V

WAVE RESISTANCE OF THIN SHIPS

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The theory of the interaction between a ship or similar body and the undulating sea through which it moves has been developed by various approximations. In particular, in developing the theory of ship waves and wave resistance one neglects viscosity in the general analytical formulation, later trying to take account of it in a patchwork fashion. Even with this simplification, mathematical progress has been made only in cases where one may linearize the boundary conditions on the water surface. The resulting theory is valid only when the disturbance of the surface is small. In situations where one is dealing with waves generated by a moving body, this means that some aspect of the body or its motion must be restricted in a suitable way. A dimensional analysis will usually show one or more parameters whose vanishing leads automatically to vanishing of the surface disturbance. In the case of steadily moving ships and related submerged bodies, the restrictions inherent in the problems treated analytically seem to fall into one of the following classes: 1. "thin" ships, approximating a vertical disc; 2. "flat" ships, approximating a horizontal disc on the free surface; 3. bodies submerged so deeply that their influence on the surface is small; 4. submerged "thin" wings of small angle of attack (in this case approach to the surface leads to a further complication). In the case of oscillating bodies some statement about smallness of amplitude is necessary.

The theory of perturbations has been extended in recent years to surface-wave problems by F. John [1], Stoker and Peters [1], Sekerzh-Zenkovich [1] and others. It was our original intention to make the appropriate expansions in each of the cases listed above, as a means of formulating the proper linearized problem as well as approximations of higher order, and then to survey the existing status of theory and experiment in each case. In the present paper, however, only the first part of this program has been attempted, the rest being left for a later time. Even so, much of the material has been gone over lightly because of the existence of several available surveys. We mention specifically papers by Hogner [2], Wigley [1, 2, 3], Weinblum [10], Havelock [10, 11], and particularly the comprehensive surveys by Lunde [1, 2]. Inui [1] has recently outlined the achievements of Japanese scientists in this field. It seems appropriate to remark that since Michell's original contribution [1], the development of the "thin"-ship theory has been carried on until recent times almost exclusively by T. H. Havelock and L. Sretenskii, the bridging of theory with experiment and design almost exclusively by W. C. S. Wigley and G. Weinblum.

I. LINEARIZATION BY PERTURBATION

Let us suppose that the y-axis is directed vertically upwards, the x-axis to the right and the z-axis coming out of the plane of the paper. We assume that the fluid is bounded partly by a free surface and possibly also by solid boundaries. When the fluid is at rest, the plane $y = 0$ will be taken to coincide with the free surface. Let the equation of the free surface be represented by $y = \eta(x, z, t)$ and the surfaces of solid boundaries by suitable equations. We assume that a velocity potential $\varphi(x, y, z, t)$ exists; here $\vec{v} = \text{grad } \varphi$. Then φ must satisfy certain well-known conditions:

$$\Delta\varphi = 0 \text{ in the region occupied by fluid} \tag{1}$$

$$\eta_x(x,z,t)\varphi_x(x,\eta(x,z,t),z,t) - \varphi_y + \eta_z\varphi_z + \eta_t = 0. \tag{2}$$

$$g\eta(x,z,t) + \varphi_t(x,\eta(x,z,t),z,t) + \tfrac{1}{2}(\text{grad } \varphi)^2 = 0. \tag{3}$$

If the equation of a solid surface is given by $F(x, y, z, t) = 0$, then on such a surface

$$F_x\varphi_x + F_y\varphi_y + F_z\varphi_z + F_t = 0, \tag{4}$$

i.e. The normal velocity of the fluid equals the normal velocity of the body. If the body is fixed, $F_t = 0$ and $\partial\varphi/\partial n = 0$.

The conditions at infinity are somewhat more troublesome; they should at the same time be physically reasonable and guarantee a unique solution. There are, however, few uniqueness theorems for the general problem. If the body starts from rest in still water, one can assume that

$$\lim_{x^2+z^2\to\infty} \eta(x,z,t) = 0. \tag{5}$$

If one wishes to consider a steady-state problem, with, say, $\eta = \eta(x\text{-}ct, z)$, or even a more general motion where the body has moved in one direction for a long time, one requires that the motion vanish far ahead of the body:

$$\lim_{x\to\infty} \eta(x,z,t) = 0. \tag{6}$$

In the case of a steadily oscillating body a "radiation" condition is usually imposed at infinity, but after linearization; this is not, however, always sufficient for a unique solution. At infinite depth one imposes:

$$\lim_{y\to-\infty} \text{grad } \varphi = 0. \tag{7}$$

Let us now suppose that some dimensionless parameter ϵ may be associated with the motion in such a way that when $\epsilon \to 0$ the motion approaches a state of rest. Some examples of the choice of ϵ will be given later. We now assume that φ and other functions entering into the description of the motion may be expanded in power series in ϵ, where φ itself starts with the first power. For the present we shall be concerned only with φ and η:

$$\varphi(x,y,z,t) = \epsilon\varphi_1(x,y,z,t) + \epsilon^2\varphi_2(x,y,z,t) + \cdots,$$

$$\eta(x,z,t) = \eta_0(x,z,t) + \epsilon\eta_1(x,z,t) + \epsilon^2\eta_2(x,z,t) + \cdots.$$

If these expansions are substituted in conditions (1) — (3) above and coefficients of the same power of ϵ grouped together and equated separately to zero, one obtains conditions to be satisfied by each of the φ_i and η_i. One finds immediately: $\eta_o \equiv 0$. Then from the coefficients of ϵ one gets the usual equations for a potential function satisfying the linearized free-surface boundary condition:

$$\Delta\varphi_1 = 0 \quad \text{for} \quad y \leq 0. \tag{8}$$

$$-\varphi_{1y}(x,0,z,t) + \eta_{1t}(x,z,t) = 0. \tag{9}$$

$$g\eta_1(x,z,t) + \varphi_{1t}(x,0,z,t) = 0. \tag{10}$$

The last two conditions combine in the usual way to give:

$$g\varphi_{1y}(x,0,z,t) + \varphi_{1tt}(x,0,z,t) = 0. \tag{11}$$

One notes that it is a result of the method that the boundary condition on φ_i is imposed on the plane $y = 0$ and not on the actual free surface.

The corresponding equations for the coefficients of ϵ^2 are still not too unwieldy to write down:

$$\Delta\varphi_2 = 0. \tag{12}$$

$$\eta_{1x}(x,z,t)\varphi_{1x}(x,0,z,t) + \eta_{1z}\varphi_{1z} - \eta_1\varphi_{1yy} - \varphi_{2y} + \eta_{2t} = 0. \tag{13}$$

$$g\eta_2(x,z,t) + \varphi_{2t}(x,0.z,t) + \eta_1\varphi_{1ty} + \tfrac{1}{2}\,(\text{grad}\ \varphi_1)^2 = 0' \tag{14}$$

After eliminating η_1 and η_2 from the last two equations and performing some manipulations one obtains a counterpart to the condition for φ_1 on $y = 0$:

$$g\varphi_{2y}(x,0,z,t) + \varphi_{2tt} = -\frac{\partial}{\partial t}(\text{grad}\ \varphi_1)^2 + \varphi_{1t}\,(\varphi_{1yy} + g^{-1}\varphi_{1tty}). \tag{15}$$

Conditions for φ_i, $i > 2$, will not be given explicitly. However, inspection of the procedure shows that

$$\Delta\varphi_i = 0, \tag{16}$$

$$g\eta_i(x,z,t) + \varphi_{it}(x,0,z,t) = A_i\{\varphi_1(x,0,z,t), \cdots, \varphi_{i-1}(x,0,z,t)\}, \tag{17}$$

$$g\varphi_{iy}(x,0,z,t) + \varphi_{itt}(x,0,z,t) = B_i\{\varphi_1(x,0,z,t), \cdots, \varphi_{i-1}(x,0,z,t)\}, \tag{18}$$

where A_i and B_i are certain functionals of the functions in braces. It may be noted that after the $\varphi_1, \ldots, \varphi_{i-1}$ have been found, the equations determining φ_i are the same mathematically as those for the linearized problem of wave motion when a prescribed surface pressure $p_o(x, z, t) = \epsilon p_{o1} + \epsilon^2 p_{o2} + \ldots$ is given. In this case the boundary condition at $y = 0$ is $\varphi_{1tt} + g\varphi_1 y + \rho^{-1}p_{o1t} = 0$, and η_1 is given by $g\eta_1 = -\varphi_{1t}(x, 0, z, t) - \rho^{-1}p_{o1}\,(x, z, t)$. For the case of steady motion, this problem has been investigated by Hogner [1] and others. However, this observation is not of great practical value since the expressions for A_i and B_i become rapidly more complicated as i goes beyond 2.

If this method is used to obtain the form of plane progressive periodic waves with $\epsilon = a/\lambda$ ($=$ the ratio of amplitude to wave length) and with constant velocity $c = c_o + \epsilon c_1 + \epsilon^2 c_2 + \ldots$, one finds

$$\eta = A[\epsilon \cos kx + \tfrac{1}{2}Ak\epsilon^2 \cos 2kx + \cdots], \qquad k = 2\pi/\lambda, \tag{19}$$

where A can be determined so that the amplitude is a. One also finds, as expected, $c_o = (g\lambda/2\pi)^{1/2}$ and $c_1 = 0$. A further step in the approximation would show $c_2 \neq 0$.

II. RECTILINEAR MOTION OF A THIN SHIP

Thin Ships. Let us now suppose that the fluid contains a ship which can be described in a coordinate system fixed in the ship by the equation $z = \pm\zeta(x,y)$. We shall assume that the ship is moving in the positive x-direction with velocity $c(t)$. The condition (4) of part I becomes

$$\zeta_x(x - \textstyle\int^t c(t)dt,y)\varphi_x(x,y,\zeta(x - \int^t c(t)dt,y),t) + \zeta_y\varphi_y - \varphi_z - c\zeta_x = 0. \tag{20}$$

We must now select a parameter ϵ. Let $2b$ be the beam and l the length of the ship. Take $\epsilon = b/l$ and write $\zeta(x,y)$ in the form $\zeta = \epsilon\zeta_1(x,y)$. the function $\zeta_1(x,y)$ (with 1 for both length and beam!) is fixed and we consider the family of forms $\epsilon\zeta_1(x,y)$. Clearly, as $\epsilon \to 0$, one approaches the problem of a flat plate of the form of the center-plane section moving lengthwise through the water. This will cause no disturbance of the water, so that the choice of ϵ fulfills the requirements of part I. This choice of ϵ also defines a "thin" ship, sometimes known as a Michell ship, namely, one which can be represented in the form $\zeta = \epsilon\zeta_1(x,y)$ with small values of ϵ.

Equation (4) may now be expanded in powers of ϵ, just as for (1), (2), and

(3), and the coefficients of the various powers of ϵ equated to zero. From the coefficient of ϵ one gets the linearized boundary condition on the ship:

$$\varphi_{1z}(x,y,0^+,t) = -c\zeta_{1x}(x - \int^t c\,dt, y). \tag{21}$$

From the coefficient of ϵ^2 one gets

$$\varphi_{2z}(x,y,0^+,t) = \zeta_{1x}(x - \int^t c\,dt, y)\varphi_{1x}(x,y,0^+,t) + \zeta_{1y}\varphi_{1y} - \zeta_1\varphi_{1zz}. \tag{22}$$

In the part of the plane $z = 0$ outside of the centerplane section of the ship both φ_{1z} and φ_{2z} must be zero, for we have assumed a symmetric ship and φ_z must be zero there. The boundary condition for φ_i, $i > 2$, will be of the form

$$\varphi_{iz}(x,y,0^+,t) = C_i\{\zeta_1, \varphi_1, \cdots, \varphi_{i-1}\}, \tag{23}$$

where C_i represents some functional of the functions in braces.

Several problems immediately occur to one, and, it seems to the author should be considered if the procedure outlined above is to be regarded as more than formal. The first concerns the class of functions to which one must restrict ζ_1. Can, for example, infinite slopes be allowed at the bow and stern? Although it does not interfere with the formal procedure for linearization, it would seem to conflict with the requirement of small surface disturbance if the infinite slope is near the water-line; on the other hand, exclusion of infinite slopes below the waterline would exclude treatment of bulbous bows. Such questions are presumably connected with the convergence of the perturbation series. As far as the author is aware, the only cases where convergence has been proved do not involve solid boundaries penetrating the surface. A further question concerns specification of the proper region of the (x, y)-plane in which condition (23) is to be satisfied. The simplest thing to do would be to take the part of the centerplane section below the undisturbed water plane. However, this would neglect the effect of the part of the hull above the waterplane on the wave-making, and this must, of course, eventually be taken into account. If the region for φ_i is to depend upon the wave profile determined from φ_{i-1} in the preceding step, as seems reasonable, then (23) shows that one must first be prepared to extend one's knowledge of φ_k, $k < i$, beyond the regions in which they are originally defined. This problem does not occur, of course, for a completely submerged "thin" body, for it arises from the meeting of two boundaries with different boundary conditions.

A point worth noting is that it results from the procedure of linearization that the boundary conditions for φ_1, insofar as they concern the ship hull, are imposed on the centerplane of the ship. Indeed, the same remark applies to φ_i for $i > 1$. Thus recent attempts to improve the linearized theory by altering the boundary conditions on the centerplane or by adding additional source distributions off the centerplane do not seem to the author to be well founded as an attempt to find a better approximation to the exact solution. Such attempts usually have as goal to satisfy the exact boundary conditions on the ship hull and the linearized boundary conditions on the free surface. But this is contrary to the physical assumptions inherent in the use of the linearized free-boundary conditions. The perturbation procedure described above does give, subject to the noted difficulties, a systematic procedure for improving simultaneously and step by step the accuracy with which the boundary conditions on both the hull and the free surface are satisfied. On the other hand, one should remain aware that even if one finds φ_2 (or further φ_i) in the preceding formulation, one is still dealing with a "thin" ship and the method is not likely to be suitable for a form which varies radically from a thin ship. For a form such as a barge a different method of linearization would be used, i.e., a different choice of ϵ would be made. Here one should consult remarks of Stoker and Peters [1, pp. 14, 15].

The force on the ship may now be computed by integrating the pressure over the wetted hull:

$$
\begin{aligned}
R &= \iint_{\text{hull}} p \cos (n,x) d\sigma = 2 \iint_S p(x,y,\zeta(x,y))\zeta_x(x,y)dxdy \\
&= -2\rho \iint_S [\varphi_t(x,y,\zeta(x,y)) + \tfrac{1}{2}(\text{grad } \varphi)^2 + gy]\zeta_x(x,y)dxdy
\end{aligned}
\tag{24}
$$

from Bernoulli's equation, where S is the projection of the wetted hull on the centerplane. We now substitute for φ and ζ their expansions in power of ε:

$$
\begin{aligned}
R &= -2\rho \iint_S [\epsilon\varphi_{1t}(x,y,0,t) + \epsilon^2\zeta_1\varphi_{1tz} + \cdots + \epsilon^2\varphi_{2t} + \cdots + \tfrac{1}{2}\epsilon^2(\text{grad } \varphi_1)^2 \\
&\qquad + \cdots + gy]\epsilon\zeta_{1x}dxdy \\
&= -2\rho\epsilon \iint_S gy\zeta_{1x}dxdy - 2\rho\epsilon^2 \iint_S \varphi_{1t}\zeta_{1x}dxdy - 2\rho\epsilon^3 \iint_S [\zeta_1\varphi_{1tz} + \varphi_{2t} \\
&\qquad + \tfrac{1}{2}(\text{grad } \varphi_1)^2]\zeta_{1x}dxdy \\
&\qquad + \cdots .
\end{aligned}
\tag{25}
$$

If one is going to approximate R by computing only as far as φ_i, then one may carry out the integrals only over S_{i-1}, the portion of the centerplane section below $y = \eta_o + \varepsilon\eta_1 + \ldots + \varepsilon^{i-1}\eta_{i-1}$. Thus, in the first approximation one need integrate only over S_0, i.e. the part of the centerplane section below $y = 0$. In this case, the first integral representing the hydrostatic force vanishes. Since this term gives the impression of being the most important, it should be noted that it really starts to contribute to R only with ε^3, for by the mean-value theorem

$$
\begin{aligned}
\iint_S y\zeta_x dxdy &= \int dx \int_0^\eta dy\, y\zeta_x = \int dx\, \zeta_x(x,y_0) \int_0^\eta dy\, y = \tfrac{1}{2} \int dx\, \eta^2\zeta_x(x,y_0) \\
&= \tfrac{1}{2}\epsilon^3 \int dx\, \eta_1{}^2\zeta_{1x}(x,y_0(x)) + \cdots .
\end{aligned}
\tag{26}
$$

In the important case of linearized boundary conditions the resistance R may be written:

$$
R = -2\rho \iint_S \varphi_t(x,0,y,t)\zeta_x(x - \textstyle\int^t c\, dt,\, y)dxdy. \tag{27}
$$

Before passing on to special cases, we observe that the method for finding the linearized boundary conditions for a thin ship generalizes to the case when several thin ships are moving through the water. In case the ships are moving on parallel paths, the linearized boundary condition to be satisfied by φ on each centerplane section $S_0{}^{(i)}$ will be:

$$
\varphi_z(x,y,z_i{}^{\pm},t) = \mp c_i\zeta_x{}^{(i)}(x - \textstyle\int^t c_i dt, y), \qquad (x,y) \text{ in } S_0{}^{(i)}. \tag{28}
$$

Since we shall generally be dealing with linearized problems, it will be convenient hereafter to drop the subscript 1, as we have done just above.

Potential and Resistance for Steady Motion. If a thin ship moves with constant velocity in the x-direction, then $\varphi(x, y, z, t) = \bar{\varphi}(x-ct, y, z,) = \bar{\varphi}(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x}$, $\bar{y}$, $\bar{z}$ are coordinates moving with the ship. Then $\varphi_t = -c\bar{\varphi}_x$, $\varphi_{tt} = c^2\bar{\varphi}_{xx}$, and the linearized boundary conditions for $\bar{\varphi}$ become, after dropping the bars over the letters:

(1) $$g\varphi_y(x,0,z) + c^2\varphi_{xx}(x,0,z) = 0, \tag{29}$$

(2) $$\varphi_z(x,y,0^{\pm}) = \pm\, c\zeta_x(x,y) \quad \text{for} \quad (x,y) \quad \text{on} \quad S_0, \tag{30}$$

(3) $$\lim_{x\to\infty} \varphi_x(x,0,z) = 0, \tag{31}$$

(4) $$\lim_{y\to-\infty} \text{grad } \varphi = 0. \tag{32}$$

One can clearly satisfy (2) by a distribution of sources over S_o with density $-c\zeta_x/2\pi$. However, (1), (3), and (4) must also be satisfied. This can be done if one can find a function $H(x, y, z; \xi, \eta, \zeta) = r^{-1} + h$, $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$, where H satisfies (1), (3), and (4) and h is harmonic for $y < 0$. This function is occasionally called a Havelock source of strength -1. A distribution of Havelock sources over S_o with density $-c\zeta_x/2\pi$ will satisfy all conditions.

The determination of H may be made in several ways. Timman and Vossers [1, 2] use double Fourier transforms, and indeed most methods seem to use them in some way. Perhaps the greatest difference in methods comes from the way in which (3) is satisfied. Havelock [6] and Sretenskii [3] introduce a "fictitious viscosity", a device originally used by Rayleigh [v. Lamb, Hydrodynamics, p 399]. Although Lamb's statement [loc. cit.], "This law of friction does not profess to be altogether a natural one; but it serves to represent in a rough way the effect of dissipative forces", is misleading in that it seems to imply that the fictitious viscosity has some connection with a physical force, its use does require one to interpret certain improper integrals properly, so as to obtain the desired solution. T. Y. Wu and C. R. DePrima in recent unpublished work have shown why this gives the correct solution, and indeed give a physical interpretation to the fictitious viscosity. Timman and Vossers [loc. cit.] and Kochin [1] confront (3) directly. Havelock in several papers [e.g., 11] has derived the function H by letting the source start from rest at time $t = 0$ and move with constant velocity; he then finds the limit as $t \to \infty$. This method has the virtue that (3) is automatically satisfied. In any case, one obtains finally the result:

$$H(x,y,z;\xi,\eta,\zeta) = \frac{1}{r} - \frac{1}{r'} - 4\nu \int_0^{\pi/2} e^{\nu(y+\eta)\sec^2\theta} \sin[\nu(x-\xi)\sec\theta] \cos[\nu(z-\zeta)\sin\theta\sec^2\theta] \sec^2\theta \, d\theta \tag{33}$$

$$- \frac{4\nu}{\pi} \int_0^{\pi/2} d\theta \int_0^{\infty} \frac{e^{\mu(y+\eta)} \cos[\mu(x-\xi)\cos\theta] \cos[\mu(z-\zeta)\sin\theta]}{\mu\cos^2\theta - \nu} d\mu,$$

where

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \quad r'^2 = (x-\xi)^2 + (y+\eta)^2 + (z-\zeta)^2, \quad \nu = g/c^2, \tag{34}$$

and the principal value is to be taken in the second integral. The velocity potential is then given by:

$$\varphi(x,y,z) = \iint_{S_o} \frac{c}{2\pi} \zeta_x(\xi,\eta) H(x,y,z;\xi,\eta,0) d\xi d\eta. \tag{35}$$

In substituting in the formula for the wave resistance, with $\varphi_t = -c\varphi_x$, only the first integral in the expression for H leads to a nonzero term. It gives the well-known Michell's integral:

$$R = \frac{4g^2\rho}{\pi c^2} \iint_{S_o} dxdy \iint_{S_o} d\xi d\eta \zeta_x(x,y)\zeta_x(\xi,\eta) \int_0^{\pi/2} d\theta \sec^3\theta e^{\nu(y+\eta)\sec^2\theta} \cos[\nu(x-\xi)\sec\theta]. \tag{36}$$

For convenience let us denote the last integral with respect to θ by $K(\nu(x-\xi), \nu(y+\eta))$. The function K occurs in the literature in various forms. For example, letting $\lambda = \sec\theta$, one gets

$$K(x,y) = \int_1^{\infty} d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} e^{\lambda^2 y} \cos\lambda x. \tag{37}$$

If one lets $\mu = \nu\lambda$, one gets

$$K(\nu x,\nu y) = \frac{1}{\nu^2}\int_{\nu}^{\infty} d\mu \frac{\mu^2}{\sqrt{\mu^2 - \nu^2}} e^{\mu^2 y/\nu^2} \cos \mu x) \tag{38}$$

Both forms occur frequently. Birkhoff and Kotik [1, 2] introduce new variables: $u = x - \xi$, $v = y + \eta$, resulting in

$$R = \frac{4g^2\rho}{\pi c^2}\int_{-L}^{L} du \int_{-2H}^{O} dv M(u,v) K(\nu u,\nu v), \tag{39}$$

where

$$M(u,v) = \int_{\max(-\frac{1}{2}L,-\frac{1}{2}L-u)}^{\min(\frac{1}{2}L,\frac{1}{2}L-u)} dx \int_{\max(-H,v)}^{\min(H+v,0)} dy\, \zeta_x(u + x, v - y)\zeta_x(x,y); \tag{40}$$

here the hull extends from $-\frac{1}{2}L$ to $\frac{1}{2}L$ and the draft is H. The obvious advantage of this form is that the data concerning the hull are concentrated in the one function M which Birkhoff and Kotik call the "hull function". Finally, by changing the order of integration and using simple properties of the exponential and cosine functions, one may obtain

$$R = \frac{4g^2\rho}{\pi c^2}\int_{1}^{\infty} d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} [P^2 + Q^2], \tag{41}$$

where

$$P = \iint_S dxdy\, \zeta_x(x,y)e^{\nu\lambda^2 y} \cos \nu\lambda x, \qquad Q = \iint_S dxdy\, \zeta_x(x,y)e^{\nu\lambda^2 y} \sin \nu\lambda x. \tag{42}$$

If $\int dx|\zeta_x(x,y)| \leqslant A$, a constant, it is easy to show that the integral for R is absolutely convergent (one gets $R \leqslant 4\pi^{-1}\rho L^2 A^2 c^2$), so that the change of order of integration is allowable [cf. Birkhoff and Kotik, 2, pp. 6-7, 19-21].

If the function $\zeta(x,y)$ is of the simple form $\zeta(x,y) = X(x)Y(y)$, where, say, $Y(0) = 1$ and thus $X(0) = b =$ half the beam, the last expression for R takes on a somewhat simpler expression insofar as one has to deal only with single integrals in the expressions for P and Q:

$$P = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx\, X'(x) \cos \nu\lambda x \cdot \int_{-H}^{0} dy\, Y(y)e^{\nu\lambda^2 y}, \tag{43}$$

and similarly for Q with sine replacing cosine. Ships which may be so represented are called "elementary ships" by Weinblum [10, p. 83] and have been treated extensively by him, Havelock, Wigley and others. In the special case when the ship is wall-sided, $Y(y) = 1$, $-H \leqslant y \leqslant 0$, and the second integral in P and Q becomes

$$(1 - \exp - \nu\lambda^2 H)/\nu\lambda^2.$$

If H is allowed to become infinite, one obtains the wave resistance for an infinitely long vertical strut:

$$R = \frac{4\rho c^2}{\pi}\iint_{-L/2}^{L/2} dx d\xi\, X'(x)X'(\xi)K(\nu(x - \xi)), \tag{44}$$

where

$$K(x) = \int_{1}^{\infty} d\lambda \frac{\cos \lambda x}{\lambda^2\sqrt{\lambda^2 - 1}} \tag{45}$$

The function $K''(x) = \frac{1}{2}\pi Y_0(x)$, where $Y_0(x)$ is Weber's Bessel function of the second kind. Indeed, if in the expression for R, one integrates by parts, once with respect to x and once with respect to ξ, and makes use of $X(-\frac{1}{2}L) = X(\frac{1}{2}L) = 0$, one obtains for the strut (cf. Pavlenko [1]):

$$R = \frac{-2\rho g^2}{c^2} \int\int_{-L/2}^{L/2} dx d\xi\, X(x)X(\xi)Y_0(\nu(x-\xi)). \tag{46}$$

Birkhoff and Kotik [2] have succeeded in reducing the general Michell integral to a form involving $Y_0(x)$. Unfortunately, a statement of the precise result requires considerable preparation, so that we refer to the original paper.

Before passing on to other aspects of Michell's integral, we should like to mention an interesting result of Stoker and Peters [1]. They assume a thin ship, but instead of assuming it towed in a fixed position with constant velocity as we essentially have done, they assume a constant thrust T along the longitudinal direction of the ship and a fixed weight. In the linearized theory this leads to a relation between T and the resulting velocity which again is just Michell's integral, i.e., the linearized theory does not distinguish between thin ships held fixed and ones free to trim. In addition, they obtain expressions for the vertical force and the moment about a transverse axis, as integrals similar in nature, but more complicated than Michell's integral. Thus both the sinkage and trim can be computed also (see also Havelock [8] and Lunde [1, app. 5]). The result as formulated by Stoker and Peters is actually more general than this in that they consider a thin ship heading into periodic infinitesimal waves and find the transient forces and moments which are to be added to the equilibrium ones mentioned above.

Limitations of the Theory. Since ships as actually designed do not seem to have the appearance of what one might expect for a "thin" ship and since they move in water and not in perfect fluids, one may reasonably ask: What are the limitations of Michell's integral in use and what useful information can be obtained from it?

An estimate by purely mathematical means of the possible error in using the linearized theory for a conventional or even a simplified hull shape has not been attempted, as far as the author is aware. One might try to make such an estimate from the second-order theory in the perturbation series, but it would likely be difficult and crude. In any case, this would still be only for a perfect fluid. Comparison with experiment remains, and has frequently been made. This leads to another type of difficulty, how to separate and measure the resistance caused by waves. If L is the length and c the velocity of a towed ship-like form, one may write.

$$R = \tfrac{1}{2}\rho c^2 L^2 C(\mathrm{Re}, F), \tag{47}$$

where $\mathrm{Re} = cL/\nu$, the Reynolds number, and $F = c/(Lg)^{1/2}$, the Froude number. Thus, from tests of a hull model, or of a set of geometrically similar models, one can obtain information about $C(\mathrm{Re}, F)$ for some range of values of these two variables (actually, for points on several lines in the (Re, F)-plane radiating from the origin), but no direct information concerning the wave resistance. Considerable ingenuity has been used in trying to make this step. All methods seem to rely on first separating the contributions of the tangential and normal components of the force on a surface-element of the hull, which gives after integration over the hull

$$C(\mathrm{Re}, F) = C_t(\mathrm{Re}, F) + C_n(\mathrm{Re}, F), \tag{48}$$

and then assuming

$$C_t = C_t(\mathrm{Re}), \qquad C_n = C_n(F). \tag{49}$$

The coefficient C_t is usually estimated from data for the frictional resistance of a flat plate of the same length and wetted area as the model at rest, although occasionally more exact methods taking account of hull curvature and wave profile are tried. C_t does of course depend upon F also since both the wave profile and the consequent local velocities near the hull will change with F. The so-called form resistance will be represented in the coefficient C_n, as well as the wave resistance, and is strongly dependent on the Reynolds number. Methods are available for estimating this part of C_n, but they will not be described here; let us call this estimated coefficient C_e. The wave resistance coefficient is then assumed to be given by

$$C_w = C - C_t - C_e. \tag{50}$$

Experimenters do not, of course, delude themselves concerning the reliability of C_w as a coefficient for comparison with a theoretically computed coefficient. However, it is sometimes difficult to know whether a theoretically predicted effect is really absent or is lost in the working-up of experimental data. One can find more comprehensive discussions of these difficulties in various places, for example, Birkhoff, Korvin-Kroukovsky and Kotik [1] and Apukhtin and Voitkunskii [1].

In view of the preceding remarks it may seem surprising how good the agreement sometimes is between the experimentally estimated and the theoretically computed wave resistance. Figure 1 from a report by Weinblum, Kendrick and M. A. Todd [1] reproduces some values of C_w computed from Michell's integral and a faired curve of the "residuary resistance coefficient" $C_r = C - C_t$ derived from experiments on a towed "friction plane" (here the coefficients are formed by dividing R by $\frac{1}{2}\rho c^2 S$, S the wetted area; this changes only the vertical scale). One would expect good agreement here, if ever, for the towed body approaches in form what one would expect of a "thin ship" (here beam-length ratio = 0.0265), an accurate estimate of C_t is possible, and C_e should be negligible. And indeed the agreement is good. Comparison of measured and computed wave profiles in the same report shows fairly good agreement at Froude numbers 0.292 and 0.326, but progressively worse as the Froude number increases. Here, however, the comparison is somewhat inconclusive since the calculations were made for a form of infinite draft.

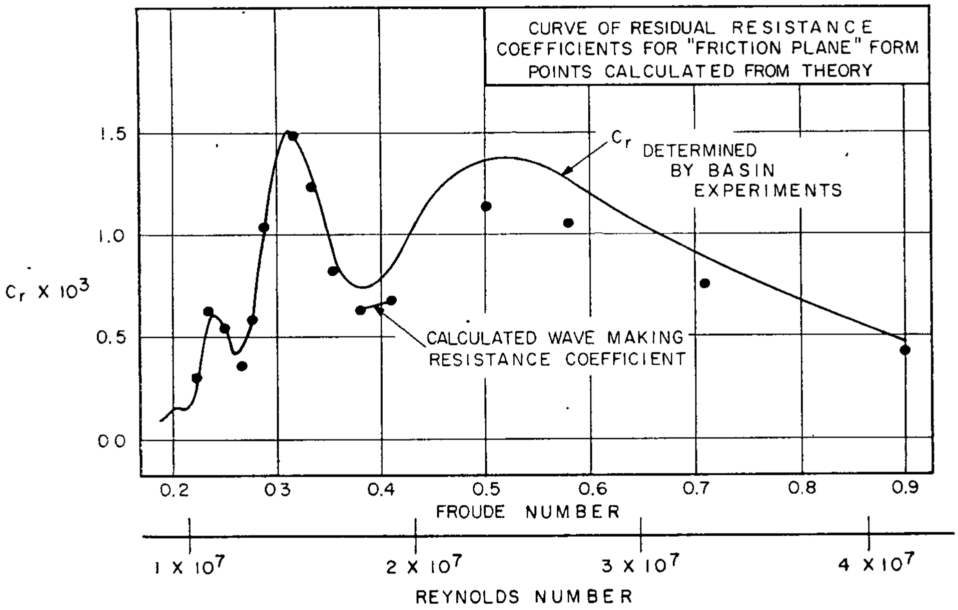

Figure 1.

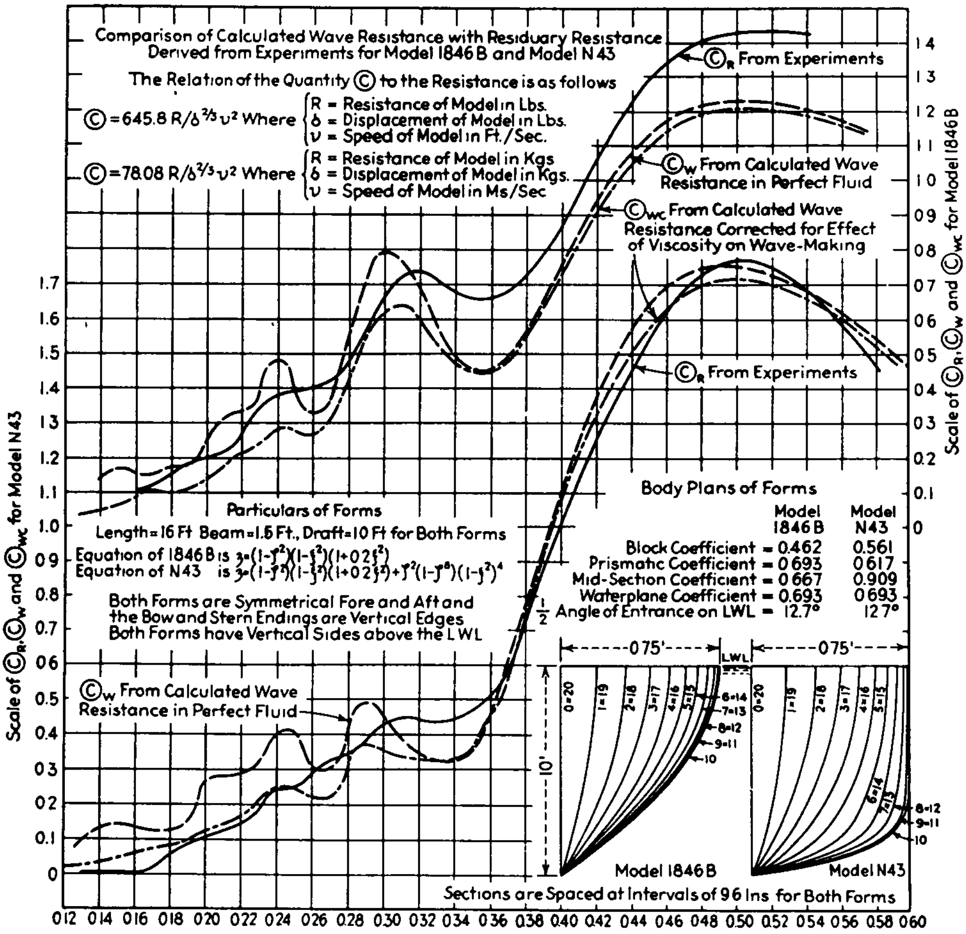

Figure 2. (From Havelock [11], by permission of the Society of Naval Architects and Marine Engineers.)

Comparisons of measured and calculated wave resistance for hull forms resembling conventional ships have been carried out by Wigley, Weinblum, Lunde, Shearer and others. For convenience in calculation most comparisons are made for "elementary" ship forms with lines which can be expressed by simple polynomials. Figure 2 is from a paper by Wigley and Lunde [2] and is also reproduced in papers by Havelock [11] and by Birkhoff, Korvin-Kroukovsky and Kotik [1] (note that the resistance coefficient is again defined differently). Figure 3 from a paper by Shearer [1] shows a comparison between computed and observed wave profiles for a given model at several Froude numbers. In each case the qualitative behavior of the measured and calculated curves is similar, and even the quantitative agreement is fairly good. Inspection of comparisons of measured and calculated wave resistance indicates that in the neighborhood of the first (counting from the right) hump the calculated curve generally lies below the measured curve if the model is free to trim but above it if it is fixed in position (the Stoker-Peters result mentioned earlier indicates that the linearized theory is the same in either case). To the left of this hump calculated values generally seem

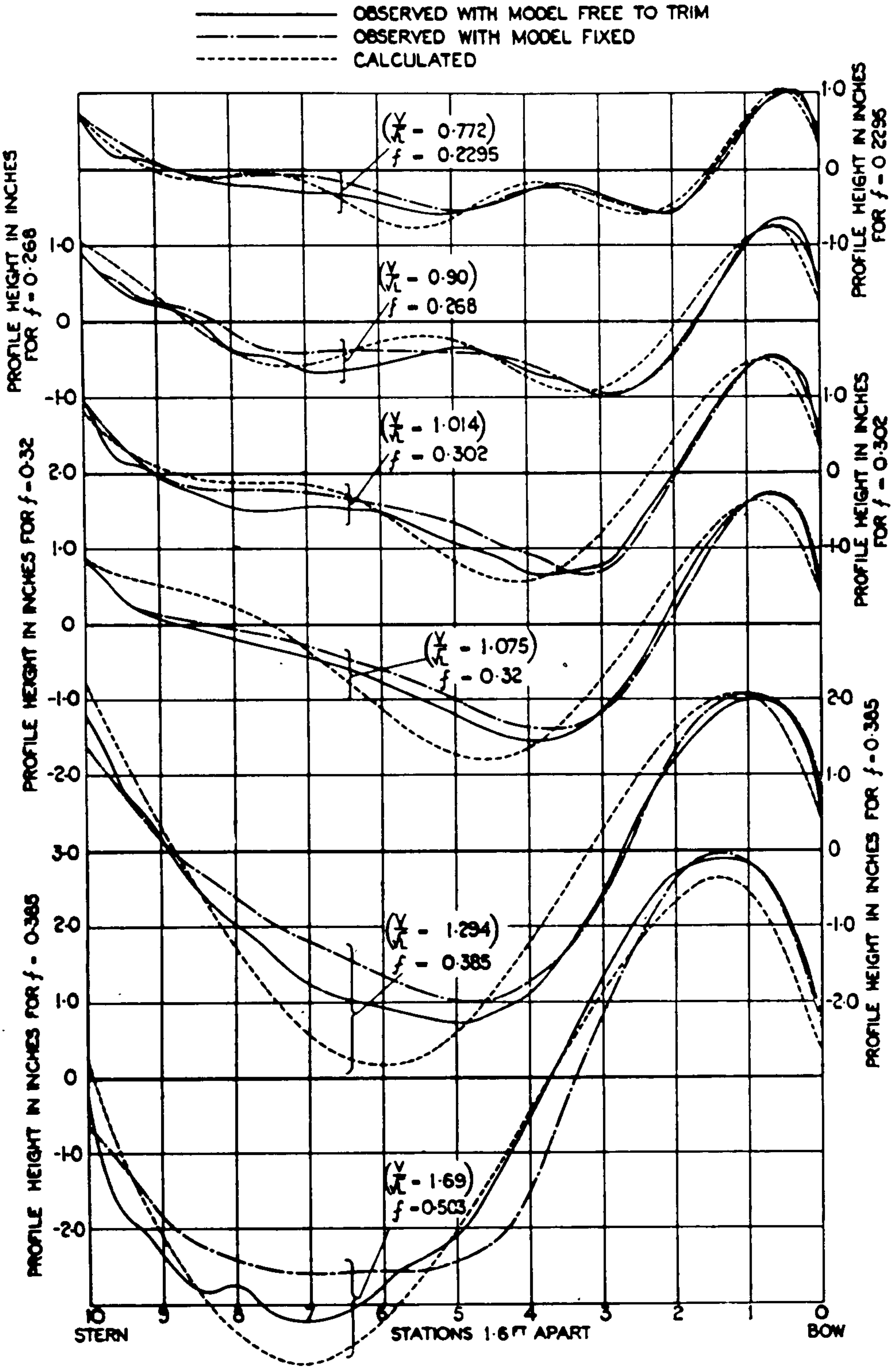

Figure 3. (From Shearer [1], by permission of the North-East Coast Institution of Engineers and Shipbuilders.) (Note that Shearer's f is our F, for Froude Number.)

to lie below measured ones for hulls of small prismatic coefficient, but to lie above them for higher prismatic coefficients and to have much more pronounced humps and (especially) hollows; these are shifted somewhat to the left of the observed ones. As the Froude number becomes smaller than about 0.20 it becomes questionable whether the residuary resistance should be compared with the calculated wave resistance.

The discrepancies are generally ascribed to neglect of the effect of viscosity on the wave formation, to the assumptions inherent in the linearization, and, of course, to the difficulty of determining experimentally something which may be called the wave resistance. There have been several attempts to take account of viscosity. An empirical method of Wigley's [8], based upon damping of the bow wave, has been used to correct the calculated curves in Figure 2. Havelock [9] has proposed computing the wave resistance for a hull enlarged at each point by the amount of the displacement thickness of the boundary layer at that point; a tail then follows the ship. This method has also been considered by Okabe and Jinnaka [1], and by V. M. Lavrent'ev [1]. Although it does damp out the oscillations at low Froude numbers in the calculated curve for C_w, it does not shift the positions of the humps and hollows appreciably. Inui [1, 3] introduces an empirically determined shift in the stern wave to take account of this. The empirical methods are perhaps objectionable because of their ad hoc character. On the other hand, they do produce better agreement with experimental results and may possibly give some idea as to where to look for reasons. Finding the causes for discrepancies seems more important here than finding empirical curves to fit experimental data.

Attempts to go beyond the linearized theory have been made by Guilloton, Okabe and Jinnaka [1] and by Inui [2]. In general, these attempts aim at trying to satisfy exactly the boundary conditions (for a perfect fluid) on the hull while retaining the linearized boundary condition on the free surface. We have commented earlier on this in connection with the derivation of the linearized theory. It seems to the author that an attempt at a complete second-order theory should be postponed temporarily. The expressions involved will be so unwieldy for computation that it may be easier to deal directly with the exact equations in machine computation. In addition, a too refined perfect-fluid theory may be out of keeping with neglect of a more fundamental treatment of the effect of viscosity.

Proposals have been made for other methods of determining experimentally the wave resistance. We mention especially one by Tulin [1].

These few remarks can not, of course, cover adequately the relation between measured and calculated wave resistance. For a more comprehensive treatment one should refer to Weinblum's excellent report [10], to Wigley [3], and to Birkhoff, Korvin-Kroukovsky and Kotik [1]. However, the agreement seems good enough to give support to proposals of Weinblum, Guilloton and others that one should reverse the present method of treating towing-test data by computing the wave resistance and considering the difference as a Reynolds-number dependent frictional-plus-form resistance. The current emphasis on methods of computing Michell's integral might seem misplaced otherwise. However, the author is not aware of published comparisons of computation and test for any ship forms in actual use.

Relation Between Hull Form and Wave Resistance. Once it has been established that the linearized theory gives a reasonable prediction of the wave resistance, it is possible to use it to investigate in a qualitative way the effect of various changes in hull shape, and even to try to find shapes of minimum wave resistance under certain restrictions. This is perhaps an even more important use of the theory than calculations for special forms, and it has occupied a great part of the efforts of Havelock, Weinblum, Wigley and others.

There are a few immediate corollaries of Michell's integral: 1. R varies as the square of the beam for a family of hulls related by

$$\zeta(x,y;\beta) = \beta\zeta(x,y;1). \qquad (51)$$

(In comparing this conclusion with experimental results given as resistance coefficients, one must take care that the coefficient is based on a length squared and not on wetted area or displacement $^{2/3}$.) 2. The resistance is independent of the direction of motion. 3. For any given hull shape the symmetrized hull,

$$\zeta_s(x,y) = \tfrac{1}{2}[\zeta(x,y) + \zeta(-x,y)], \tag{52}$$

has less resistance than ζ if $\zeta_s \neq \zeta$. (In fact, the symmetrization can be carried out with respect to any plane perpendicular to the x-axis, so that even a pair of ships in tandem can be so produced.) The displacement remains, of course, the same. Wigley [4, 5] and Weinblum [3] have made experiments to test the validity of these conclusions. 1. is satisfied approximately; however, an empirical determination of n in a law $R \sim B^n$ would show n somewhat smaller than 2 (see Weinblum [10], p. 27ff.). As might be expected, conclusions 2. and 3. are not well satisfied. The reason lies in the neglect of viscosity (see also Weinblum, loc. cit., pp. 61-65).

The behavior of the wave resistance for small and large values of the Froude number $F = c/(gL)^{1/2}$ can be found as an asymptotic expansion. Wigley [9] has derived results of this sort; Inui [1, pp. 67-68] finds several terms in the expansion for small F and a special class of hulls and uses it in some computations; Kotik, in unpublished work, derives the expansion for a very general class of hulls and includes the effect of sharp angles in the hull. A method for obtaining such an expansion can be outlined fairly easily for an "elementary" ship. Let us introduce dimensionless variables by measuring lengths in terms of L and let $f = gL/c^2 = F^{-2}$. Then we may write

$$R = \tfrac{1}{2}\rho c^2 L^2 \cdot \frac{8}{\pi} f^2 \int_1^\infty d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} [P^2 + Q^2], \tag{53}$$

$$P = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx\, X'(x) \cos f\lambda x \cdot \int_{-H/L}^{0} dy\, Y(y) e^{-f\lambda^2 y} = C(f\lambda) \cdot W(f\lambda^2) \tag{54}$$

and similarly for $Q = S(f\lambda) \cdot W(f\lambda^2)$ with sine replacing cosine. If $X(x)$ has no corners, i.e. if X' has no discontinuities, integration of $C(f\lambda)$ by parts yields:

$$C(f\lambda) = \frac{1}{f\lambda}[X'(\tfrac{1}{2}) + X'(-\tfrac{1}{2})] \sin \tfrac{1}{2} f\lambda - \frac{1}{f\lambda} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx\, X''(x) \sin f\lambda x, \tag{55}$$

where the last term is $O(1/f^2\lambda^2)$ (i.e. $f\lambda$ times the integral remains bounded as $f\lambda \to \infty$) if X'' is of bounded variation (a standard result from the theory of Fourier coefficients). Similarly, one may obtain

$$S(f\lambda) = \frac{1}{f\lambda}[-X'(\tfrac{1}{2}) + X'(-\tfrac{1}{2})] \cos \tfrac{1}{2} f\lambda + \frac{1}{f\lambda} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx\, X''(x) \cos f\lambda x \tag{56}$$

and

$$W(f\lambda^2) = \frac{1}{f\lambda^2}[Y(0) - Y(-H/L) e^{-f\lambda^2 H/L}] - \frac{1}{f\lambda^2} \int_{-H/L}^{0} dy\, Y'(y) e^{f\lambda^2 y}, \tag{57}$$

where the term in $W(f\lambda^2)$ is $O(1/f^2\lambda^4)$ if Y' is bounded. In order to carry the asymptotic expansion further, one performs further integrations by parts, assuming that the higher derivatives behave properly. Finally, we note that if $X(x)$ had corners at, say, $x_1, x_2, \ldots, x_n$,

$$-\tfrac{1}{2} < x_1 < x_2 < \cdots < x_{n-1} < \tfrac{1}{2}, \tag{58}$$

with jumps in X' given by

$$\Delta X'(x_\iota) = X'(x_\iota^+) - X'(x_\iota^-), \tag{59}$$

then, letting

$$x_0 = -\tfrac{1}{2}, \qquad x_n = \tfrac{1}{2}, \qquad \Delta X'(-\tfrac{1}{2}) = X'(-\tfrac{1}{2}), \qquad \Delta X'(\tfrac{1}{2}) = -X'(\tfrac{1}{2}), \tag{60}$$

the first terms in the expansions for $C(f\lambda)$ and $S(f\lambda)$ become

$$-\frac{1}{f\lambda}\sum \Delta X'(x_k)\sin f\lambda x_k \quad \text{and} \quad \frac{1}{f\lambda}\sum \Delta X'(x_k)\cos f\lambda x_k. \tag{61}$$

The effect of corners in $Y(y)$ will not show up until another partial integration is performed and thus their effect on the wave resistance will be of a higher order. The term $Y(-H/L)$ in the expansion of $W(f\lambda^2)$ will ordinarily be zero, but would presumably be kept if one were dealing with a sharply cut off strut. Forming $P^2 + Q^2$ for the case where X is smooth and $Y(-H/L) = 0$, one finds

$$\frac{1}{f^4\lambda^6}\left\{X'^2(\tfrac{1}{2}) + X'^2(-\tfrac{1}{2}) - 2X'(\tfrac{1}{2})X'(-\tfrac{1}{2})\cos f\lambda + O\left(\frac{1}{f\lambda}\right)\right\} \cdot \left\{Y(0) + O\left(\frac{1}{f\lambda^2}\right)\right\}. \tag{62}$$

Then the resistance coefficient $C_w = R/\tfrac{1}{2}\rho c^2 L^2$ is

$$C_w = \frac{8}{\pi}\frac{Y^2(0)}{f^2}\left\{[X'^2(\tfrac{1}{2}) + X'^2(-\tfrac{1}{2})]\int_1^\infty d\lambda\,\frac{1}{\lambda^4\sqrt{\lambda^2-1}} - 2X'(\tfrac{1}{2})X'(-\tfrac{1}{2})\int_1^\infty d\lambda\,\frac{\cos f\lambda}{\lambda^4\sqrt{\lambda^2-1}}\right\} + O\left(\frac{1}{f^3}\right). \tag{63}$$

For large values of f, one has the asymptotic expansion (see, e.g., Erdélyi [1], pp. 46-51)

$$\int_1^\infty d\lambda\,\frac{\cos f\lambda}{\lambda^4\sqrt{\lambda^2-1}} = \sqrt{\frac{\pi}{2f}}\cos(f + \tfrac{1}{4}\pi) + O(1/f); \tag{64}$$

the first integral in C_w has value $\tfrac{2}{3}$. Hence, and returning to the usual Froude number,

$$C_w = \frac{16}{3\pi}Y^2(0)[X'^2(\tfrac{1}{2}) + X'^2(-\tfrac{1}{2})]F^4 - 8\sqrt{\frac{2}{\pi}}\,Y^2(0)X'(\tfrac{1}{2})X'(-\tfrac{1}{2})F^5\cos(F^{-2} + \tfrac{1}{4}\pi) + O(F^{-6}). \tag{65}$$

In case $X(x)$ has corners, the expansion becomes

$$C_w = \frac{16}{3\pi}Y^2(0)\sum[\Delta X'(x_i)]^2F^4 - 8\sqrt{\frac{2}{\pi}}\sum_{i=j}\frac{\Delta X'(\tfrac{1}{2})\Delta X'(-\tfrac{1}{2})}{|x_i - x_j|^{\frac{1}{2}}}F^5\cos(F^{-2}|x_i - x_j| + \tfrac{1}{4}\pi) + O(F^{-6}) \tag{66}$$

The foregoing is only a caricature of Kotik's result, which does not assume an "elementary" ship and carries the expansion much further.

Unfortunately, the Froude-number region in which this expansion has its greatest validity is one in which the wave resistance is not of great practical importance and also one where the influence of viscosity on wave making may be presumed to be great. However, it does give information of the sort one hopes to get from a theoretical investigation, a statement concerning the dependence of wave resistance on some simple parameter of the form, in this case the tangents of the angles at the bow and stern at the waterline. If a sharp turn at the shoulder may be taken to approximate a corner, the second formula indicates that this also has an important wave-making effect at small Froude numbers.

At the other extreme of infinite Froude numbers, or small f, an asymptotic expansion in the general case does not seem to work out as neatly. However, by working with upper and lower approximating functions, one can see that the expansion starts out like

$$C_w = Af^2 + Bf^2 \log f + \cdots, \tag{67}$$

which is also confirmed by Havelock in his discussion of Wigley [9]. The coefficient A appears to be proportional to the square of the volume divided by the distance between the surface and the centroid of the midship section. In view of the fact that the method used is awkward and can almost certainly be improved, it would be unpleasant to commit it to paper. In the case of the long vertical strut one may write out the asymptotic expansion immediately from the known expansion of $Y_o(x)$ for small x (see below). It is of the same form as for the ship of finite draft and begins with the term

$$\frac{8}{\pi} f^2\{\gamma + \log \tfrac{1}{2}f\}\{\int dx\, \zeta(x)\}^2, \qquad \gamma = \text{Euler's Constant} \tag{68}$$

Observation indicates that the higher the Froude number, the thinner a "thin" ship should be in order not to violate the condition of small disturbance of the water. Thus the applicability of an expansion at infinite Froude number to normal ship forms seems dubious.

In the Froude number range 0.2 to 0.6, important in practice, studies of influence of hull form have to be chiefly computational or experimental. Havelock [1-5] has made a systematic study of the effect of varying various aspects of the hull form; many of his results are summarized in [10] and in Wigley [1]. His computations have included varying the form of the cross-section of an infinitely long vertical strut while keeping its area constant, adding parallel middle-body to such a strut, and cutting the strut off at various depths. Wigley [4-10] and Weinblum [2-9] have made extensive investigations, both theoretical and experimental, of the interrelation of ship form and wave resistance, with the aim of both testing the predictions of Michell's integral and of explaining theoretically known facts about ship forms. The results don't lend themselves to a brief summary; one should consult the original papers and also summaries by the authors themselves, e.g., Wigley [1, 2, 3], Weinblum [3, 10]. The benefits to be gained from further studies of this sort have certainly not been exhausted. Newly developed computational methods and tables to facilitate them will make such studies much easier (see Guilloton [1] and Weinblum [11]).

It would be a gross exaggeration to state that these studies have had any substantial effect on ship design. However, most naval architects are aware of their existence and that they do give insight into the wave-making properties of ship forms. As shown by both Wigley [7] and Weinblum [7], the advantages to be obtained by use of a bulbous bow under certain conditions could have been predicted theoretically from Michell's integral. Unfortunately, the theory followed the discovery in this case, but one can hope that it will also happen the other way occasionally.

Ships of Minimum Wave Resistance. Once a theoretical expression for the wave resistance in terms of ship form is found, it is natural to try to use this expression to deduce forms of minimum wave resistance in case such exist. One must, of course, take care in formulating such a problem in order that a solution will exist. For example, it is not sufficient to fix just the volume and ask for a form of minimum resistance, for by distributing the volume deeper and deeper the resistance becomes less and less without reaching a minimum. A solution, if it exists, will of course depend upon the Froude number. This problem has been treated extensively by Weinblum and also by Pavlenko [1] and Sretenskii [1].

If one fixes the centerplane section of the ship and its volume, and uses Michell's integral for the resistance, the mathematical problem is to minimize a quadratic form subject to a linear constraint. This leads to the following integral equation for $\zeta_x(x,y)$:

$$\iint_{S_0} d\xi\, d\eta\, \zeta_x(\xi,\eta) K(f(x-\xi), f(y+\eta)) = kx, \; (x,y) \text{ in } S_0. \tag{69}$$

A solution ζ_x is sought such that

$$\iint_{S_0} d\xi\, d\eta\, \xi\zeta_x(\xi,\eta) = V, \qquad V = \text{fixed volume.} \tag{70}$$

The latter condition will determine the constant k in the integral equation. Other constraints could replace this one, or be added to it. For example, one could fix $\zeta_x(1/2,0)$, $\zeta(0,y)$, etc. The integral equation is of the first kind, and these are notoriously difficult.

Sretenskii [1] deals with the analogous problem for the infinite vertical strut (although he states that his result is valid for "elementary" ship), and claims to prove

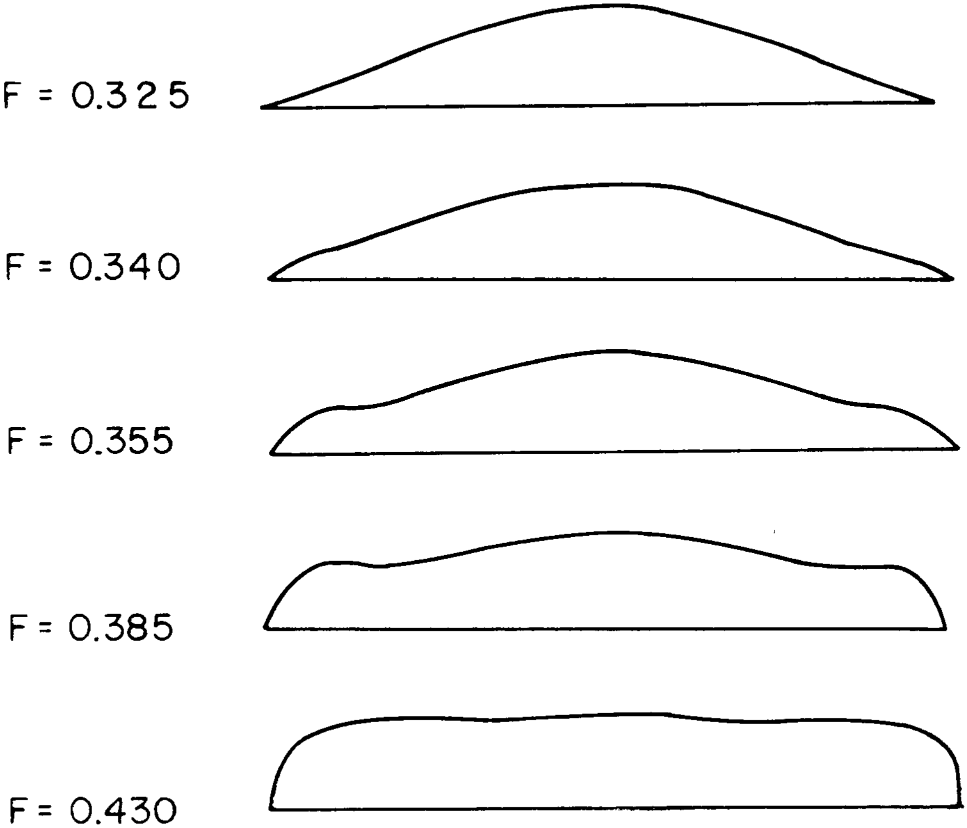

Figure 4.

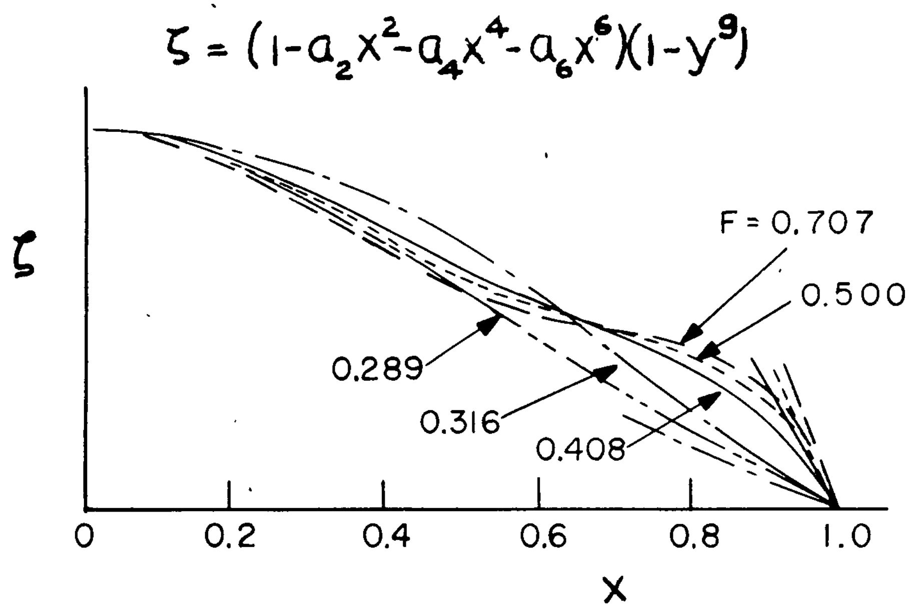

Figure 5.

that no solution can exist among square-integrable functions. It seems to the author that what Sretenskii takes as eigenfunctions of the kernel are not really such and that his proof is not valid. Pavlenko also deals with the strut, side-steps the question of existence, integrates by parts with respect to x (remembering that $\zeta(\pm\frac{1}{2}) = 0$) and treats numerically the resulting integral equation:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\,\zeta(\xi)Y_0(f|x - \xi|) = k, \qquad |x| \leq \tfrac{1}{2}, \tag{71}$$

where

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\,\zeta(\xi) = A = \text{fixed cross-section area}, \qquad \zeta(\pm\tfrac{1}{2}) = 0. \tag{72}$$

Pavlenko replaces the integral equation by a set of linear algebraic equations and solves these. Figure 4 shows the resulting strut cross-sections for several Froude numbers. It is somewhat amusing that for Froude number less than 0.325 Pavlenko got negative ordinates near the ends. Weinblum [1] considered "elementary" ships represented by the product of two polynomials, so that his computed minimal forms are for a very restricted class of functions. However, they show the same general behavior as functions of Froude number as do Pavlenko's. Figure 5 shows the water-plane sections for various Froude numbers.

According to a recent paper by Dörr [1], the solution of the strut integral equation, if it exists, can be expressed as a series of Mathieu functions ce_{2n}. In fact, Dörr has shown that

$$\text{ce}_n(\beta;f^2) = \lambda_n \int_0^{\pi} d\alpha\,\text{ce}_n(\alpha)Y_0(f|\cos\alpha - \cos\beta|). \tag{73}$$

Then, if one expands the constant k in a series in ce_n (actually, only ce_{2n} are needed)

$$k = \sum_{n=0}^{\infty} a_n\,\text{ce}_n(\beta;f^2) \tag{74}$$

and lets

$$G(\beta;f) = \sum_{n=0}^{\infty} a_n\lambda_n \,\mathrm{ce}_n(\beta;f^2), \tag{75}$$

the solution of the integral equation is given by

$$\zeta(x;f) = \frac{2}{\sqrt{1-4x^2}}\,G(\mathrm{arc\,cos}\,-\,2x;f) \tag{76}$$

Unless G has compensating behavior at $x = \pm\frac{1}{2}$, the solution has a singularity at these points which deprives it of physical meaning; and, of course, the condition $\zeta(\pm\frac{1}{2}) = 0$ is not satisfied. This seems indeed to be the case for sufficiently small f (large Froude number), either from a study of the asymptotic expansions of λ_n and ce_n or from the development of Y_0 about the origin:

$$Y_0(x) = \frac{2}{\pi}\log\frac{x}{2}\left[1-\left(\frac{x}{2}\right)^2+\cdots\right]+\frac{1}{\pi}\left[2\gamma-\left(\frac{x}{2}\right)^2(2\gamma-2)+\cdots\right]. \tag{77}$$

The latter leads to the simpler integral equation for small f

$$\frac{2}{\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\,\zeta(\xi)\log \tfrac{1}{2}f\delta|x-\xi| = b, \qquad |x| \le \tfrac{1}{2}, \qquad \log\delta = \gamma = \text{Euler's Constant}, \tag{78}$$

and eventually to the solution

$$\zeta(x) = \frac{A}{\pi\sqrt{1-4x^2}} \tag{79}$$

independent of f. It is not known at present whether this behavior persists for small Froude number. The matter should be investigated. The asymptotic expansion of Michell's integral for small Froude number which was given earlier indicates that an optimum form should have cusps at bow and stern and elsewhere avoid discontinuities in slope and curvature.

Establishing the existence of a solution in a physical problem may seem like a luxury. However, unless a physically acceptable solution exists, Weinblum's investigations of optimal forms among restricted classes of functions such as polynomials lose much of their significance; for the underlying assumption is that these are approximations to an exact solution. This applies even more strongly to Pavlenko's approach. One must, of course, remain aware of the fact that ships are not necessarily "thin" and that they do not sail in perfect fluids, so that the practical implications of such studies are not necessarily important. However, they may put one in possession of the techniques for finding, say, optimal forms for the forward half of a ship when the after half is given.

Several Ships and Ships Near Walls or in Canals. The wave-resistance integral applies equally well to two or more ships in a row if they all travel at the same speed. For let

$$\zeta(x,y) = \zeta_1(x,y) + \zeta_2(x,y), \tag{80}$$

where $\zeta_1(x,y) = 0$ except on S_1 and $\zeta_2(x,y) = 0$ except on S_2, where the centerplane sections S_1 and S_2 are at a distance d between centers. Treating ζ as a single ship, one may compute, say,

$$P(\lambda;d) = \iint_{S_1} dxdy\, \zeta_{1x}(x,y)e^{\nu\lambda^2 y} \cos \nu\lambda x + \iint_{S_2} dxdy\, \zeta_{2x}(x,y)e^{\nu\lambda^2 y} \cos \nu\lambda x \tag{81}$$
$$= \iint_{S_1} dxdy\, \zeta_{1x}(x,y)e^{\nu\lambda^2 y} \cos \nu\lambda x + \iint_{\bar{S}_2} d\bar{x}dy\, \zeta_{2x}(\bar{x} - d,y)e^{\nu\lambda^2 y} \cos \nu\lambda(\bar{x} - d),$$

where in the second integral we have shifted to local coordinates centered in S_2. Expansion of $\cos \nu\lambda(\bar{x}-d)$ leads to the following:

$$P(\lambda;d) = P_1(\lambda) + P_2(\lambda) \cos \nu\lambda d - Q_2(\lambda) \sin \nu d\lambda. \tag{82}$$

Similarly,

$$Q(\lambda;d) = Q_1(\lambda) + Q_2(\lambda) \cos \nu\lambda d + Q_2(\lambda) \sin \nu d\lambda. \tag{83}$$

The subscripts refer, of course, to the individual ship. Substitution in the resistance integral leads to

$$R = R_1 + R_2 + 2\frac{4\rho g^2}{\pi c^2}\int_1^\infty d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} \tag{84}$$
$$\cdot [(P_1P_2 + Q_1Q_2) \cos \nu d\lambda - (P_1Q_2 - Q_1P_2) \sin \nu d\lambda] = R_1 + R_2 + R_{\text{int}}(d).$$

If the ships are identical, $P_1Q_2 - Q_1P_2 = 0$ and

$$R = 2R_1 + 2\frac{4\rho g^2}{\pi c^2}\int_1^\infty d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} [P_1^2 + Q_1^2] \cos \nu d\lambda \leq 4R_1. \tag{85}$$

It follows from theorems on Fourier transforms that $R_{\text{int}} \to 0$ as $d \to \infty$. However, one can also find the asymptotic expansion for large d:

$$R_{\text{int}}(d) \approx \frac{4\rho g^2}{\pi c^2}\sqrt{\frac{2\pi\nu}{d}} \cos (\nu d + \tfrac{1}{4}\pi)[P_1^2(1) + Q_1^2(1)]. \tag{86}$$

Thus, at some separations d, the total resistance is less than that of the ships separately. A similar expansion holds for the case when the ships are not identical. At the other extreme, $d = 0$, the expression for R verifies the quadratic dependence on beam (a statement which has more mathematical than physical meaning). Kostyukov [1] has also treated the problem of a caravan of n ships, but without the above asymptotic expansion.

The case of two ships moving abreast, or of one parallel to a wall, is somewhat more complicated. The boundary condition which should be fulfilled on the center-plane sections of each ship is, according to the linearized theory, that $\varphi_n = \mp c\zeta_{ix}$, $i = 1, 2$. However, a source distribution of density $- c\zeta_{ix}/2\pi$ over S_i will not satisfy this exactly but will approximate it at small values of the ratio beam/separation (which can presumably be used as a further expansion parameter). The same difficulty does not arise in the case of a ship moving down the center of a deep canal. Both problems are treated by Lunde [1] and the details will not be repeated here. The second one was considered first by Sretenskii [2 and 3, chap. II]. Numerical computations are scanty. However, Sretenskii has computed the resistance for a vertical strut of cross-section $\zeta(x) = B(1 - x^2/L^2)$, $|x| \leqslant L$, for a number of values of gb/c^2 and $2L/b$, b the canal width; he gives both tables and a graph of $\pi R/64\rho gB^2L$. For the case of the strut Keldysh and Sedov [1] obtain the theoretically interesting result

$$\lim_{c\to\infty} R = 2\rho gA^2/b, \tag{87}$$

where A is the cross-sectional area.

Water of Finite Depth. The theory of "thin" ships has been extended by Sretenskii [3, Chap. IV] to motion in water of finite depth h, and he has obtained resistance integrals in terms of ship form which are analogous to Michell's. In [4] he has made numerical computations for the form

$$\zeta(x,y) = B(1 - x^2/L^2)(1 + y/H) \tag{88}$$

for $h = L$ and several values of H/L ranging from 0.025 to 0.3. For comparison the calculations were repeated for $h = \infty$ and the same values of H/L. The resulting

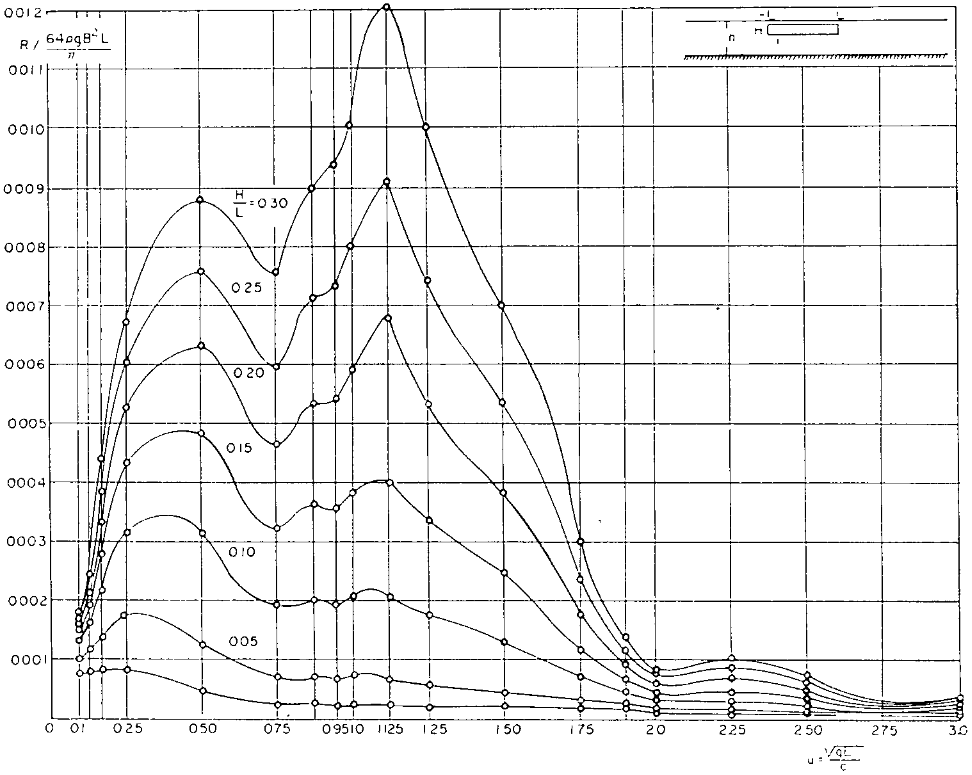

Figure 6.

graphs are shown in Figures 6 and 7. For the case of canals of finite depth the problem was solved independently by Keldysh and Sedov [1] and by Sretenskii [3, Chap. IV]. The author is not aware of any published computations for special shapes; they would obviously be important in estimating corrections to towing-tank test data. Wigley [3] has computed the resistance of the submerged body generated by a source and sink separated by $2l$ and located at a depth $0.0375h$ in a canal of width $b = 2.5h$; graphs are given of the ratio of the resistance to the resistance in unrestricted water for $c/\sqrt{gh} = \sqrt{2}$ and $\frac{1}{2}\sqrt{2}$ and for values of $2l/b$ between 0 and 1.8. Lunde has informed the author that he has partially completed computations for an "elementary" ship form. The theory for finite depth is given in detail in Lunde [1] and will not be repeated here. The resistance integral for the case of an infinite ocean of finite depth h is given by

$$R = \frac{2\rho g c}{\pi} \int_{\mu_h}^{\infty} d\mu \sqrt{\frac{\mu}{\mu - \nu \tanh \mu h}}\, [P^2 + Q^2], \tag{89}$$

where

$$P = \iint_S dxdy\, \zeta_x(x,y)\, \frac{\cosh \mu(y+h)}{\cosh \mu h} \cos x\sqrt{\nu\mu \tanh \mu h} \tag{90}$$

and Q is a similar function where sine replaces cosine; here μ_h is the nonzero solution of $\mu = \nu \tanh \mu h$ if such exists, otherwise zero. The latter case occurs if $c^2/gh \geqslant 1$. As $h \to \infty$, $\tanh \mu h \to 1$, $\mu_h \to \nu$, and R approaches Michell's integral in the form it takes if one makes the substitution $\mu = \nu\lambda^2$.

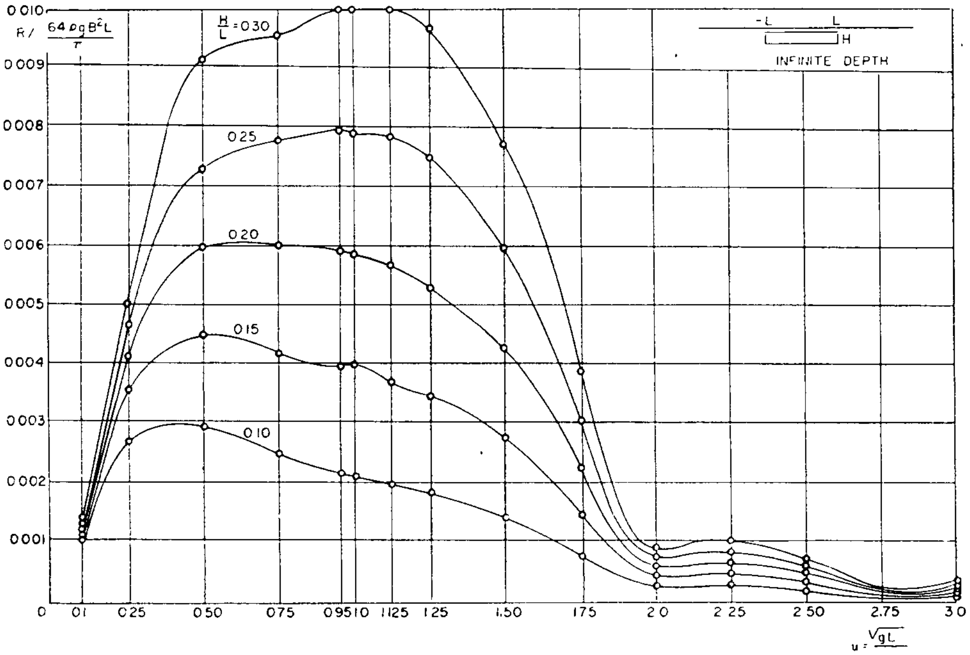

Figure 7.

Accelerated Motion. The velocity potential for a submerged source moving with velocity $c(t)$ has been given Sretenskii [3], and by a different method by Havelock [e.g., 11]. From this one can find the wave resistance of a thin ship as before. A detailed exposition is given in Sretenskii [3] and Lunde [1]. The result is:

$$\begin{aligned} R = {} & \frac{\rho c'(t)}{\pi} \iint_S dxdy \iint_S d\xi d\eta\, \zeta_x(x,y)\zeta_x(\xi,\eta) \{[(x-\xi)^2 + (y-\eta)^2]^{-1/2} \\ & - [(x-\xi)^2 + (y+\eta)^2]^{-1/2}\} + \frac{2\rho g}{\pi} \iint_S dxdy \iint_S d\xi d\eta\, \zeta_x(x,y)\zeta_x(\xi,\eta) \\ & \cdot \int_0^\infty dk\, k e^{k(y+\eta)} \int_0^t d\tau\, c(\tau) \cos\sqrt{gk}\,(t-\tau) J_0(k(x-\xi) + k\int_\tau^t d\tau'\, c(\tau')). \end{aligned} \tag{91}$$

Lunde [1, 3] has extended this to the case of accelerated motion in a canal of finite depth. The first summand in the formula above is an added-mass term. If one takes the case of a ship starting suddenly from rest at constant velocity c, this term vanishes and the second summand approaches Michell's integral as $t \rightarrow \infty$. Since towing-tank tests approximate this situation, it would be of interest to have computations of the transient part of this for a ship-like form. Lunde informs the author that such are in the case of a ship starting suddenly from rest at constant velocity c, this term vanishes progress. The author has obtained asymptotic expansions for large t.

Other unsteady-state problems which would be of interest but which have not been attacked as far as the author is aware are motion over a stepped bottom or approaching a beach. The steady-state problem of motion parallel to a beach or step has also not been treated. Hansen [1] has considered the case of a pressure point moving parallel to a beach, but, although this gives a qualitative notion of the form of the waves, it doesn't relate them to the geometry of the ship or to the forces on it.

Ship Waves. In concluison we should like to call the reader's attention to the fact that we have seldom discussed the actual waves produced by a ship. There seems to have been little investigation of the form of the free surface away from the ship as it relates to the form of the ship. The studies of Hogner [2] and others on waves generated by moving pressure distributions give qualitative information about ship waves, but not in relation to a specific hull geometry.

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PRSL — Proceedings of the Royal Society of London. Series A.
Trans. INA — Transactions of the Institution of Naval Architects (London).
Trans. NECIES — Transactions of the North-East Coast Institute of Engineers and Shipbuilders (Newcastle-upon-Tyne).
Trans. SNAME — Transactions of the Society of Naval Architects and Marine Engineers (New York).
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DISCUSSION

T. B. Benjamin

I should like to mention briefly some aspects of wave resistance which I have examined recently, both theoretically and by way of some experiments with the water channel belonging to the Engineering Department of Cambridge University. An account of this work has already appeared[1], but so recently that I may be justified in outlining some of it now. It seems to be appropriate to the present discussion, although I fear it is rather far removed from Professor Wehausen's topic.

The study was confined to two-dimensional flow along a straight horizontal channel, and was concerned with the effects of placing various rigid obstacles in the

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[1] J. Fluid Mech. **1**, 227. (1956)

stream. In particular, I attempted to clarify the different circumstances under which on the one hand, the development of a drag on the obstacle is accompanied by the formation of a train of periodic waves on the downstream side and, on the other hand, the receding stream converges towards a uniform state. As Professor Lighthill explained this morning, a great many different flow phenomena possible in a straight channel can be summarized very neatly in the energy/momentum diagram * which was put on the blackboard this morning. It is still there, and we may conveniently refer to it now.

The diagram shows that whenever a slightly subcritical stream (i.e., one for which the Froude number is slightly less than unity) is obstructed, the reduction of momentum associated with the drag on the obstacle gives rise to a train of waves on the downstream side. Increasing the drag, by deeper immersion of the obstacle, causes the amplitude and length of the waves to become larger; and eventually the wave train resembles a succession of 'solitary waves.'

However, a different state of affairs arises when the undisturbed stream has a Froude number less than about 0.8, the value at which the solitary wave of maximum height (i.e. the sharp-crested form of the wave) is formed by reducing the momentum of the stream as much as is possible. In this lower range of Froude numbers, the resistance experienced by an obstacle cannot, except for very small values, be ascribed to wave formation alone. The effect of gradually increasing the resistance is first to produce periodic waves of maximum height, and then to produce breaking waves. In other words, the reduction in the momentum of the stream must eventually be accompanied by a loss of energy. Nevertheless, by increasing the resistance sufficiently it is possible to bring about once again a steady lossless flow on the downstream side. In this extreme case, the receding stream converges steadily towards a uniform supercritical state, and waves are entirely absent; this is exemplified by the flow under a vertical sluice-gate, although such a flow is possible under an obstacle of any shape which is adequately immersed.

These effects were clearly demonstrated in some experiments in a horizontal water channel with glass sides. A metal plate was supported from a horizontal axle, and spanned the whole width of the channel. The plate could be swung down broadside to the stream, and was balanced in such a way that the drag due to contact with the water could be fixed at any desired value. The observed effects of gradually lowering the plate into a subcritical stream with Froude number less than 0.8 was as follows. First, a smooth train of periodic waves was formed, whose amplitude and length increased as the drag was increased. Then the leading wave broke at its crest, accompanied by a reduction in the amplitude of the waves in the rear, so that the front of the wave train resembled a turbulent 'hydraulic jump.' On a further increase of the drag, the wave train was swept downstream out of the channel, and the flow at a distance behind the plate became approximately uniform; thus, the maximum drag occurred for a uniform supercritical flow downstream, as indicated by the theoretical considerations summarized above.

To account theoretically for the case of maximum drag, where waves are absent, it is convenient to invert the problem; that is, the depth and Froude number far downstream are specified, so that the drag is determined by a simple momentum relation between the uniform flows up and downstream, and the position of the obstacle remains to be found. In my paper I showed that the profile of the converging flow in the rear of the obstacle is determined uniquely by the ultimate conditions reached far downstream; that is to say, the flow pattern is independent of the shape of the obstacle. However, this conclusion does not apply to the region where the free surface first springs clear of the solid boundary, and where its curvature is large. This general wake profile has a simple approximate analytic expression in which the Froude number is a parameter, and it may be seen that this expression is the same as a well-known equation for

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* Figure 9 of Prof. Lighthill's paper in this volume.

the soltary wave, although this wave cannot itself occur under the circumstances considered.

To find the shape of the free surface very close to the separation point, other methods have to be used. In the example of a vertical sluice-gate, I was able to apply a fairly straight-forward method based on conformal transformation whereby the free-surface profile in the hodograph plane is approximated by an arc of an ellipse. A similar method is also adequate when the obstacle is an inclined plane. It appears that these methods will provide estimates of the drag with an accuracy considerably better than 1%. The results of these calculations compare favourably with the solutions which Southwell and Vaisey[2] obtained by relaxation methods.

Although this work has no direct bearing on the problem of the resistance of ships in deep water, it may possibly have applications to other problems within the sphere of this symposium. For instance, it seems to be relevant to the performance of hydroplanes and seaplane floats in shallow water.

J. J. Stoker

Both of the lecturers this afternoon have made reference to the work of Arthur Peters and myself about the motion of ships in a seaway. I thought I might try to tell you what we had in mind to do, although it is impossible to give any details.

For this audience the best way to put it would probably be to say that what we did was to formulate a theory which is a thoroughgoing generalization of the classical theory of Michell. We make the same basic assumptions as in that theory. Those are principally that we have a perfect fluid, so that the turbulent wake is ignored, for example; and then on top of that we make an assumption that is necessary to linearize the problem, namely that the ship is slender in some sense so that it is possible for it to have a forward motion with finite velocity, and still create waves of small amplitude. The result is a theory which is a generalization of the Michell-Havelock theory, in the sense that the latter results when we make the restrictive assumptions that are necessary for that case. What we have done, in effect, is to give up the strong assumption that the ship is held fixed in space, while the water streams past. That is what Michell assumed, and the one quantity he computed is the wave resistance, the force necessary to maintain the body in that position.

We assume that the body floats freely, calculating its motion from the pressure force of the water, treating the combined system of ship and sea without making any a priori assumptions with respect to the coupling of the various degrees of freedom of the ship or the coupling of the motion of the ship and the motion of the water.

In developing the theory our procedure was to carry out a formal development of all quantities with respect to a small parameter, which was a thickness-length ratio of the ship's hull. Depending upon the manner in which this parameter is chosen, various theories arise. If, for example, the ship is regarded as small in the beam-length ratio, our generalization of Michell's theory arises. If it is the draft-length ratio that is assumed small, still a different theory arises which might be called a theory of planing ships. For a yacht-shaped hull, which is a combination of the previous two cases, still a third theory is derived.

In the first case, i.e. the case of a ship with a thin vertical section—a ship of Michell's type—it turns out that a more general problem than that of Michell can be solved explicitly. That is the problem of motion of the ship when it is assumed that it moves only in the vertical longitudinal plane through waves which are symmetrical with respect to that plane. The amplitude of the oscillations, the trim, and the wave resistance can be calculated by explicitly given integrals.

However, it is an inescapable consequence of this generalization of Michell's theory that the pitching and heaving modes of oscillation are not damped. The reason for that is simple: the ship is like a knife-blade placed vertically in the water, and small

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[2] Phil. Trans. A, **240**, 117.

vertical oscillations can create only waves with amplitudes which are of higher than first order in the thickness of the blade. Since the only mechanism for energy dissipation is the creation of waves which carry off energy to infinity, it is clear that no damping of such oscillations is to be expected. Since these modes of oscillation seem actually to be damped when observations are made for actual ship's hulls, it is indicated that Michell's assumptions are not always the appropriate ones, even though they do lead to fairly reasonable results for the wave resistance. However, two other forms of our theory are available, both of which would lead to damping in the pitching and heaving modes of oscillation.

Unfortunately, only for ships of Michell's type are there explicit solutions available for the complicated boundary problems in potential theory posed by our basic theory—and only then for the restricted type of motions mentioned above. We have formulated the problems in terms of integral equations over the hull of the ship, but unfortunately it turns out that the integral equations have singular kernels, and are in general of unconventional types so that even their numerical solution with modern computing equipment presents problems of some difficulty.

R. F. Chisnell

I wish to describe an unsuccessful attempt to use a Ferranti Mark I computer at Manchester on the problem of wave resistance of non-thin ships. As this problem is still one of the unsolved problems urgently requiring attention, I mention this attempt as firstly, there may well be others contemplating a similar program who would like to hear of the difficulties encountered; and secondly, with the expected arrival in Manchester of a Ferranti Mark II machine the attack on this problem may be continued, and we would be pleased to hear from people interested in this work.

A ship of arbitrary hull shape is assumed to move with constant velocity into an undisturbed ideal fluid. The free surface condition is linearized, and may be expressed in terms of a velocity potential function as,

$$\phi_{xx} + \frac{1}{F^2}\phi_y = 0, \qquad \text{on} \qquad y = 0, \tag{92}$$

where x is the direction of motion, y normal to the plane of the free surface and F the Froude number of the motion. On the hull the normal velocity ϕ_n must satisfy

$$\phi_n = F\cos(n,x) \tag{93}$$

n being normal to the hull.

It is the existence of these two separate boundary conditions that makes the problem of finding the appropriate solution of Laplace's equation so difficult.

The method employed was to represent the ship as a distribution of Kelvin sources over its wetted area. A Kelvin source has a source-type singularity and also satisfies the free surface condition (1). The second boundary condition leads to an integral equation to be satisfied by the strength of the Kelvin source distribution.

For numerical work the continuous distribution is replaced by a sufficiently fine Kelvin point source distribution, the integral equation becoming a set of linear equations for the strengths of the point sources. The majority of terms in these equations are velocities due to Kelvin sources at the location of another of the sources. Part of the modification necessary to convert the velocity potential of a source into that of a Kelvin source is a double integration over a semi-infinite rectangle. The integrand is fierce, containing a curve of singularities. One to two hours computing time was envisaged for each integral and of the order of a hundred integrals. The reliability of the machine, which was the first production model, was not great enough to perform these integrations. The possibility of performing the integration in several smaller stages was considered, but as it would result in an appreciable increase in running time it was decided to postpone the work till the arrival of the Mark II machine.

M. C. Eames

While much attention has been given in recent years to refinement of the linearized theory of wave resistance as initially formulated by Michell and developed by Havelock and others, and now there is evidence of some success in Russia and Japan of handling the formidable mathematics of non-linearized solutions, the fact remains that comparatively little has been done in the opposite direction. By this is meant the simplification of solution to the point where first approximations to the wave resistance of an actual ship form may be carried out in the preliminary design stages, calculated by naval architects as opposed to mathematicians (or complex computing machines).

It is, of course, unlikely that an accurate wave resistance calculation for a practical ship form will ever be reduced to the simplicity of, for example, a stability calculation, but the present writer believes that an approximation which is sufficiently close to yield a useful result in a comparative sense, is possible.

An approach which was developed by the present writer in 1952 will be briefly outlined by way of illustration. It should be clearly pointed out however, that the technique remains largely unproven, since other commitments have so far prevented the writer from concluding the work.

The ship is treated in a number of horizontal layers, the layer including the waterline being by far the most significant. By direct substitution of ordinates from the lines plan as coefficients in simple polynomials, these layers are then represented mathematically with a fair degree of accuracy. A refinement consists of adding to the ordinates in the after body so as to take into account the growth of the boundary layer according to the ideas expressed by Havelock in his 1948 paper to the Institution of Naval Architects.

The linearized Michell solution, in one of the forms developed by Havelock can be written directly in terms of relatively simple integrals involving these layer polynomials, as has been shown in a particular case by Weinblum, who has also tabulated values for one of the resulting integrals.

The other integral has been tabulated by the present writer and the Weinblum table extended to cover the range of practical interest. (The polynomials used by the writer involve higher powers than those considered by Weinblum, thereby increasing the accuracy of representation of the ship form.)

The procedure is then purely automatic and can be conveniently arranged in a tabular form of calculation well within the capabilities of a calculations-draughtsman.

Consideration of some work by Emerson applying Havelock's well known source and sink compartmentisation of the ship has shown that very few vertical subdivisions of the ship are necessary to give a satisfactory result, provided the layer containing the waterline is kept narrow. Thus the amount of arithmetic involved in the method proposed by the writer need not be excessive, and one obvious advantage of the technique over that of the source and sink calculation is that systematic variations of hull lines can be more readily performed and traced through the calculation so that their significance on the final result is more clearly understood.

Finally it might be remarked that if, for initial estimates, a two layer horizontal subdivision is adopted, it is possible to modify the method so that the only ordinates required to perform the calculation are those of the waterline, and those of the curve of section areas—the two basic curves from which line plans are usually derived. This immediately suggests a technique for rapidly comparing fundamental differences of hull form from the wave resistance point of view at the very outset of a design, which might be of particular value where other requirements dictated a rather radical departure from "normal" ship forms.

It is not suggested that calculations of the form outlined above will ever reach a state whereby they would replace the techniques of the model experiment, but it is felt that they could constitute a useful control on such work, and in many cases serve to reduce the number of models required by suggesting the most promising hull forms thereby reducing the range of possibilities.