looked at the 3/4-power singularity I thought this singularity may be ruled out by the argument that the energy is not integrable at the leading edge. Though this statement is true, I was not too happy with this answer. Then I looked for other physical requirements and found a satisfactory one. I imposed another condition, namely that the pressure outside of the solid body and the cavity wake must not be less than the pressure in the cavity. That is,

$$u = \mathrm{Rl}\, w = -\frac{1}{2} C_p \leq 0 \qquad \text{in the flow field.}$$

Suppose that the solution of w has, in the neighborhood of the leading edge, the expansion

$$w = iAz^{-3/4} + iBz^{-1/4} + O(z^{1/4})$$

where A, B are two real coefficients so that $u = 0$ on the cavity. Then, with $z = re^{i\theta}$, we have near $r = 0$,

$$w = Ar^{-3/4} \sin(3\theta/4) + Br^{-1/4} \sin(\theta/4) + O(r^{1/4}) .$$

From this result we notice that the term $\sin(3\theta/r)$ will change sign, whereas the term $\sin(\theta/4)$ will not, as θ changes from 0 to 2π. Since the first term violates this physical condition on the minimum pressure, we must therefore have $A = 0$.

My own viewpoint with respect to the existence of the flow source is as follows. Suppose we have a two-dimensional cylinder and let it pulsate with its radius R as a function of t in a uniform stream of velocity U; then, the velocity potential is

$$\phi(r,\theta,t) = U(r + R^2/r)\cos\theta + R\dot{R}\log r ,$$

so that we have now a source of strength $2\pi R\dot{R}$ which depends on the rate of change of the cross section of the cylinder. For the problem of unsteady cavity flows, however, it seems that the physical requirement must be imposed that the pressure be finite at infinity. Otherwise, we would require an infinite amount of energy, and hence an infinite time, to create such a flow. Thus the question arises: Why don't we let the cavity volume grow and have an infinite pressure at infinity? At the first sight it seems to me that the affirmative is not the case. One flow model which avoids the flow source at infinity is that, when the volume of the cavity near the body changes, there will be a wake which becomes thinner or fatter in the opposite sense.

In regard to the philosophical question about the physical background of these flow models and their agreement with experiments, I have the following point of view: We realize that all the wake flows are the end product of the real fluid effect. In order to solve the problem in an easy way, however, we want to keep the potential problem as a possible approximation by making some mathematical assumptions, which we call mathematical models. If any mathematical model gives a good approximation of the flow quantities near the solid body, so that we can predict very accurately the total hydrodynamic forces on the body, then, as far as I am concerned, the model should be quite acceptable. However, we should not expect that the simple model is also capable of providing a good description of the complicated wake flow downstream. The problem of the wake flow in the wake is entirely different from those considered here and I believe one cannot obtain a good result without considering the viscous effect, vortex shedding, the turbulent mixing, and so forth.

Several experimental results have been available for a few special cases of the cavity flow past a flat plate inclined at a small angle. These results give substantial support to these mathematical models. With respect to the present linearized model, there are certain features which are different from the nonlinear cases. Here, again, the validity of this model depends on its agreement with future experiments.

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