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WAVE SCATTERING DUE TO TURBULENCE

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SUMMARY

When a sound wave is incident upon a part of the medium which contains small irregularities, scattered waves are set up. The irregularities, which may be either variations of velocity or of physical properties such as temperature, are here assumed to be random as a result of the action of turbulence in the medium. The basic equations that describe the propagation of sound in both air and water containing various kinds of non-uniformity are derived. After a general exposition of the theory of scattering in continuous media, the scattering cross-section for each of these cases is obtained, and is found to depend on the spectrum of the spatial distribution of the relevant property of the medium.

It happens occasionally in science that a problem suddenly becomes important and interesting in several different contexts simultaneously. This is exactly what has happened in the case of propagation of waves through a medium whose physical properties are not quite uniform and vary from point to point in a random fashion as a result of the presence of turbulence. This problem has sprung into prominence in the post-war years in no fewer than four different physical settings.

One setting concerns the propagation of radio waves of about one metre wavelength through the atmosphere, and has led to the spectacular development of a new method of long-range communication, which employs scattering of the radio waves through a small angle by small random variations of electron density in the ionosphere. Another concerns the old problem of the scintillation of starlight; that the twinkling of stars is due to variations of refractive index in the atmosphere was known to Newton, but only recently have attempts been made to analyse the phenomenon quantitatively and to relate it to the turbulence in the atmosphere. In these two contexts the waves are electro-magnetic, whereas in the other two I have in mind sound waves are involved.

Most people here will already know of the current interest in various aspects of the propagation of sound in the ocean. If the ocean were perfectly uniform, ordinary acoustic theory would tell us all that we need to know in order to design and operate equipment for sound-ranging and sound-detection. But the ocean is not uniform, and contains currents which convect the sound and variations of temperature and of salt concentration which refract the sound. These non-uniformities are made random by the ever-present turbulence, and are broken down in scale so that there are always some Fourier components with wave-lengths comparable with that of the incident sound. As a consequence, sound signals are subject to large fluctuations in amplitude, to scattering about their original direction, and to rapid attenuation. This is not by any means a new problem, but the needs of naval warfare gave a great stimulus to its investigation, and interest in it has remained active during the post-war years.

Finally, there is the allied problem of propagation of sound in the atmosphere.

The effect of non-uniformities in the atmosphere came under notice during the first world war in connection with devices for the detection of enemy aircraft, but these devices were not successful and the problem was not examined fully. Interest in it has more recently been revived, as a part of a comprehensive investigation of the noise produced by jet engines. Instead of wanting to pick up the noise of an aircraft, we now wish to suppress or avoid it, but in either case we require a knowledge of the effect of random variations of the properties of the air on the sound waves passing through it.

Thus radio engineers, astronomers, oceanographers and aerodynamicists all have good reasons for wanting to know more about the same basic problem. So far as I can tell from a limited acquaintance with the separate literatures, they seem on the whole to have pursued the problem in their own ways, without making much use of advances made in other physical contexts. There is a need, I think, for an investigation which is framed in as general a manner as possible, and which brings together all the available results. The general aspects of the subject are especially suitable as material for a survey at the present time, and, although this lecture does not provide the time that would be needed for a proper account, I hope at least to arouse your interest and to give you the right orientation.

In order to keep fairly close to the theme of this symposium, I propose to confine my attention to sound waves, thereby excluding the applications of the theory to radio communication and stellar scintillation. (The analysis required in these latter applications is essentially similar, such differences as there are being mostly those arising from the need to specify electromagnetic waves by a vector, instead of a scalar, quantity.)

When reading the published papers on the propagation of sound waves in a non-uniform medium, I have noticed that many of the authors assume intuitively the validity of a certain equation describing the propagation of sound, and then proceed to investigate some special feature of this equation, such as the nature of the scattering. This procedure has its dangers, since mistakes in the form of the governing equations cannot be detected by the usual checks on consistency of the analysis. Some authors have assumed, for instance, that any non-uniformity of the physical properties of the medium is equivalent to a variation of the phase velocity, that is, of refractive index; others have assumed that the effect of turbulent convection with velocity $\boldsymbol{u}$ in the medium can be represented by replacing $\partial/\partial t$ in the conventional acoustic wave equation by $\partial/\partial t + \boldsymbol{u} \cdot \nabla$. Both of these ideas are plausible, but actually they are too simple, and lead to correct results only in special circumstances. In view of the evident differences of opinion about the equations on which the whole problem is based, I shall spend a large part of my time on a careful derivation of the governing equations appropriate to each kind of non-uniformity of the medium. I regret that there will not be time to get to grips with the observational data about scattering of sound.

There are three essentially different aspects of a theoretical treatment of the problem of propagation of sound waves through a non-uniform medium in turbulent motion:

a. the derivation of the equations describing the propagation of sound waves in these circumstances;

b. the analysis of the scattering and of other consequences of the non-uniformity;

c. the determination, from turbulence theory or experiment, of those statistical properties of the medium that the analysis in b. has shown to be relevant.

Consideration of this last aspect would take us into the complexities of turbulence theory, and must be omitted for lack of time. I shall begin with an analysis of scattering in general form, since talk about the form of governing equations is not interesting until one knows the kind of use to which they are to be put.

The scattered wave: general theory

We shall see later that the propagation of sound waves can be described, in all the situations under consideration, by a linear modified wave-equation of the form

$$\nabla^2\psi - \frac{1}{c_0^2}\frac{\partial^2\psi}{\partial t^2} = \sum_n P_n(\zeta)\, D_n(\psi). \tag{1}$$

(It is desirable to use a complete wave theory, rather than the methods of geometrical optics, in a discussion of scattering, because the relative scattered acoustic power is greatest under conditions for which a treatment by geometrical optics would not be valid.) ψ is here a scalar wave-function, such as the fluctuating pressure or density, the exact interpretation being left unspecified for the moment. c_0 is a constant, D_n is a linear differential or integral operator with respect to space or time, and $P_n(\zeta)$ is a function of the group of variables ζ that specify the (departures from the mean of the) relevant local properties of the medium. The summation is over all terms of the type shown, the usual number being not more than two. The variables ζ, of which temperature is a typical example, are themselves random functions of position and time as a result of the action of turbulent motion in the medium. The physical problems that we have in mind are such that the medium is only slightly non-uniform, and P_n is small compared with unity when all quantities are made dimensionless with the aid of parameters characteristic of the sound wave.

We wish to find a solution of this equation representing the passage of a sound wave, generated by given external means, through a region in which the random variables ζ are known in the statistical sense. It is important for the theoretician that he be allowed to assume that the incident sound wave has a frequency which is high by comparison with the fractional rates of change of the variables ζ, and fortunately this is nearly always true in practice. The consequence of this assumption is that ψ is effectively the only quantity in the above equation that depends on t, and, since ψ occurs linearly in all terms, the dependence on t plays only a minor role in the problem.

Since the right-hand side of the equation is small, whereas terms on the left-hand side are not, it is appropriate to seek a solution by a perturbation procedure. The first approximation to ψ, represented by ψ_0, is given by

$$\nabla^2\psi_0 - \frac{1}{c_0^2}\frac{\partial^2\psi_0}{\partial t^2} = 0. \tag{2}$$

We need consider only one Fourier component (with respect to t) of the incident wave, and we shall suppose the incident wave that is generated externally to be harmonic, with frequency ω, and to be approximately plane; thus

$$\psi_0 = A e^{i(\boldsymbol{\kappa}\cdot x - \omega t)}, \tag{3}$$

where

$$|\boldsymbol{\kappa}| = \kappa = \frac{\omega}{c_0} = \frac{2\pi}{\lambda}. \tag{4}$$

The second approximation to ψ is then $\psi_0 + \psi_1$, where

$$\nabla^2\psi_1 - \frac{1}{c_0^2}\frac{\partial^2\psi_1}{\partial t^2} = \sum_n P_n(\zeta)\, D_n(\psi_0). \tag{5}$$

The linear operations represented by D_n can be carried out explicitly, thus allowing the equation for ψ_1 to be written as

$$\nabla^2\psi_1 - \frac{1}{c_0^2}\frac{\partial^2\psi_1}{\partial t^2} = Ae^{i(\mathbf{K}\cdot\boldsymbol{x}-\omega t)}Q(\boldsymbol{x}) \tag{6}$$

say, where $Q(\boldsymbol{x})$ is independent of t. The formal solution of this equation as a "retarded potential" is

$$\psi_1(\boldsymbol{x},t) = -\frac{A}{4\pi}\int e^{i(\mathbf{K}\cdot\boldsymbol{y}-\omega t)+ik|\boldsymbol{x}-\boldsymbol{y}|}\frac{Q(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\boldsymbol{y}. \tag{7}$$

The sound wave represented by ψ_1 is the first approximation to the effect of the non-uniformity of the medium on the incident wave ψ_o. ψ_1 has the same form as the wave-function appropriate to a volume distribution of acoustic sources in a uniform medium, each source being of frequency ω and the density of source strength at position $\boldsymbol{y}$ being proportional to $AQ(\boldsymbol{y})$. It is as if the incident wave "polarized" each volume element of the medium with a strength proportional to $|\psi_o|$ and to a quantity representing the local departure of the medium from uniformity, the radiation from these "poles" then being, to this same order of approximation, the same as if the medium were uniform. The usual physical interpretation is that ψ_1 represents the collection of singly-scattered waves, that is, those waves that would be set up in a uniform medium by the incidence of the wave ψ_0 on single isolated scattering elements.

It is not often possible to go to the higher approximation $\psi_o + \psi_1 + \psi_2$, and most of the available theoretical work concerns single scattering. It is therefore important that we should know when $\psi_o + \psi_1$ is a valid approximation to the wave-function. In the case of a plane wave passing through a medium such that the velocity of sound has different, but uniform, values inside and outside a region of volume V, it has been established that $\psi_o + \psi_1$ is a good approximation to the wave-function provided $\varepsilon V^{1/3}/\lambda \ll 1$, where ε is a measure of the magnitude of the fractional variation in the velocity of sound. When the properties of the medium are random, having values above the ambient level in some parts of V, and below it in others, the requirement will be weaker and seems likely to be of the form $\varepsilon(V^{1/3}L)^{1/2}/\lambda \ll 1$, where L is a length characterizing the scale of variation of the properties of the medium. A numerical example will help to convey the significance of this criterion. Suppose sound waves of wave-length 10 cm pass through either water or air in which the temperature varies, with an r.m.s. fluctuation of 1 °C and a length scale of a metre. Single scattering will then be a reasonable approximation provided the region containing the non-uniformities has linear dimensions small compared with a kilometre. The restriction on the size of the volume V can be thought of as ensuring that the chances of a scattered wave being itself scattered before leaving V are slight.

The wave-function that describes the waves associated with a point singularity of some kind takes a simpler form at large distances from the singularity. The wave motion is purely radiative at these large distances, with an intensity that is a measure of the rate of output of energy from the source. There is thus both analytical convenience and physical significance in a consideration of the radiation fields of the scattered waves emanating from the various elements of the volume V in which the medium is non-uniform. If we wish to describe the fluctuations in sound intensity at some point within the scattering volume V, we must of course take into account the "near" fields of the scattered waves from neighbouring elements. Investigations of this kind have been made for certain choices of the governing quation (Ellison 1951, Mintzer 1953 & 1954). In order to keep this lecture within bounds, only the algebraically simpler quantities depending on the radiation fields will be considered.

A point whose distance from V is large compared with the linear dimensions of V will be in the radiation field of all the scattering elements, and at such a point

$$
\begin{aligned}
\psi_1(\boldsymbol{x},t) &\sim -\frac{A}{4\pi x}\int_V e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{y}-\omega t)+i\kappa x(1-\boldsymbol{x}\cdot\boldsymbol{y}/x^2)}Q(\boldsymbol{y})\,d\boldsymbol{y} \\
&= -\frac{A}{4\pi x}e^{i(\kappa x-\omega t)}\int_V e^{i\boldsymbol{k}\cdot\boldsymbol{y}}Q(\boldsymbol{y})\,d\boldsymbol{y} \qquad (8) \\
&= -\frac{A}{4\pi x}e^{i(\kappa x-\omega t)}\Gamma(\boldsymbol{k})
\end{aligned}
$$

where Γ is the Fourier transform of the function Q that describes the distribution of the relevant property of the medium, and $\boldsymbol{k} = \boldsymbol{\kappa} - \kappa\boldsymbol{l}$ (where $\boldsymbol{l} = \boldsymbol{x}/x$) is the so-called scattering vector that bisects the direction of the incident wave and that of $\boldsymbol{x}$ reversed (see figure). The scattered wave at a considerable distance from V thus consists of a spherical wave of frequency ω propagating radially outwards with an amplitude proportional to that of the incident wave and to the function $\Gamma(\boldsymbol{k})$. The appearance of $\Gamma(\boldsymbol{k})$ in the analysis is to be expected from considerations of the mutual interference of the scattered waves from the different volume elements. The Fourier component of $Q(\boldsymbol{y})$ with vector wave-number $\boldsymbol{k} = \boldsymbol{\kappa} - \kappa\boldsymbol{l}$ is the only one for which the phases of the scattered waves from the various volume elements combine in such a way as to produce a sinusoidal variation of ψ_1 along a line drawn from V in the direction $\boldsymbol{l}$; all other Fourier components of the distribution of the relevant property of the medium produce scattered waves which annul each other at distance points in the direction $\boldsymbol{l}$.

Now the properties of the medium vary randomly as a result of the action of turbulence, and the function $\Gamma(\boldsymbol{k})$, and consequently $\psi_1(\boldsymbol{x}, t)$, take different values from one realization to another. Only the statistical properties of $\Gamma(\boldsymbol{k})$ and $\psi_1(\boldsymbol{x}, t)$

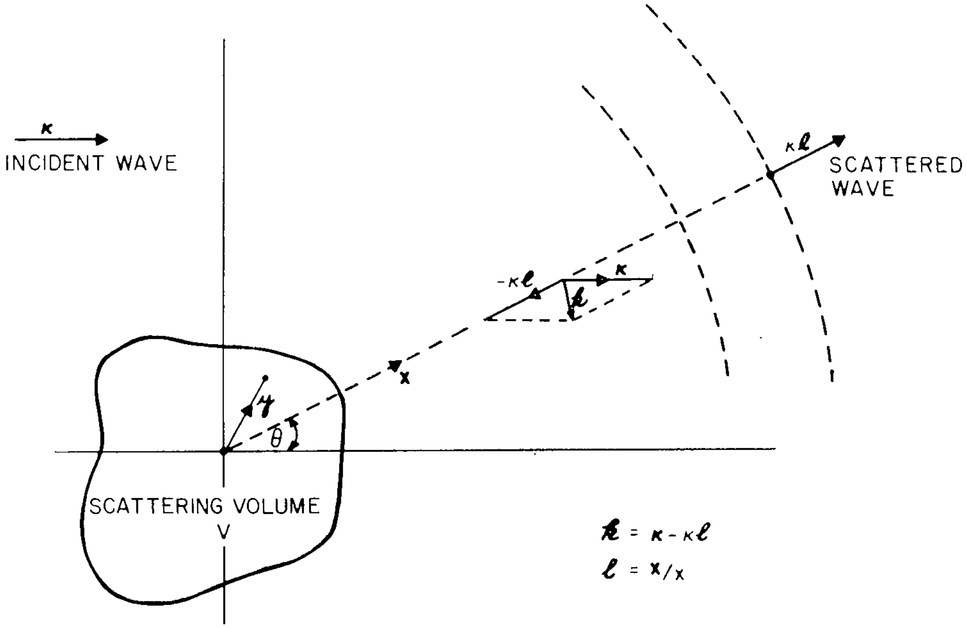

Figure 1

are relevant, and we therefore calculate the mean square of ψ_1, this being the most significant of these properties. The mean intensity of the scattered wave at position $\boldsymbol{x}$, expressed as a fraction of the intensity of the incident wave, is

$$\frac{\overline{\psi_1\psi_1^*}}{A^2} = \frac{\overline{\Gamma(\boldsymbol{k})\Gamma^*(\boldsymbol{k})}}{16\pi^2 x^2}$$

or

$$= \frac{1}{16\pi^2 x^2}\int_V\int_V \overline{Q(\boldsymbol{y})Q(\boldsymbol{z})}\; e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{z})}\, d\boldsymbol{y}\, d\boldsymbol{z}. \tag{9}$$

When the physical properties of the medium are approximately stationary random functions of position, and the volume V is large enough to contain many fluctuations of Q, this relation can be simplified to

$$\begin{aligned} \frac{\overline{\psi_1\psi_1^*}}{A^2} &\approx \frac{V}{16\pi^2 x^2}\int_V \overline{Q(\boldsymbol{y})Q(\boldsymbol{y}+\boldsymbol{\xi})}\; e^{-i\boldsymbol{k}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} \\ &\approx \frac{\pi V}{2x^2}\,\Phi(\boldsymbol{k}), \end{aligned} \tag{10}$$

where $\Phi(\boldsymbol{k})$ is the function usually referred to as the spectrum of the quantity Q.

The form of this expression for the mean intensity of the scattered wave at position $\boldsymbol{x}$ is worthy of note, not only for its simplicity. It reveals the prominent role played by the spectrum of the relevant property of the medium, showing that this is one of the functions that must be supplied from turbulence theory. A good many authors have analysed the scattering of sound and radio waves in terms of distributions of temperature, etc., in physical space. This makes the mathematical work cumbersome, and conceals the dependence of the intensity of the scattered wave on a very particular aspect of the turbulence. If, for instance, we wish to calculate the scattering of incident waves of high frequency, in directions not too close to that of the incident wave, we see from the above formula that a knowledge of the spectrum function $\Phi(\boldsymbol{k})$ at large wave-numbers is required. Quite large variations in the form of $\Phi(\boldsymbol{k})$ at large k correspond to relatively slight changes in the form of the correlation function $\overline{Q(\boldsymbol{y})\ Q(\boldsymbol{y}+\boldsymbol{\xi})}$ near $\boldsymbol{\xi} = 0$, so that the latter function would need to be known very accurately indeed if it were to be used in a calculation of the scattering.

The flux of energy in the scattered wave in direction $\boldsymbol{l}$ is sometimes expressed in terms of a quantity $\sigma(\boldsymbol{l})$ called the "scattering cross-section." The interpretation of our wave-function ψ has been left unspecified, but it will bear the same relation to the energy flux for both the incident and scattered waves.

Thus $\sigma(\boldsymbol{l})$ = (mean flux of energy in scattered wave per unit solid angle in direction $\boldsymbol{l}$ per unit scattering volume) ÷ (energy flux in incident wave, at position of scattering volume, per unit area of wave-front)

$$\begin{aligned} &= \frac{x^2}{VA^2}\,\overline{\psi_1(\boldsymbol{x},t)\psi_1^*(\boldsymbol{x},t)} \\ &= \frac{\pi}{2}\,\Phi(\boldsymbol{k}). \end{aligned} \tag{11}$$

The fraction of the energy of the incident wave that is lost by scattering, per unit length traversed by the incident wave-front, that is, the logarithmic decrement of the attenuation by scattering, is then

$$\begin{aligned}\gamma &= \int \sigma(\boldsymbol{l})\, d\boldsymbol{l} \\ &= \frac{\pi}{2}\int \Phi(\boldsymbol{\kappa} - \kappa\boldsymbol{l})\, d\boldsymbol{l} \qquad (12)\end{aligned}$$

where the integration is over all directions of the unit vector $\boldsymbol{l}$.

The special case of sound of very high frequency is worth noting, in view of its common use in under-water instruments and devices. If $\boldsymbol{\kappa}$ is considerably larger than the reciprocal of the length scale of the inhomogeneities of the medium, $\Phi(\boldsymbol{\kappa} - \kappa\boldsymbol{l})$ will be small for all directions of $\boldsymbol{l}$ except those close to the direction of $\boldsymbol{\kappa}$. Most of the energy of the scattered wave is thus thrown forwards, at small values of the scattering angle θ. To the first order in the angle of the cone containing most of the scattered energy,

$$\gamma = \frac{\pi}{2\kappa^2}\int \Phi(\boldsymbol{m})\, d\boldsymbol{m} \qquad (13)$$

where $\boldsymbol{m}$ is a vector in the plane perpendicular to $\boldsymbol{\kappa}$ and the (two-dimensional) integration is over all values of $\boldsymbol{m}$. If we choose the x_1-axis to be parallel to $\boldsymbol{\kappa}$, this can be written as

$$\begin{aligned}\gamma &\quad \frac{\pi}{2\kappa^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi(o,m_2,m_3)\, dm_2\, dm_3 \\ &= \frac{1}{2\kappa^2}\int_{o}^{\infty} \overline{Q(y_1+\xi_1,y_2,y_3)Q(y_1,y_2,y_3)}\, d\xi_1\,, \qquad (14)\end{aligned}$$

showing the dependence of γ, in this case, on a kind of integral scale of the quantity Q.

To proceed further we must know more about the right-hand side of the equation that was adopted at the beginning of this discussion, and we therefore turn now to a consideration of the equations that describe the propagation of sound waves. There are two important cases, first a medium of variable physical properties which is stationary except for the disturbance caused by the incident wave, and second a medium of uniform physical properties in turbulent motion. These two cases often occur together in practice, but separate consideration is desirable so that we can see which is likely to be the more important effect.

Stationary medium of variable properties

Provided we can neglect the effects of molecular viscosity, which usually plays the minor role of a small damping agent in these problems, the exact equations describing any kind of fluid motion are

$$\begin{aligned}\frac{D\boldsymbol{v}}{Dt} &= -\frac{1}{\rho}\nabla p\,, \\ \frac{D\rho}{Dt} &= -\rho\nabla\cdot\boldsymbol{v}, \end{aligned} \qquad (15)$$

where $\boldsymbol{v}$, ρ and p are the local velocity, density and pressure. Since the changes in the state of a material element of fluid are adiabatic (in the absence of molecular transfer effects), p and ρ are related by

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} = -\rho c^2 \nabla \cdot \boldsymbol{v} \tag{16}$$

where c is a parameter describing one of the physical properties of the material element under consideration and is equal to the phase velocity that sound waves would have in a uniform medium having the same physical properties as that element. Thus we have two equations for $\boldsymbol{v}$ and p, the form of the equations showing that the only two physical properties of the medium that are relevant are the local inertia per unit volume ρ, and the local phase velocity of sound waves c.

Our interest is in sound waves of small amplitude, and we therefore linearize in the departures from the undisturbed state in which $\boldsymbol{v}$ and ∇p are zero, obtaining

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} &= -\frac{1}{\rho} \nabla p \\ \frac{\partial p}{\partial t} &= -\rho c^2 \nabla \cdot \boldsymbol{v} \end{aligned} \tag{17}$$

where ρ and c are now symbols denoting the local density and acoustic phase velocity of the medium in the undisturbed state, and are independent of t according to the assumption explained earlier. The equation for p alone is

$$\frac{\partial^2 p}{\partial t^2} - \rho c^2 \nabla \cdot \left(\frac{\nabla p}{\rho} \right) = 0, \tag{18}$$

while that for $\boldsymbol{v}$ is

$$\frac{\partial^2 \boldsymbol{v}}{\partial t^2} - \frac{1}{\rho} \nabla (\rho c^2 \nabla \cdot \boldsymbol{v}) = 0. \tag{19}$$

The velocity field $\boldsymbol{v}$ is not irrotational when ρ varies with position, but it is clearly possible to define a quasi-potential such that

$$\boldsymbol{v} = \frac{1}{\rho} \nabla \varphi, \tag{20}$$

the equation for φ then being the same as that for p. If the length scale on which ρ varies is large compared with the wave-length of the incident sound wave, the equation for p may, for some purposes, be reduced approximately to

$$\frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p = 0, \tag{21}$$

which is the ordinary wave-equation with variable phase velocity; however, the Fourier components of the distribution of ρ that are most effective in scattering energy through a large angle are just those with length-scale comparable with the wave-length, so that we shall not make use of this approximation.

If the variations of ρ and c in the medium are relatively small, we can write

$$\rho = \rho_0 + \rho_1, \qquad c = c_0 + c_1 \tag{22}$$

where ρ_1, c_1 are local departures from the uniform mean values ρ_0, c_0, and proceed to linearize in ρ_1 and c_1. The equation for p then becomes

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = (\nabla p)\cdot\left(\nabla\frac{\rho_1}{\rho_0}\right) - \frac{2c_1}{c_0}\nabla^2 p, \tag{23}$$

which is now in the standard form adopted earlier in the lecture, p taking the part of the wave-function ψ.

The relative importance of the two terms on the right-hand side of this equation depends on the nature of the medium, and some discussion of the properties of the two common cases of air and water is needed. In the case of air, variations of ρ and c are likely to be attributable to variations of either temperature or water-vapor concentration, or both. It happens that the ratio of the specific heats of moist air is about the same as that of dry air, so that, for stationary air at uniform pressure, variations of c and ρ are related by

$$c^2 \propto \rho^{-1}, \qquad i.e. \qquad \frac{2c_1}{c_0} = -\frac{\rho_1}{\rho_0}, \tag{24}$$

irrespective of whether the variations of ρ and c are a consequence of variations of temperature or of water-vapor concentration. Of the two possible causes of variations in ρ, variations in temperature seem likely to the more important in the atmosphere, since the difference in density between completely dry air and saturated air at 10°C is no more than that produced by a variation of temperature of about 1.5°C.

For air, then, the governing equation becomes

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = (\nabla p)\cdot\left(\nabla\frac{\rho_1}{\rho_0}\right) + \frac{\rho_1}{\rho_0}\nabla^2 p, \tag{25}$$

and from this point onwards the analysis of the scattering follows the path already laid down. The function $Q(\boldsymbol{x})$ of the general analysis is here given by

$$Q(x) = i\boldsymbol{\kappa}\cdot\nabla\frac{\rho_1}{\rho_0} - \kappa^2\frac{\rho_1}{\rho_0}, \tag{26}$$

and its Fourier transform is

$$\begin{aligned}\Gamma(\boldsymbol{k}) &= (\boldsymbol{\kappa}\cdot\boldsymbol{k} - \kappa^2)\Lambda(\boldsymbol{k}) \\ &= -\kappa^2\cos\theta\,\Lambda(\boldsymbol{k}),\end{aligned} \tag{27}$$

where $\Lambda(\boldsymbol{k})$ is the Fourier transform of $\rho_1(\boldsymbol{x})/\rho_0$. The function $\sigma(\boldsymbol{l})$ that describes the directional distribution of intensity of the scattered wave is

$$\sigma(\boldsymbol{l}) = \frac{\pi}{2}\kappa^4\cos^2\theta\,\Psi(\boldsymbol{k}), \tag{28}$$

where $\Psi(\boldsymbol{k})$ is the spectrum of the relative density fluctuations $\rho_1(\boldsymbol{x})/\rho_0$ in the sense in which $\Phi(\boldsymbol{k})$ was defined to be the spectrum of $Q(\boldsymbol{x})$.

The method of obtaining more specific results for the case of high frequency sound has already been described; it will be seen to show that the total scattered energy flux is then proportional to κ^2. Another particular case of some interest is that in which the distribution of density is statistically isotropic (as is always likely to be a valid approximation, in the atmosphere, for the small-scale components, according to the universal equilibrium theory). $\Psi(\boldsymbol{k})$ then depends only on the vector magnitude

$$k = 2\kappa \sin \frac{\theta}{2}, \tag{29}$$

and the decrement of the incident wave becomes

$$\begin{aligned}\int \sigma(\boldsymbol{l}) dl &= \pi^2\kappa^4 \int_0^\pi \cos^2\theta\, \Psi\left(2\kappa \sin \frac{\theta}{2}\right) \sin\theta\, d\theta \\ &= 2\pi^2\kappa^2 \int_0^{2\kappa} s\left(1 - \frac{s^2}{2\kappa^2}\right)^2 \Psi(s)\, ds.\end{aligned} \tag{30}$$

A similar set of results can be obtained for water. Variations of ρ and c will arise here from variations of temperature and of salt concentration (and possibly from the presence of air bubbles, in some circumstances). From data given in the textbook by A. B. Wood (1932), it can be calculated that, for a fluctuation δT in absolute temperature under normal conditions,

$$\frac{\delta\rho}{\rho} \approx -0.06\frac{\delta T}{T}, \qquad \frac{\delta c}{c} \approx 0.80\frac{\delta T}{T}, \tag{31}$$

and, for a fluctuation δC about the normal level of salt concentration (roughly 0.035 gm salt/cm^3 water),

$$\frac{\delta\rho}{\rho} \approx 0.025\frac{\delta C}{C}, \qquad \frac{\delta c}{c} \approx 0.025\frac{\delta C}{C}. \tag{32}$$

The relative importance of all these separate influences on the scattering will probably depend on the prevailing conditions, but it seems likely that temperature fluctuations in the sea will often be more effective in changing ρ and c than fluctuations in salt concentration, and that temperature fluctuations will be relevant to sound scattering primarily through their influence on c. At any rate, I shall assume so here for illustrative purposes.

With this assumption, we can put $\rho_1/\rho_0 \approx 0$ and $c_1/c_0 \approx 0.8\dfrac{T_1}{T_0}$, and the equation for the sound pressure becomes

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -1.6\frac{T_1}{T_0}\nabla^2 p, \tag{33}$$

which is simply the ordinary wave equation with non-uniform phase velocity. We can now proceed to apply the general analysis of scattering once again. The function $Q(\boldsymbol{x})$ is here given by

$$Q(\boldsymbol{x}) = 1.6\kappa^2 \frac{T_1(\boldsymbol{x})}{T_0}, \tag{34}$$

and the associated scattering cross-section is

$$\sigma(\boldsymbol{l}) = 1.28\pi\kappa^4\Omega(\boldsymbol{k}), \tag{35}$$

where $\Omega(\boldsymbol{k})$ is the spectrum of the relative fluctuations in absolute temperature $T_1(\boldsymbol{x})/T_0$ in the sense in which $\Phi(\boldsymbol{k})$ and $\Psi(\boldsymbol{k})$ are the spectra of $Q(\boldsymbol{x})$ and $\rho_1(\boldsymbol{x})/\rho_0$.

The only difference between the scattering cross-sections for stationary air and sea-water, apart from the difference in the numerical factors, lies in the appearance of the $\cos^2\theta$ factor in the former but not in the latter. In the case of sea-water, the scattered energy flux is fairly evenly distributed over all directions, but in the case of air there is a minimum at the lateral plane $\theta = \frac{\pi}{2}$. The expressions for the total energy flux $\int \sigma(\boldsymbol{l})\,d\boldsymbol{l}$ in the cases of air and water should not differ by more than a factor of order unity as a result of the presence or absence of the $\cos^2\theta$. Much more substantial differences can occur as a result of differences between the forms, and the magnitudes, of the spectra of density fluctuations in the one case and temperature fluctuations in the other. However, these differences cannot be described in a general way since they depend on particular considerations such as the value of the sound frequency.

Uniform medium in turbulent motion

A discussion of the effect of turbulent fluctuations of velocity on the propagation of sound waves encounters many difficulties, and a good deal of care is necessary in the use of approximations. Many papers have been written on the problem, and three people have independently derived the cross-section for single scattering correctly. The first was Blokhintzev in 1945, who made use of the equation for the propagation of sound in turbulent flow derived by Obukhoff two years earlier, and the others were Lighthill and Kraichnan in 1953. These people used completely different kinds of argument, and it takes a good deal of work to show that in fact they used the same assumptions and arrived at the same results. None of these pieces of work seems yet to have been assimilated into the literature of the subject. By standing on the shoulders of these pioneers I have devised my own derivation, which I believe to be simpler and freer from logical imperfections. The result is important enough to warrant continued attempts to improve the argument.

One or two numbers will help us first to see what is perturbing what, in this combination of sound and turbulence. The r.m.s. particle velocity in a plane sound wave of N decibels sound level in air is about $5 \times 10^{-6} \times 10^{N/20}$ cm/sec, which comes to 0.5 cm/sec for a very loud noise of 100 decibels. On the other hand, the r.m.s. of the turbulent velocity fluctuations in the lower levels of the atmosphere is often in the neighbourhood of 50 cm/sec. Thus the fluid velocity in the turbulence field alone is very much bigger than that in the sound field alone. However, this is not necessarily true also of the corresponding fluctuations in density and pressure, since density variations in a sound wave increase as the first power of the accompanying particle velocity whereas in a turbulent motion they increase as the second power. Use of the above figures for the atmosphere shows that the density fluctuations in the sound and turbulence fields alone are comparable in magnitude. Similar calculations for underwater sound waves, having intensities like those commonly used in connection with ranging equipment, show that here again the sound waves are likely to be accompanied by particle velocities much smaller than, and density fluctuations comparable to, those produced by turbulence in the sea.

Despite this approximate parity of the separate density variations, the incident sound wave cannot have much effect on the turbulent velocity field. The shearing motions of the turbulence are determined primarily by the boundary conditions on velocity, and are not affected appreciably by superimposed relative density variations of order M_t^2 (where M_t is the Mach number of the turbulence alone)—as indeed can be seen from the fact that the velocity distribution in the pure turbulence is known to be approximately independent of M_t. Thus the velocity distribution in the combined incident sound and turbulence field is approximately the same as that in the turbulence alone.

We shall find it convenient to write the density, pressure and velocity in the combined incident sound and turbulence field in the following form:

$$\rho = \rho_0 + \rho_t + \rho_s, \qquad p = p_0 + p_t + p_s, \qquad \boldsymbol{U} = \boldsymbol{u} + \boldsymbol{v}, \tag{36}$$

where ρ_0 and p_0 are the uniform density and pressure in the completely undisturbed medium, and $\rho_0 + \rho_t$, $p_0 + p_t$, $\boldsymbol{u}$ are *exactly* the values of the density, pressure and velocity in the turbulent field in the absence of the incident sound wave. From what has already been said, $|\boldsymbol{v}| \leqslant |\boldsymbol{u}|$ although $|\rho_s|$ and $|p_s|$ may be comparable with $|\rho_t|$ and $|p_t|$. Moreover, since the Mach numbers of the incident sound wave alone (M_s) and of the turbulence alone (M_t) are both small, we can safely assume that

$$|\,\rho_t + \rho_s\,| \ll \rho_0, \qquad |\,p_t + p_s\,| \ll p_0. \tag{37}$$

We shall ignore again the effects of molecular conductivity, so that the motions are adiabatic and

$$\begin{aligned} \frac{p_s}{p_0} &= \left(1 + \frac{\rho_t + \rho_s}{\rho_0}\right)^{\gamma} - \left(1 + \frac{\rho_t}{\rho_0}\right)^{\gamma} \\ &\approx \frac{\gamma\rho_s}{\rho_0} + \tfrac{1}{2}\gamma(\gamma - 1)\left[\left(\frac{\rho_t + \rho_s}{\rho_0}\right)^2 - \left(\frac{\rho_t}{\rho_0}\right)^2\right]. \end{aligned} \tag{38}$$

These are the approximations that we can now use in the equations of motion.

The equations for the quantities ρ, p, $\boldsymbol{U}$ relating to the combined field can be written as

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho U_i)}{\partial x_i} = 0, \tag{39}$$

$$\frac{\partial(\rho U_i)}{\partial t} + \frac{\partial(\rho U_i U_j)}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0, \tag{40}$$

without approximation, apart from the neglect of the effects of viscosity. A combined form of these equations is

$$\frac{\partial^2\rho}{\partial t^2} - \nabla^2 p - \frac{\partial^2(\rho U_i U_j)}{\partial x_i \partial x_j} = 0. \tag{41}$$

This equation is satisfied when ρ, p, $\boldsymbol{U}$ are given by (36), and also when

$$\rho = \rho_0 + \rho_t, \qquad p = p\boldsymbol{U} + p_t, \qquad \boldsymbol{U} = \boldsymbol{u}, \tag{42}$$

so that

$$\frac{\partial^2\rho_s}{\partial t^2} - \nabla^2 p_s - \frac{\partial^2}{\partial x_i \partial x_j}[\rho_s(u_i + v_i)(u_j + v_j) + (\rho_0 + \rho_t)(2u_i v_j + v_i v_j)] = 0, \tag{43}$$

again without approximation. ρ_s and p_s are the increments in density and pressure in the combined field due to the incident sound wave, and they are unlikely to be smaller than the corresponding increments in the absence of the turbulence. Thus ρ_s/ρ_0 and p_s/p_0 are at least of order M_s. Examination of the magnitudes of the different terms of this equation in the light of the above permissible approximations, together with a little (uncontentious) guesswork about the magnitudes of the spatial derivatives of $\boldsymbol{u}$, then shows that to a first approximation the equation reduces to

$$\nabla^2\frac{\rho_s}{\rho_0} - \frac{1}{c_0{}^2}\frac{\partial^2(\rho_s/\rho_0)}{\partial t^2} = 0. \tag{44}$$

The boundary conditions on ρ_s/ρ_0 are those appropriate to the circumstances of generation of the incident sound wave. Thus to a first approximation ρ_s, p_s and $\boldsymbol{v}$ have the same values as for the incident sound wave in a stationary medium.

It appears, then, that the interaction between the sound and turbulence fields is weak, and that they each perturb the other. This lack of coupling is essentially a consequence of the big difference in the time scales on which changes take place in the two fields.

The equation that describes the perturbation of the incident sound wave due to the presence of the turbulence is obtained by going to the second approximation of the above exact equation for ρ_s; thus

$$\begin{aligned}\nabla^2(\rho_s/\rho_0) - \frac{1}{c_0^2}\frac{\partial^2(\rho_s/\rho_0)}{\partial t^2} &= -\frac{2}{c_0^2}\frac{\partial^2(u_i v_j)}{\partial x_i \partial x_j}, \\ &= -\frac{2}{c_0^2}\frac{\partial}{\partial x_j}\left(u_i\frac{\partial v_j}{\partial x_i}\right)\end{aligned} \tag{45}$$

to a consistent approximation. Terms on the left-hand side of this equation are of order M_s/λ^2, that on the right-hand side is of order $M_s M_t/\lambda^2$ (assuming, for the sake of definiteness, that the spatial derivatives of $\boldsymbol{u}$ have a magnitude of an order equal to that of the r.m.s. of $\boldsymbol{u}$ divided by λ), and the largest term of those neglected is of order M_s^2/λ or $M_s M_t^2/\lambda^2$, whichever happens to be the larger.

It should be noted that this equation describing the changes in the incident sound wave contains approximations additional to the conventional assumptions of linearized sound theory, unlike the corresponding equations obtained earlier for the various cases of a non-uniform stationary medium. These additional approximations may have consequences for the perturbation procedure on which the scattering analysis is based. While the equation certainly allows the investigation of single scattering, an investigation of multiple scattering from the same equation would need further justification.

For the investigation of single scattering, $\boldsymbol{v}$ on the right-hand side can be expressed in terms of ρ_s/ρ_0 by the ordinary lineary acoustic equation

$$\partial\boldsymbol{v}/\partial t = -c_0^2\nabla(\rho_s/\rho_0),$$

and so, for an incident harmonic plane wave of the usual form, the equation becomes

$$\nabla^2(\rho_s/\rho_0) - \frac{1}{c_0^2}\frac{\partial^2(\rho_s/\rho_0)}{\partial t^2} = \left(\frac{2u_i}{c_0}\kappa\kappa_i - \frac{2i}{c_0}\frac{\partial u_i}{\partial x_j}\frac{\kappa_i\kappa_j}{\kappa}\right)\frac{\rho_s}{\rho_0}. \tag{46}$$

This equation is in the standard form used earlier for the general analysis of scattering. The function $Q(\boldsymbol{x})$ introduced there is here given by

$$Q(\boldsymbol{x}) = \frac{2u_i}{c_0}\kappa\kappa_i - \frac{2i}{c_0}\frac{\partial u_i}{\partial x_j}\frac{\kappa_i\kappa_j}{\kappa}, \tag{47}$$

and the scattering cross-section is

$$\sigma(\boldsymbol{l}) = 2\pi\kappa^4\cos^2\theta\,\frac{\kappa_i\kappa_j F_{ij}(\boldsymbol{k})}{\kappa^2 c_0^2}, \tag{48}$$

where $F_{ij}(\boldsymbol{k})$ is the spectrum of the turbulent velocity $u_i(\boldsymbol{x})$ in the sense in which $\Phi(\boldsymbol{k})$ was defined to be the spectrum of $Q(\boldsymbol{x})$. $\dfrac{\kappa_i\kappa_j}{\kappa^2}F_{ij}(\boldsymbol{k})$ is simply the spectrum of the component of turbulent velocity in the direction of propagation of the incident sound wave, showing that only one component of the turbulent velocity is relevant to single-scattering of a plane wave. This expression for $\sigma(\boldsymbol{l})$ is effectively the same as the results obtained by Blokhintzev, by Lighthill, and by Kraichnan.

It will be noticed that the single-scattering due to a turbulent velocity field $\boldsymbol{u}(\boldsymbol{x})$ in a medium of uniform physical properties is exactly the same as that due to a density variation $\rho_1(\boldsymbol{x})$ in stationary air provided

$$\frac{\boldsymbol{\kappa}\cdot\boldsymbol{u}(\boldsymbol{x})}{\kappa c_0} = -\tfrac{1}{2}\frac{\rho_1(\boldsymbol{x})}{\rho_0}. \tag{49}$$

This is not an entirely unexpected result, since the two sides of this relation represent the local fractional increases in apparent phase velocity of the incident wave as a result of the non-uniformity of the medium, the increase occurring in one case by bodily convection and in the other by a change in the physical properties of the medium. However, lest this result should begin to seem obvious, may I remind you that it holds only for single-scattering and only for the case of air, for which variations in inertia and elasticity are connected in a particular way. It should be noted, moreover, that $\boldsymbol{\kappa}\cdot\boldsymbol{u}/\kappa c_0$ and $-\rho_1/2\rho_0$ would not have identical distributions (even statistically) in two real cases, since one quantity is subject to the kinematical requirements of the continuity condition and the other is not. A comparison of the relative importance of the scattering effects of turbulent velocities and of variations of density or temperature, as they occur in reality, would require a consideration of turbulent spectra, which lies outside the scope of this lecture. So far as I know, an analytical comparison of this kind has not yet been made.

Particular cases can now be discussed as before. When the turbulent motion is isotropic, we have

$$F_{ij}(\boldsymbol{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\frac{E(k)}{4\pi k^2}, \tag{50}$$

where $E(k)$ is the usual energy spectrum function, and hence

$$\sigma(\boldsymbol{l}) = \frac{\kappa^2 \cos^2\theta}{8\tan^2\theta/2}\,\frac{E(2\kappa\sin\theta/2)}{c_0^2}. \tag{51}$$

When k is very small, $E(k)$ is proportional to k^4 and so σ falls to zero at the straight-ahead direction $\theta = 0$ (as is also true when the turbulence is not isotropic). For an incident sound wave of very high frequency, and turbulence which is not necessarily isotropic, the total scattered power is given approximately by

$$\int \sigma(\boldsymbol{l})\,d\boldsymbol{l} = 2\pi\kappa_i\kappa_j \int \frac{F_{ij}(\boldsymbol{m})}{c_0^2}\,d\boldsymbol{m}\,, \tag{52}$$

where $\boldsymbol{m}$ is a vector in the plane at right angles to $\boldsymbol{\kappa}$. This expression can be written as

$$\int \sigma(\boldsymbol{l})\,d\boldsymbol{l} = 2\kappa^2\frac{\overline{u^2_\kappa}}{c_0^2}L_\kappa\,, \tag{53}$$

where $\overline{u_\kappa^2}$ stands for the mean-square of the turbulent velocity component parallel to $\boldsymbol{\kappa}$, and L_κ is the integral scale in this same direction, defined in the usual manner.

This brings me to the end of my account of scattering of sound waves due to turbulence. It is not a particularly dramatic end, and no startling conclusions or new ideas have been evolved. What I have tried to do is to present reasonably reliable analysis of the process of single-scattering of sound waves in some important cases. It took me a long time to make sense out of the large and confused literature on the problem, and I hope that the account I have given will save other people from having to spend the same amount of time.

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DISCUSSION

D. Mintzer

I would like to make a few comments about a possible new experimental technique which may be useful in analyzing turbulence. As shown in Dr. Batchelor's paper, and in previous published works, we now have a good understanding of the scattering of sound waves by turbulence. This leads me to believe that we should start thinking of sound scattering as a technique for determining correlation functions in media which are randomly inhomogeneous. It has been shown from scattering experiments that one can determine values for some of the variables of interest, such as the scale of turbulence (and characteristic times, in time-varying problems). This may turn out to be a very useful technique, especially in large-scale turbulence, such as in the ocean and the atmosphere.

As a simple example, you can show that by taking the differential scattering cross-sections in two directions close to each other, you can obtain information about the derivatives of the energy spectrum of the turbulence.

R. H. Kraichnan

There are two subjects I should like to discuss briefly in connection with Dr. Batchelor's interesting talk: the domain of validity of sound-turbulence scattering theory based on the first Born approximation, and the effect on the scattering process of the time-dependence of the turbulence field.

As Dr. Batchelor pointed out, the errors implied in the first Born (single-scattering) approximation are not wholly separable from the errors involved in arriving at the usual isentropic inhomogeneous wave-equation for the scattering of sound by turbulence. The equations of conservation of mass and momentum do not alone provide a complete description of the dynamics of a compressible fluid; they must be supplemented by an expression for the internal energy function of the fluid and a law of entropy flow and production. A general consequence is a multi-valued relationship between density and pressure in the fluid. If one makes the usual adiabatic approximation, a single-valued relationship is established between pressure and density, and there results a complete set of equations for the fluid motion in which the entropy does not appear explicitly.

The magnitude of the errors introduced both by an isentropic approximation and by later use of a single-scattering approximation to solve the associated wave-equation, may be estimated from the *exact* inhomogeneous wave equation for the pressure field, the right side of which will involve density, pressure and velocity. One may transform this equation to the corresponding integral equation and determine the contributions of successive orders of an appropriate iteration procedure, using the other equations of motion for relations between pressure, density, and velocity.

For examination of a scattering process, the appropriate starting point for the iteration could be zeroth order density, pressure, and velocity fields which are the sums of the fields corresponding to an infinitesimal sound wave and very low Mach number turbulence, together with the entropy field associated with these infinitesimal disturbances. It is to be noted that the infinitesimal sound wave represents a solution of the *homogeneous* wave-equation; because the integral equation for the pressure field provides for explicit introduction of this homogeneous part of the solution (automatically maintaining proper boundary conditions at infinity during iteration), it is a more suitable instrument for investigating orders of magnitudes of correction terms than the differential equation. for which boundary conditions at infinity must be taken account of at every step.

Now, if the amplitudes of sound wave and turbulence are *sufficiently small*, and if the volume in which scattering takes place is *sufficiently small*, the first Born approximation—the solution resulting from a single iteration starting with the zeroth order fields described—may be justified (R. H. Kraichnan, J. Acoust. Soc. Am. 25, 1096 (1953)). This implies that departures from isentropic conditions, and what is loosely termed "multiple scattering" will both have small effect on the scattered wave, for both these phenomena here show up in higher steps of the iteration procedure.

The expression "sufficiently small" which I have used requires discussion. It is notoriously difficult in physics to give compact and generally applicable sufficient conditions for the validity of the first Born approximation. An obvious (but sometimes ignored) necessary condition can be stated however: The difference between the exact field and the zeroth order field must be small compared to both throughout the region of scattering. When an applicable criterion of this sort is not satisfied, and the first Born approximation breaks down, it is frequently meaningful to ask to what extent the breakdown is due to nonisentropic conditions or to inappropriate procedure in solving the isentropic equations. The two effects (each represented by terms in the iteration expansion of the source function for the pressure field) are often fairly distinct physically.

For a simple example, consider a plane sound wave propagating through a very large region of homogeneous turbulence. If the turbulence is very weak, the error in the first Born approximation may be ascribable principally to a non-isentropic effect—the attenuation of the sound wave by viscosity. If the turbulence is stronger, however, the attenuation of the plane sound wave may be due principally to energy scattered out of the wave by the turbulence, and an accurate description of this could be obtained by applying better scattering approximations to the isentropic equations. In either of these simple cases, provided there is small attenuation within the correlation length of the turbulence, a better description may be obtained by using a "doctored" Born approximation in which the zeroth order sound field is exponentially damped. (This may be considered the first Born approximation to an inhomogeneous wave equation in which the homogeneous part contains a damping term.)

On the other hand, in the interaction of intense sound waves with strong turbulence, as in a high-speed turbulent jet, it is not very meaningful to speak separately of non-isentropic effects and multiple scattering effects. In this case, in fact, the distinction between sound and turbulence is unclear, although one may distinguish a transverse velocity and a longitudinal velocity. What is most important to keep in mind is that generalizations about the validity of a particular scattering approximation are risky, and the self-consistency of a given approximation should be investigated for the application at hand. For example, in certain applications the scattering due to turbulent density fluctuations may be negligible, in others not.

Dr. Batchelor noted some similarities between scattering by random scalar inhomogeneities (scattering of radio and light waves by turbulent refractive index fluctuations) and the scattering of sound by turbulence. I should like to stress that there are essential differences in the range of validity of the Born approximation in the two cases. In the scalar case, given a size of the scattering volume, the first Born approximation can remain valid at indefinitely high frequency if the scalar index fluctuations are small. In the sound-scattering case, the interaction between sound and turbulence involves quadratically the *gradients* of the velocities, thereby leading to a strong divergence of the first Born approximation cross-section toward short sound wave-lengths. By the criterion mentioned previously, this approximation is therefore not valid for indefinitely short wave-lengths; for example, it cannot properly be applied to the scattering of energy out of a shock-wave traversing turbulence. In some cases, the sound-scattering at very short wavelengths is satisfactorily described by a "physical optics" approximation in which the corrugation of coherent wavefronts is computed. Here again, *safe* criteria of validity are not easy to state.

Now I should like to. turn briefly to the effects of time-variation of the turbulence. Although in most cases where electromagnetic or sound waves are scattered by turbulence the characteristic frequencies of the turbulence are much smaller than the wave frequency, it does not follow that the effect of the time variation is unimportant. Slow time dependence will not alter the time-average scattering cross-sections, but it will alter the time-structure and spectrum of the scattered radiation. In general, the scattered radiation will be smeared over a frequency bandwidth range the order of the characteristic frequencies of the turbulence, and it will exhibit fading times the order of the characteristic period of the turbulence. In the scattering of sound by turbulence the spreading in frequency of the first Born approximation scattered wave has an interesting angular dependence; the energy scattered in nearly forward directions is very little spread in frequency, compared to radiation at larger scattering angles.

Further effects of the time-structure of the turbulence include phenomena of energy exchange between wave field and turbulence. For electromagnetic waves, these effects are not of much interest in nonionized media, but they may be significant for the propagation of sound through turbulence, especially in large regions. The energy exchange has two aspects: radiation of broad-band acoustic noise by the turbulence, and transformation of energy from the incident sound field into turbulent kinetic energy (that is to say, the scattering is not exactly "elastic," or conservative). The relative magnitude of the two effects depends on the ratio of turbulence and sound particle velocity Mach numbers, and on the spectral structure of the fields.

Both of these effects tend to degrade an initially monochromatic sound field by increasing the ratio of energy in diffuse spectrum to that in the line spectrum. For strong enough turbulence, the noise radiation may actually contain much more energy than the scattered sound wave, although the latter may still be distinguishable because of its sharper frequency structure. When Mach numbers of both sound and turbulence are high, the distinction between scattered and radiated energy is not very meaningful.

Finally I should like to comment on Dr. Mintzer's proposal for investigation of turbulence structure by scattering experiments. It is of interest in this connection that different information would be obtainable in principle by the scattering of electromagnetic or acoustic waves in the same wavelength range, because the coupling is scalar in one case and tensor in the other.

M. J. Lighthill

I hadn't actually prepared anything, but I had a feeling Dr. Batchelor would say some interesting things, and I was right.

There is this question about any differences between the approach he has described and the approaches that Dr. Kraichnan and myself derived in 1953. I found it a little difficult to see just what the differences were, in the mathematical details of what he showed us and the thing that I did.

But I would like just to mention something which isn't connected with mathematical details, but rather with a physical interpretation which I found helped me, though it won't necessarily help everyone. That was this.

When Lord Rayleigh worked out the explanation of the blue colour of the sky, he showed that this was due to variations of refractive index. Later, the theory of refractive index was worked out, and it was seen that this was due to polarization of dipole molecules in the direction of the incident beam. So we can say that this scattering of light (with some preference for the shorter waves) is due to polarization of molecules in a random sort of way, with some sort of spatial correlation.

The point of view from which I found it helpful to look at the problem we have just heard about was this. We have the following equation which Dr. Batchelor used—I deliberately leave out any question of variation of properties and just leave in the terms

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla^2 \rho = \frac{\partial^2}{\partial x_i \partial x_j} (\rho u_i u_j). \tag{1}$$

We have this equation, and this equation also came out in the theory of aerodynamic noise, and one said that the sound produced by turbulence was effectively that due to the distribution of quadrupoles indicated on the right of (1), namely quadrupoles of strength

$$\rho u_i u_j \tag{2}$$

per unit volume.

Now, Dr. Batchelor pointed out that in the presence of an incident wave the term inside the second derivative in (1) would become

$$\rho(u_i + v_i)(u_j + v_j), \tag{3}$$

where v_i is the velocity field of the incident sound wave. We can write (3), the new quadrupole strength per unit volume, as

$$\rho(u_i u_j + u_i v_j + v_i u_j + v_i v_j). \tag{4}$$

Now, here you have got four lots of quadrupoles, and we can interpret them physically as follows. The first term (see (2)) represents the sound that is being generated by the turbulence itself, which we are not concerned with. The middle two terms represent the scattered sound due to the interaction of the turbulence and the sound beam. And the last term $\rho v_i v_j$ represents the effect of the sound wave on itself, which we know is a steepening, tending to produce shock waves, and so on, but will be very small in the case of ordinary weak waves.

The middle two terms

$$\rho(u_i v_j + v_i u_j) \tag{5}$$

are the part we are interested in, and it struck me that a way of looking at this part would be to say that here you had quadrupoles that were polarized in the direction of the incident sound beam v_i, you see. Well, only one axis of each quadrupole was polarized—in the first term of (5) it is the v_j, in the second the v_i. But you have an analogy in this to the case of light scattering.

I may, perhaps, conclude with some remarks concerning multiple scattering. If β is the proportion of sound energy scattered per unit distance, then at a distance l from the source the scattered sound will be a substantial proportion of the whole if βl is not small. In this case one must, and may, consider multiple scattering by successive use of results from single-scattering theory.

If βl is actually large, then most of the sound that is present will have been scattered many times, and may be thought of as having described a crooked path like the "random walks" treated by statisticians.

The particular kind of random walk which is involved has been insufficiently studied. However, to one simple question about it there is a rather simple answer.

If we ask, what change of direction on the average will a ray of sound undergo (due to multiple scattering) in travelling over a *total* distance (no longer the distance from the source "as the crow flies", but somewhat greater), then we have first that a proportion

$$e^{-\beta l}\frac{(\beta l)^n}{n!}\ \text{(Poisson's distribution)} \tag{6}$$

of the energy carried by such rays will have been scattered n times. During this process their *direction* may be regarded as performing a random walk (on the unit sphere). If $\frac{1}{2}p(n, \theta) \sin\theta\, d\theta$ is the probability that after n scatterings the direction will make an angle between θ and $\theta + d\theta$ with the initial direction (where the factor $\frac{1}{2}\sin\theta$ has been inserted to ensure that for a uniform directional distribution $p = 1$), then p satisfies approximately the partial differential equation (of diffusion)

$$\frac{\partial p}{\partial n} = \tfrac{1}{4}\overline{\omega^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right) \tag{7}$$

where $\overline{\omega^2}$ is the mean square deviation in direction at each scattering. The solution of (7) such that the direction is initially $\theta = 0$ is

$$p = \sum_{m=o}^{\infty}(2m + 1)P_m(\cos\theta)e^{-\frac{1}{4}\overline{\omega^2}m(m+1)n} \tag{8}$$

Summing the product of (6) and (8) for $n = 0$ to ∞, we obtain the distribution of direction after propagation through a distance l, as

$$p = \sum_{m=o}^{\infty}(2m + 1)P_m(\cos\theta)e^{-\beta l\left(1-e^{-\frac{1}{4}\overline{\omega^2}\, m(m+1)}\right)} \tag{9}$$

$$= 1 + 3\cos\theta e^{-\beta l(1-\frac{1}{2}\overline{\omega^2})} + \cdots$$

Equation (9) shows that to make the sound directionally uniform ($p = 1$) needs a distance

$$l \gg \frac{1}{\beta\left(1 - e^{-\frac{1}{2}\overline{\omega^2}}\right)} \doteq \frac{2}{\beta\overline{\omega^2}}. \tag{10}$$

For homogeneous isotropic turbulence this condition may be written approximately as

$$l \gg \frac{2a^2}{\pi\int_o^K kE(k)\, dk}, \tag{11}$$

where a is the velocity of sound, κ the radian wave-number of the incident wave, k that of the turbulence, and $E(k)$ its three-dimensional energy spectrum.

G. K. Batchelor

I think that Professor Lighthill's remarks about double and multiple scattering take it for granted that an investigation by the usual iterative procedure is permissible. In the particular case of sound waves scattered by turbulent motion of the medium, it is not yet proven that this procedure is valid beyond the stage of single scattering. To say again in one sentence what I said in the lecture, if one takes the quantities appropriate to the scattered wave, and inserts them into the right-hand side of the basic equation, one may obtain terms that are of the same order of magnitude as those that have already been neglected in deriving the equation; in this way, an inconsistency may arise.

R. Kraichnan

One of the problems may be that it is a very broad problem, trying to decide which higher order effects are important where. Take the specific problem Lighthill brought up: a region which receives no single scattering at all. Then it is quite possibly safe to use the double scattering approximation to find out what happens there. But if you consider a region which can receive both multiple and single scattering, and where the scattered energy accumulates from large volumes, you must take account of slight deviations from single scattering results, and this requires taking into account neglected terms in the differential equation. Then, there are a lot of high order terms that come in.

It is extremely complicated to separate all of these high order effects.

G. K. Batchelor

May I add a word concerning Professor Lighthill's remark earlier that he wasn't quite clear why I preferred my derivation to the ones that he and Dr. Kraichnan have given? I am sure this needs more informal argument over a table, rather than discussion here, but I can perhaps give a qualitative description of the reason.

There are really two reasons. The more obvious one concerns Professor Lighthill's presentation, and arises from the history of his work on scattering. His work was done when he was in full pursuit of his theory of the generation of aerodynamic noise, which, as everybody now recognizes, has made a fundamental advance in our understanding of that problem. An essential and important part of that theory is the recognition that the generation of noise by turbulence is equivalent to the action of quadrupole acoustic sources. Then, when he came to write his paper on the scattering of sound by turbulence, he worked out the analysis within the framework of the theory of aerodynamic noise, and interpreted the scattered sound in terms of the action of quadrupole sources.

I avoided that presentation because I think it makes for unnecessary complication. In this problem of scattering, it is not really necessary to know that the turbulent fluctuations in velocity are equivalent in their action to quadrupole sources of sound. The multi-pole character of the equivalent sources does not play an important part in scattering theory, essentially because the wavelength of the sound is here comparable with the length-scale of the turbulence, whereas the sound generated by turbulence has a much larger wave-length.

I think that probably Professor Lighthill will agree with that.

M. J. Lighthill

Yes.

G. K. Batchelor

The second reason applies mostly to Dr. Kraichnan's work. I felt that he was not entirely rigorous about his decomposition of the motion into turbulence on the one hand and sound waves on the other.

The basic exact equation for a motion in which the total pressure, density and velocity are p, ρ and $\boldsymbol{U}$ is

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 p = \frac{\partial^2(\rho U_i U_j)}{\partial x_i \partial x_j}.$$

We wish to obtain from this equation an estimate of the effect of a given field of turbulence on an incident sound wave, and a perturbation procedure corresponding to the different orders of scattering. One cannot simply assume, as Dr. Kraichnan seems to have done, that the density and pressure fluctuations are due mainly to the incident sound wave, and that the term on the right-hand side of this equation is small compared with either of the other two terms when values of p, ρ, $\boldsymbol{U}$ appropriate to the combined turbulence-sound field are substituted in the equation. This is not a correct procedure because, as I showed with the help of some typical values, the fluctuations in p and ρ due to the turbulence alone may be as large as those due to the incident sound wave alone. For the combined turbulence-sound field, the term on the right-hand side may be as large as the other two terms and one cannot begin the perturbation procedure by neglecting the right-hand side.

An equation whose solution has the incident sound wave as its first approximation can be obtained only by inserting in the above equation first the quantities appropriate to the combined turbulence-sound field and second those exactly appropriate to the turbulence alone, and by subtracting the two resulting equations. Only in this way can the unwanted density and pressure variations arising from the turbulence alone be removed from the problem. Of course, one is now left with dependent variables which cannot immediately be given a physical interpretation, since they are simply the increments in pressure, density and velocity that result from the incidence of the sound wave on the existing turbulent motion; however, as I showed in my lecture, the first approximation to this new equation does indeed describe the incident sound wave, and the way is open for an investigation of sound scattering due to the turbulence by the usual perturbation procedure (at least as far as the stage of single scattering). In this way, it seems to me, procedure is made a little more rigorous, and a little clearer, than in the two existing presentations.

R. Kraichnan

I think the whole idea of the decomposition into "turbulence" and "sound" is intrinsically non-rigorous, and at least in the formal structure of the theory, I tried to avoid that by making a rigorous decomposition into transverse and longitudinal velocity fields. If you do that, and work not with the differential equation where the fact is perhaps obscured that the Laplacian operator and time operator largely cancel each other—but with the integral formulation of that equation and consider explicitly the contribution of the homogeneous solution, you can rigorously establish a scale of order of magnitude of the different terms involved. This may have been done sketchily in my paper, but it can be done quite rigorously, and I think that is the fundamental way to attack the problem of rigor.

From The Floor

I would like to throw one more question at Dr. Batchelor or any other gentleman, and that is how much formal analysis they might have done about the time dependent case.

If you want to get the broadening of a single line, you have to get into the space-time correlation of the fluctuations in order to get the line broadening in the scattered energy.

Have you done anything about that?

G. K. Batchelor

I think Dr. Kraichnan is the only one to have given thought to that.

R. Kraichnan

I did this formally, but as to what exactly the time dependence is, we don't know too much about that. Guesses about that are probably less secure than about the space-correlation.

From the Floor

You have the formula with the space-time correlation?

R. Kraichnan

Yes, a four dimensional treatment.

From the Floor

I am not going to attempt to discuss Dr. Batchelor's valuable contribution to the study of wave scattering due to turbulence, but there is a practical observation I should like to make, and I don't think it is entirely irrelevant.

It concerns the interpretation of the twinkling of sound transmitted in the ocean. In the sea, of course, we have a moving boundary, the surface of the sea, and a fixed surface, the bottom, and it may well be that these have a greater effect on the twinkling under certain conditions than does the volume scattering.

D. Mintzer

It is true that at long ranges, where you may have reflections from the surface, this will play an important role; however, there are many cases, including several reported in the literature, in which the sound wave goes completely through the volume without touching the surface; in this case the simple formulation is correct.

I will make another comment, concerning time correlation problems, which someone previously mentioned. The scattering of sound in a time-varying randomly-inhomogeneous medium has been done for the case of scalar inhomogeneities.