Random Processes

Robert N. Miller

College of Oceanic and Atmospheric Sciences

Oregon State University

Outline

- The Random Walk
- Stochastic Differential Equations
- White Noise
- Numerical Considerations
- The Stochastic Calculus
- Evolution of the Probability Density Function

I saw under the sun that the race is not to the swift, nor the battle to the strong..., but time and chance happen to them all.

--Ecclesiastes, 9:11

For a more recent reference, see Jazwinski, 1970

The Random Walk

- Start at $x = 0$
- In every fixed interval Δt move some random distance Δx_j
- The Δx_j are independent Gaussian random variables with variance v
- In time $T = N\Delta t$ move a distance:

$$X = \sum_{i=1}^{T/\Delta t} \Delta x_i; \langle X \rangle = 0$$

The Random Walk

- The mean square displacement is given by:

$$\langle X^2 \rangle = vT / \Delta t$$

- If we choose v=$\sigma^2\Delta$t, we get:

$$\langle X^2 \rangle = \sigma^2 T$$

independent of Δt.

Stochastic Differential Equations (SDE's)

- A simple example:

$$\frac{dx}{dt} = ax + W$$

where "W" denotes some noise process.

What shall we use for W?

A Simple SDE

We want the properties:

- $\langle W(t_i)\, W(t_j) \rangle = \sigma^2 \delta(t_i - t_j)$
- Each $W(t_i)$ is a Gaussian random variable with mean 0

This is a "*white noise process*"

White Noise Processes

- A white noise process is the formal derivative of a random walk
- If b is a random walk, then we should have:

$$\frac{1}{\Delta t}\int_{t}^{t+\Delta t}\dot{b}\,dt = \frac{b(t+\Delta t)-b(t)}{\Delta t}$$

- But RMS $(b(t+\Delta t)-b(t)) \approx \sqrt{\Delta t}$ so db/dt doesn't exist!

Statistics of Solutions to the Langevin Equation

- A formal calculation leads to:

$$< x^2 >= e^{2at} x_0^2 + \sigma^2 \frac{e^{2at} - 1}{2a}$$

Numerical Considerations

- Stochastic differential equations such as our earlier example must be modeled with special numerical techniques, e.g.:

$$x_{j+1} = (1 + a\Delta t)x_j + \sqrt{\Delta t}\, w_j$$

- w_j is obtained from a random number generator.

Numerical Considerations

- The variance of x after N steps is:

$$\mathrm{var}(x_N) = \Delta t \sigma^2 \frac{(1 + a\Delta t)^{2N} - 1}{(1 + a\Delta t)^2 - 1}$$

- Which leads to the right answer as $\Delta t \to 0$
- The presence of the √ indicates that special numerical techniques are required for SDE's

The Stochastic Calculus

- It turns out that white noise processes aren't even *integrable*, let alone *differentiable*.
- Treatment of stochastic differential equations requires the re-invention of ordinary calculus.

The Stochastic Calculus

- We want:

$$dx = f(x,t)dt + g(x,t)db$$

to mean:

$$x(t_1) - x(t_0) = \int_{t_0}^{t_1} f(x,t)dt + \int_{t_0}^{t_1} g(x,t)db(t)$$

Stochastic Calculus

- How to define the second integral? One answer: evaluate the function at the beginning of the interval:

$$\int g(x,t)\,db = \lim \sum g(x_j, t_j)(b_{j+1} - b_j)$$

- This is the *Îto integral*

The Îto Integral

- Nice properties:

$$\left\langle \int g(x,t)\,db \right\rangle = 0$$

$$\left\langle \int f\,db \int g\,db \right\rangle = \sigma^2 \int \langle fg \rangle\, dt$$

The Îto Integral

- When *x* is a solution to

$$dx = f(x,t)dt + g(x,t)db$$

- We find that, for ϕ(x,t):

$$d\phi = \phi_t dt + \phi_x dx + \phi_{xx} g^2 dt$$

- So we need an extra term in the chain rule!

A Numerical Example: Particle Dispersion in an Idealized Jet

- Consider the following simple model for an idealized quasigeostrophic jet:

$$\psi(x, y) = -\tanh(y) + a\sec^2(y)\cos(kx) + cy$$

$$a = .01;\ c = \frac{1}{3}\left(1 + \sqrt{1 - \frac{3}{2}\beta}\right);\ k = \sqrt{2\left(1 + \sqrt{1 - \frac{3}{2}\beta}\right)};$$

$$0 \le \beta \le \frac{2}{3}$$

- From Brannan et al., *Physica D* 1999

Particle Dispersion in a Jet

- We model the path of a particle moving with the jet and subject to random noise by:

$$dx = -\psi_y dt + \sigma db_1$$

$$dy = \psi_x dt + \sigma db_2$$

$$\sigma^2 = .005$$

Particle Dispersion in a Jet

Title:
ptrace_25.eps
Creator:
MATLAB, The Mathworks, Inc.
Preview:
This EPS picture was not saved
with a preview included in it.
Comment:
This EPS picture will print to a
PostScript printer, but not to
other types of printers.

Sample paths of 5 particles

Evolution of the Probability Density Function

- The probability density function *P* of a solution of a stochastic differential equation evolves according to an advection diffusion equation:

$$P_t + \nabla \cdot (fP) = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} [(gg^T)_{ij} P]$$

- This is the Fokker-Planck equation

Evolution of the Probability Density Function

- The probability density function (PDF) for the particle in jet example is a PDE in 2 space dimensions, and can be solved with finite difference or finite element methods; see Brannan et al.
- The calculated PDF could be used to supply a prior for data assimilation by Bayesian estimation