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# Development of Turbulent Biological Closure Parameterizations

Louis Goodman
School for Marine Science and Technology (SMAST)
University of Massachusetts Dartmouth
706 South Rodney French Blvd
New Bedford, MA 02744
phone: (508) 910-6375 fax: (508) 910-6376 email: lgoodman@umassd.edu

Award Number: N000140810631
http://www.smast.umassd.edu/Turbulence/

## LONG-TERM GOAL

The long-term goals of this project are:

(1) to develop a theoretical framework to quantify turbulence induced NPZ interactions.

(2) to apply the theory to develop parameterizations to be used in realistic environmental physical biological coupling numerical models.

## OBJECTIVES

Connect the Goodman and Robinson (2008) statistically based pdf theory to Advection Diffusion Reaction (ADR) modeling of NPZ interaction.

## APPROACH

A nonlinear model for biological and physical dynamical interactions in a laminar flow field being upwelled into the mixed layer Robinson (1999) (Fig 1) has been extended to turbulent flow (Goodman and Robinson, 2007). The approach of the Goodman and Robinson theory has been to develop a probability density function (pdf) for the turbulent displacement field and use that to calculate the turbulence induced biological interaction (TIBI) terms, i.e. $< P_i N_j >, < P_i Z_j >, < Z_i N_j >$, where $N_i, P_i, Z_i$ are the i$^{th}$ component of a field of different nutrients, phytoplankton, and zooplankton embedded in the turbulent field. Contrast the TIBI terms with the biological turbulent flux terms $< \vec{u}' N_i >, < \vec{u}' P_i >, < \vec{u}' Z_i >$. The formalism for modeling the latter type of terms are well developed, typically involving some type of eddy diffusivity or higher order closure such as Mellor and Yamada, (1982). However, at present, no biodynamical basis for closure of the TIBI terms has been developed.

The current approach to handling the TIBI terms is either: (1) to ignore them by setting $< (..)_i (..)_j > = < (..)_i > < (..)_j >$ in an advection diffusion reaction (ADR) equation approach (Donaghay, and Osborn, 1997), or (2) to perform a numerical simulation for the turbulent displacement field and explicitly calculate the TIBI terms. The former, as we have shown (Goodman and Robinson, 2007), can result in a large overestimate of the effect of turbulence on these interactions, while the latter is very limited in its domain size, suffers from difficulty in imposing realistic boundary condition at the laminar turbulent interface, and can only reveal significant physics of the TIBI terms with a large number of repetitive runs, which prohibit the size of the computational domain to very limited environmental scenarios. In addition, turbulence numerical models such as LES and DNS are difficult to embed in larger regional scale biophysical coupling models. What is needed to be useful in the larger scale physical biological coupling models is development of realistic parameterizations of the TIBI terms, analogously to the development of turbulent flux parameterizations used in regional and large scale ocean circulation models. In addition development of such parameterizations will lead to new physical/ biological insights into the role of the TIBI terms.

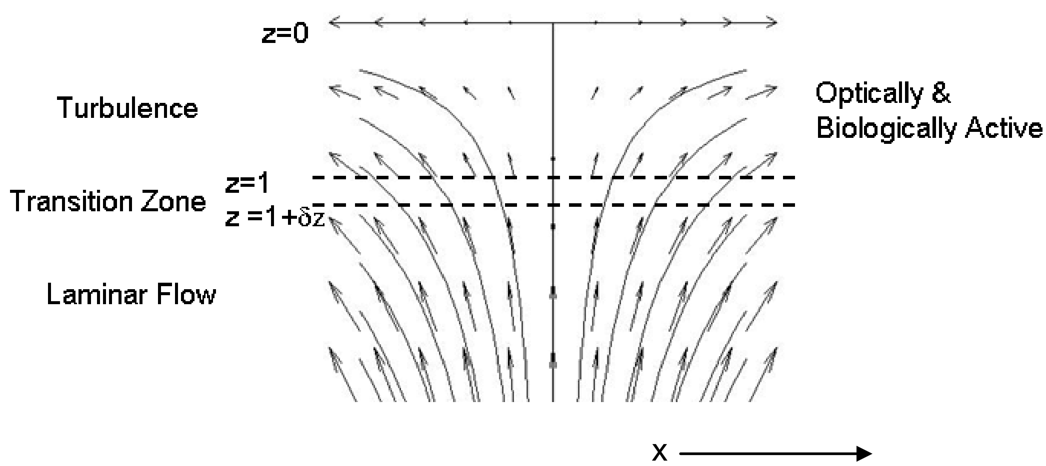


***Figure 1. Upwelling flow field into an optically and biologically active mixed layer used in the Goodman and Robinson (2007) biodynamical model. Note as $\alpha$ increases or as the uptake time increases relative to the advection time***

**WORK COMPLETED**

We have made a major breakthrough and have obtained an exact solution to the ADR equation for realistic boundary conditions using the Goodman and Robinson pdf theory approach. This work is now published in the recent Issue of Dynamics of Atmospheres and Oceans dedictaed to Allan Robinson who passed away in 2010. (Goodman ,L., 2011)

To illustrate an application of the theory and the role of TIBI term in the ADR equation a simple example, that of upwelling of seed nutrients and phytoplankton into a turbulent optically active mixed layer is presented.

Consider the simple example used in previous manuscripts by Robinson (1999) and Goodman and Robinson( 2008), namely, that of a linear strain upwelling of seed nutrients and phytoplankton into an optically active turbulent mixed layer. The equations for this model are

$$\frac{\mathrm{d}\tilde{\mathrm{P}}}{\mathrm{dt}} = (\frac{\partial\tilde{\mathrm{P}}}{\partial \mathrm{t}})_z + \tilde{w}(\frac{\partial\tilde{\mathrm{P}}}{\partial z})_t \;=\; \beta\tilde{N}\tilde{P} \tag{1a}$$

$$\frac{\mathrm{d}\tilde{N}}{\mathrm{dt}} = (\frac{\partial\tilde{N}}{\partial \mathrm{t}})_z + \tilde{w}(\frac{\partial\tilde{N}}{\partial z})_t \;= \text{-}\beta\tilde{N}\tilde{P}, \tag{1b}$$

where $\tilde{\mathrm{P}}, \tilde{\mathrm{N}}$ are the phytoplankton, nutrient linear densities (units of $\mathrm{m}^{-3}$), respectively; β is the nutrient uptake rate; $\tilde{w} = -\alpha z$ is the vertical velocity with a constant strain rate ,−α; $z$ is taken as positive downward. The turbulent mixed layer is located between z = 0 and z = D. The mean flow field is two dimensional and incompressible. It is assumed that one dimensional turbulent mixing dominates, i.e. turbulent scales in the horizontal are much larger than that in the vertical. Adding (14a) and (14b) result in the total biomass density being conserved in a Largangian coordinate system, i.e.

$$M_0 = \tilde{N} + \tilde{P}, \tag{2}$$

where $M_0 = N_0 + P_0$ is the biomass density at $t = t_0$. Normalizing the variables in equations (1a) and (1b) using the mixed layer depth D, total biomass density, $M_0$, and the nutrient uptake time $\tau = \frac{1}{\beta M_0}$ yields

$$\frac{\tilde{P}}{M_0} \to P; \quad \frac{\tilde{N}}{M_0} \to N; \; P + N = 1; \; t\tau \to t \;; \; \frac{\alpha}{\tau} \to \alpha; \frac{z}{D} \to z$$

and results in the normalized set of equations

$$\frac{dP}{dt} = (\frac{\partial P}{\partial t})_z - az(\frac{\partial P}{\partial z})_t \quad = \quad PN \tag{3a}$$

$$\frac{dN}{dt} = (\frac{\partial N}{\partial t})_z - az(\frac{\partial N}{\partial z})_t \quad = \; -PN, \tag{3b}$$

which have the solutions

$$P = P(Z, t, t_0) = \frac{P_0}{P_0 + N_0 \exp[-(t - t_0)]}, \tag{4a}$$

$$N = 1 - P, \tag{4b}$$

with $P_0, N_0$ the normalized phytoplankton and nutrient seed densities, respectively, at $t = t_0$. Consider the PDF equation for a linear strain rate mean flow and a constant vertical eddy diffusivity, κ. . The advection diffusion (AD) equation which we assume the turbulence displacement probability density function $\hat{\rho}$ satisfies is given by

$$\frac{\partial\hat{\rho}}{\partial t} - \alpha z\frac{\partial\hat{\rho}}{\partial z} - \kappa\frac{\partial^2\hat{\rho}}{\partial z^2} = 0. \tag{5}$$

For the mixed layer boundary conditions we assume that the total flux of material (the sum of the advective and turbulent components) vanish at the base of the mixed layer and both the advective and turbulent flux vanish at the top of the mixed layer, whence

$$\alpha z\hat{\rho} + \kappa\frac{\partial\hat{\rho}}{\partial z} = \kappa\frac{\partial\hat{\rho}}{\partial z} = 0 \quad \text{at } z = 0, \tag{6a}$$

$$\alpha z\hat{\rho} + \kappa\frac{\partial\hat{\rho}}{\partial z} = 0 \qquad\qquad \text{at } z = 1. \tag{6b}$$

Equations (6a) and (6b) insure that a normalized solution of $\hat{\rho}$ can be interpreted as a PDF. Equation (5) with boundary conditions (6a) and (6b) also describes the evolution of a scalar non interacting density field, $\hat{\rho}$. To see the latter, consider a solution to (5) of the form

$$\hat{\rho} = \int_0^\infty d\tilde{\gamma}K(\tilde{\gamma})\exp(-\tilde{\gamma}t) + \rho_0,$$

where $\rho_0$ is the density at z =1, t = $t_0$ and which, upon substitution into (5), yields

$$\tilde{\gamma}K + \alpha z\frac{\partial K}{\partial z} + \kappa\frac{\partial^2 K}{\partial z^2} = 0, \tag{7}$$

with the boundary conditions (6) becoming

$$\frac{\partial K}{\partial z} = 0 \quad \text{at } z = 0, \tag{8a}$$

$$\alpha K + \kappa\frac{\partial K}{\partial z} = 0 \quad \text{at } z = 1. \tag{8b}$$

Let

$$K = G\exp(-\frac{Pe}{4}z^2),$$

where the turbulent Peclet number is given by

$$Pe = \frac{\alpha}{\kappa} \;.$$

Substitution of $K$ into equation (7) yields

$$Pe[\gamma - \frac{1}{2} - Pe\frac{z^2}{4}]G + \frac{\partial^2 G}{\partial z^2} = 0, \tag{9}$$

where $\tilde{\gamma} = \alpha\gamma$. With the above substitution, $G$ satisfies the boundary conditions

$$\frac{\partial G}{\partial z} = 0 \qquad \text{at } z = 0, \tag{10a}$$

$$\frac{Pe}{2}G + \frac{\partial G}{\partial z} = 0 \quad \text{at } z = 1. \tag{10b}$$

It is straightforward to show that Eq. (9) with boundary conditions (10) result in eigenfunction solutions $G = G_m$, with associated eigenvalues, $\gamma = \gamma_m$, $m = 1, 2, \ldots n, \ldots$ Note that $G_m$ is orthonormal,

$$\int_0^1 dz G_m G_n = \delta_{mn}.$$

It is also straightforward to show that

$$G_1 = \frac{1}{c_1}\exp(-\frac{Pe}{4}z^2),$$

$$c_1^2 = \int_0^1 dz (G_1)^2,$$

$$\gamma_1 = 1.$$

Using these eigenfunctions and eigenvalues yields the solution

$$\hat{\rho} = \rho_0[1 - \sum_{m=1}^{m=\infty} A_m G_m c_1 G_1 \exp(-\alpha\gamma_m(t - t_0)], \tag{11}$$

where

$$A_m = \int_0^1 dz' G_m(z') \exp(\frac{Pe}{4} z'^2).$$

Equation (11) describes the temporal and spatial evolution of the density field, $\hat{\rho}$, which is given by $\hat{\rho} = \rho_0$ at t = $t_0$ and z =1. Note that when $t = \infty$, $\hat{\rho} = \rho_0$, as expected.
To obtain the PDF, we rewrite (11) as

$$\hat{\rho} = \int_{-\infty}^{t} dt' H(t_0, t')(-\frac{\partial \hat{\rho}}{\partial t'}) = \int_{-\infty}^{t} dt' \{\rho_0 H(t_0, t')\} \{\frac{1}{\rho_0}(-\frac{\partial \hat{\rho}}{\partial t'})\}, \quad (12)$$

with the Heaviside function $H$ defined by

$$H(t_0, t') = \begin{bmatrix} 1 & t' \geq t_0 \\ 0 & t' < t_0 \end{bmatrix}.$$

Eq. (12) can then be interpreted as the prescription for obtaining the average density $\hat{\rho} = \overline{\rho}$ from the initial density $\rho = \rho_0 H(t_0, t')$ using the PDF, $\hat{F} = \frac{1}{\rho_0}(-\frac{\partial \hat{\rho}}{\partial t'})$ .

Changing the independent random variable $t'$ to $\tilde{z}$ according to $\tilde{z}=\exp(-\alpha(t-t')$ , results in

$$\overline{\rho} = \int_0^1 d\tilde{z} \rho(\tilde{z}) \frac{\hat{F}(\tilde{z})}{\frac{d\tilde{z}}{dt'}} = \int_0^1 d\tilde{z} \rho(\tilde{z}) F(\tilde{z}), \quad (13a)$$

$$F = \sum_{m=1}^{m=\infty} \gamma_m A_m G_m c_1 G_1 \tilde{z}^{(\gamma_m - 1)} = \frac{\partial Q}{\partial \tilde{z}} \quad (13b)$$

with

$$Q = \sum_{m=1}^{m=\infty} A_m G_m c_1 G_1 \tilde{z}^{\gamma_m}.$$

Note that *F,* as a PDF, has the proper normalization

$$\int_{-\infty}^{t} dt' F(t') = \int_0^1 d\tilde{z} F(\tilde{z}) = \int_0^1 d\tilde{z} \frac{\partial Q}{\partial \tilde{z}} = Q(1) - Q(0) = 1.$$

We can interpret $\tilde{z}$ as the Lagrangian position of a fluid particle being upwelled by the linear strain vertical velocity, w = -□z, into the turbulent mixed layer at time t, whose initial position at z = 1 was t' . Ensemble averages of P and PN are then given by

$$< P >= \int_{-\infty}^{t} dt' P(t')F(t') = \int_{0}^{1} d\tilde{z} P(\tilde{z})F(\tilde{z}), \qquad (14a)$$

$$< PN >= \int_{-\infty}^{t} dt' P(t')N(t')F(t') = \int_{0}^{1} d\tilde{z} P(\tilde{z})N(\tilde{z})F(\tilde{z}). \qquad (14b)$$

Using equation (12 b) with (14a) and (14 b), it is straightforward to show that the mean phytoplankton density $\overline{P}$ satisfies the ADR equation

$$\begin{matrix} (I) & (II) & (\mathrm{III}) & (\mathrm{IV}) \end{matrix}$$

$$\frac{\partial \overline{P}}{\partial t} - \alpha z \frac{\partial \overline{P}}{\partial z} - \kappa \frac{\partial^2 \overline{P}}{\partial z^2} = R, \qquad (15)$$

with boundary conditions

$$\frac{\partial \overline{P}}{\partial z} = 0 \quad \text{at } z = 0, \qquad (16a)$$

$$Pe\overline{P} + \frac{\partial \overline{P}}{\partial z} = PeP_0 \quad \text{at } z = 0. \qquad (16b)$$

The "reaction" term, R, is given by

$$R =< PN >= R_0 + R_T;$$

with $R_0 = \overline{P}\overline{N}$ and $R_T =< P'N' >$, the latter the TIBI term. Thus, we have formally solved the ADR equation (15) without recourse to dropping the TIBI term.

To examine the role of the TIBI term, $R_T$, we will compare the results out lined above to that of obtaining a solution to the ADR equation (15) with no TIBI term, i.e. $R = R_0 = \overline{P}\overline{N}$. To obtain numerical results we use as input parameters typical oceanic values of

$$\tilde{\alpha} \sim 10^{-5} - 10^{-6} \ \sec^{-1},$$

$$\tilde{\kappa} \sim 10^{-2} - 10^{-4} \ \frac{\mathrm{m}^2}{\sec},$$

$$D \sim 10 - 50 \ \mathrm{m}$$

(Large, 1994; Goodman and Robinson, 2008). This yields a range of turbulent Peclet numbers between $.01 < Pe < 10$. Note that $Pe$ is independent of the nutrient uptake time, τ . . If we use one day as a characteristic nutrient uptake time and the larger value of the linear strain rate of $\tilde{\alpha}$ □ $10^{-5}$, this results in the normalized strain rate $\alpha$ □ $.1$ (Robinson, 1999).

In figure 2 vertical profiles of $\overline{P}$ are shown for the steady state ($t \to \infty, \frac{\partial \overline{P}}{\partial t} = 0$) for $0 \leq Pe \leq \infty$ and $\alpha = .1$. The solid color coded lines are Case A, the solution for <P> using equations (13b) and (14a), TIBI term included. The dashed color coded lines are solutions to the ADR equation (15) with the TIBI term neglected, $R_T = 0$, Case B, using the same range of $Pe$ and $\alpha = .1$. The red solid line of figure 2 corresponds to no turbulence, pure advection, $Pe = \infty$. We see a very significant difference in the limiting value of $Pe = 0$ for the two cases. Figure 2 shows that neglecting the TIBI term over a wide range of $Pe$, up to $Pe = 10$, where advection dominates turbulence, results in a large over estimate of $\overline{P}$. Compare the dashed with the solid lines. Thus, turbulence in this simple model tends to limit phytoplankton growth over a very wide range of Peclet numbers.

To understand this result we present in figures (3a),(3b), (3c), and (3d) , for the steady state cases presented in figure1, the contribution of the three steady state terms of equation (15) - advection, term II, blue lines; turbulent diffusion, term III, green lines; and growth rate (reaction), R, term IV, the black lines. As in figure 2, the solid lines refer to Case A, the dashed lines, Case B.

Note that the advection term II, blue line, is always negative and indicates an upward flux of material. The turbulent diffusion term, III, green line, can be of either sign, with negative indicating a downward flux of material, and positive, an upward flux. The growth rate term, III, black line, is always positive. The three terms must balance in the steady state.

In Fig 3d, the no turbulence, pure advection (red line) regime, where $R_T = 0$, advection balances growth rate. For the turbulence dominated regime of $Pe = 0$, Figure 3a, turbulent diffusion balances growth rate for both case A, including the TIBI term, and Case B, no TIBI term. In Figures 2 and 3a, where $Pe = 0$, note the constant vertical distribution of $\overline{P}$ and constant vertical distribution of terms III and IV of the ADR equation (15). This results from vertically uniform mixing in the steady state. Turbulence strongly dominates advection, which is reflected in the vertically uniform PDF, *F =1.* However, this result only depends on the relative value of the distribution of terms III and IV of the ADR equation (15). This results from vertically uniform mixing in the steady state. Turbulence strongly dominates advection, which is reflected in the vertically uniform PDF, *F =1.* However, this result only depends on the relative value of the turbulent time scale to advective time scale as given by the definition of the turbulent Peclet number,

$$Pe = \frac{\left(\frac{\tilde{\kappa}}{D^2}\right)^{-1}}{\tilde{\alpha}^{-1}} = \frac{\tilde{\alpha} D^2}{\tilde{\kappa}},$$

and not on the absolute intensity of the turbulent field, i.e. value of $\tilde{\kappa}$ □alone. We also see in Figures 3a and 3b for turbulent Peclet numbers $Pe = 0,1$ that for Case A, TIBI term included, (solid green line), at all depths, turbulent diffusion fluxes phytoplankton toward the base of the mixed layer. This downward flux of phytoplankton interacts with the incoming upward moving seed nutrients and the growth rate (black lines). However at $Pe = 10$ there is a change in sign of turbulent diffusion with depth at $z \approx .65$. This results in an upward turbulent flux of phytoplankton for $z \gtrsim .65$. The upward flux diminishes the growth rate in that depth range, as indicated in Figure 3c. Compare in Figure 3c the black solid line to the black dashed line, the former being with TIBI, the latter no TIBI. Thus, the TIBI term affects growth rate through its feed back with the turbulent diffusion term. The advection term is also affected when the TIBI term is included. Relative to not including TIBI, it remains negative, but upwardly transporting less material when the diffusion term III is positive, and more material when the diffusion term is negative. These effects are also present for smaller values of *Pe*, where turbulence dominates, and result in the different total integrated phytoplankton growth in the TIBI versus non TIBI case. See Figure 2, blue and green solid and dashed lines, respectively.

**RESULTS**

In the late 1990s Allan Robinson developed a theory of NPZ interaction in a laminar upwelling flow field. His approach was to use the advection reaction equation (AR) and obtain solutions for the evolution of N, P, Z in a Lagrangian coordinate system (Robinson, 1997, 1999). The Robinson theory was extended to turbulent flow by Goodman and Robinson (2008) by using a probability density function (PDF) on the solution to the AR equation. The PDF employed was associated with a random walk undergoing perfect reflection at the top and bottom of the mixed layer. Bayes' theorem was used to express the PDF in Eulerian coordinates.

A simple bilinear NP turbulent upwelling interaction case was examined with this approach. It was shown that the key non dimensional parameter describing the evolution of the primary production was the turbulent Peclet number, the ratio of the advective to turbulent time scale. It was also observed that the Turbulence Induced Biodynamical Interaction (TIBI) term could not, in general, be neglected. The TIBI term arises from the effect of turbulence on the non linear part of biodynamical interaction and is distinct from that of turbulent mixing.
However, the PDF used by Goodman and Robinson (2008) did not satisfy the AR equation nor did the resulting $\overline{N}, \overline{P}, \overline{Z}$ satisfy the advective diffusion reaction (ADR) equation . This resulted in an inability of that approach to be used to examine the role of the TIBI in the commonly used ADR models of NPZ interaction.

Allan and I, just prior to his untimely death, developed an approach , given here and in Goodman (2011) to obtain a PDF which does satisfy the AD equation and also results in the mean biodynamical state variables $\overline{N}, \overline{P}, \overline{Z}$ satisfying the ADR equation. This approach was applied to the bilinear NP interaction example considered in previous manuscripts (Robinson (1999) and Goodman and Robinson (2008), with particular emphasis on the role of the TIBI term. Except for extremely high Peclet numbers, *Pe* >> 10, which correspond to advection effects dominating that of turbulence, the TIBI term results in a very significant contribution to the mean phytoplankton profile $\overline{P}$. Neglect of the TIBI term, as seen in figure 2, results in an overestimate of $\overline{P}$ with the overestimate increasing with decreasing *Pe*. Not including the TIBI term in the ADR equation also greatly alters the overall role that turbulence plays in determining the mean phytoplankton profile and in contributing to total phytoplankton production. As indicated in figure 2, for the limiting case of $Pe = \infty$ , not including TIBI term, results in the mean total phytoplankton production in the steady state, $P_T = \int_0^1 dz\overline{P}$ , being overestimated by approximately 40 %. Also, as shown in Figures (2) and (3) , the vertical distribution of $\overline{P}$ and the three terms of the steady state ADR equation (14) are altered in the vertical over the intermediate range of Peclet numbers $1 < Pe < 10$. In particular the turbulent diffusion term (green lines, solid, no turbulence case, dashed lines, turbulent cases) show a sign reversal at depth for $1 < Pe < 10$.

The original theoretical framework developed by Allan Robinson lead to a PDF approach to modeling the TIBI term and most recently to a PDF which satisfies the AR equation and with $\overline{N}, \overline{P}, \overline{Z}$ satisfying the ADR equation. This allows a quantification of the role of TIBI in the ADR equation approach and a prescription on how to proceed with more complicated and realistic NPZ models in turbulent flows.

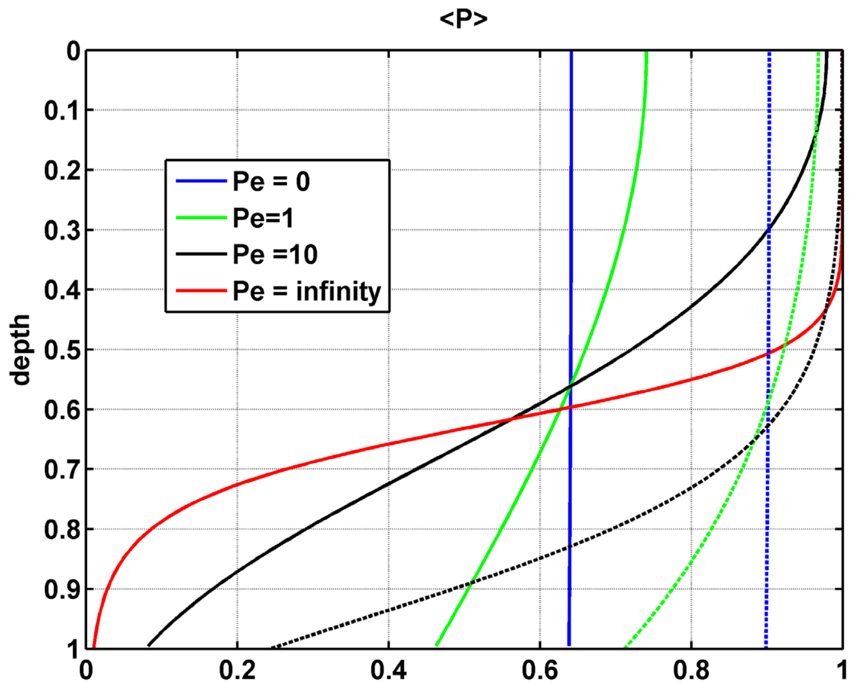


***Figure 2 Steady state ($t \to \infty, \frac{\partial \overline{P}}{\partial t} = 0$) vertical profiles of <P> for the Peclet number,Pe, indicated in the figure, using $\alpha$ =.1. . Case A, the solid color coded lines are the solutions for <P> obtained by using (27a) with (26b) as the PDF, TIBI term included. Case B, the dashed color coded lines, the solution to the ADR equation (28) with the TIBI term neglected, $R = R_0$.***

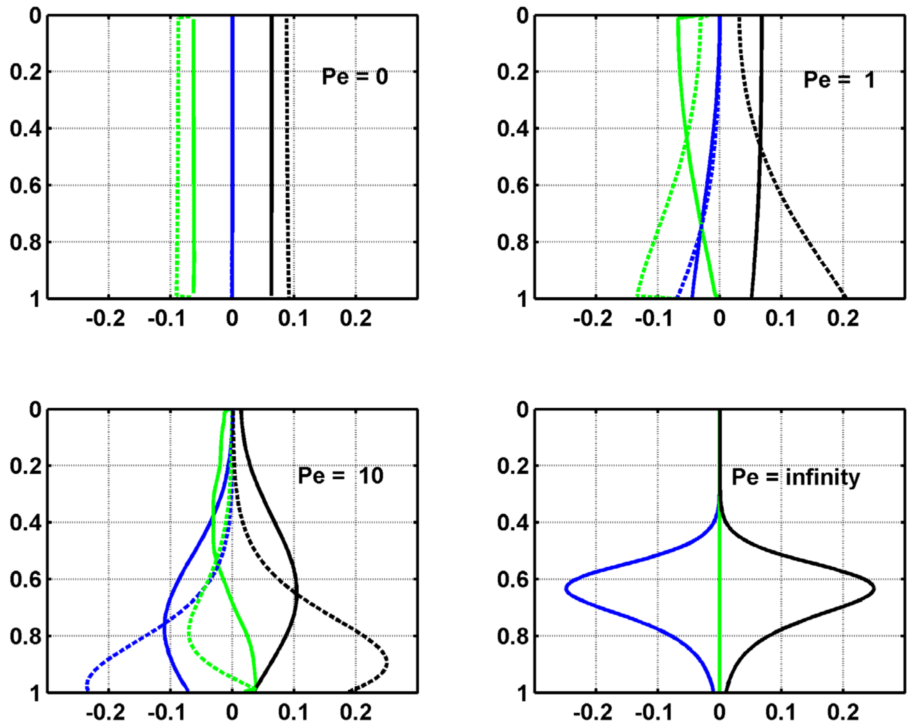


***Figure 3 Steady state ($t \rightarrow \infty, \frac{\partial \overline{P}}{\partial t} = 0$) contribution of the terms of equation (28)- advection, term II, blue lines; turbulent diffusion, term III, green lines; and the growth rate (reaction) , R, term IV, black lines. As in figure 1, the solid lines refer to Case A, the dashed lines, Case B. Note that: the advection term II, blue line, is always negative and indicates an upward flux of material; the turbulent diffusion term, III, green line, can be either sign with negative a downward flux of material and positive an upward flux; the reaction (growth rate) term, III, black line, is always positive. The three terms must balance in the steady state. Four cases, each of different Peclet number, Pe, are shown.***

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