



Random Processes

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Outline

- ◆ The Random Walk
- ◆ Stochastic Differential Equations
- ◆ White Noise
- ◆ Numerical Considerations
- ◆ The Stochastic Calculus
- ◆ Evolution of the Probability Density Function



*I saw under the sun that the race
is not to the swift, nor the battle
to the strong..., but time and
chance happen to them all.*

--Ecclesiastes, 9:11

For a more recent reference, see
Jazwinski, 1970



The Random Walk

- ◆ Start at $x = 0$
- ◆ In every fixed interval Δt move some random distance Δx_j
- ◆ The Δx_j are independent Gaussian random variables with variance v
- ◆ In time $T = N\Delta t$ move a distance:

$$X = \sum_{i=1}^{T/\Delta t} \Delta x_i; \langle X \rangle = 0$$



The Random Walk

- ◆ The mean square displacement is given by:

$$\langle X^2 \rangle = vT / \Delta t$$

- ◆ If we choose $v = \sigma^2 \Delta t$, we get:

$$\langle X^2 \rangle = \sigma^2 T$$

independent of Δt .



Stochastic Differential Equations (SDE's)

- ◆ A simple example:

$$\frac{dx}{dt} = ax + W$$

where “W” denotes some noise process.
What shall we use for W?



A Simple SDE

We want the properties:

- ◆ $\langle W(t_i) W(t_j) \rangle = \sigma^2 \delta(t_i - t_j)$
- ◆ Each $W(t_i)$ is a Gaussian random variable with mean 0

This is a “*white noise process*”



White Noise Processes

- ◆ A white noise process is the formal derivative of a random walk
- ◆ If b is a random walk, then we should have:

$$\frac{1}{\Delta t} \int_t^{t+\Delta t} \dot{b} dt = \frac{b(t + \Delta t) - b(t)}{\Delta t}$$

- ◆ But RMS $(b(t + \Delta t) - b(t)) \approx \sqrt{\Delta t}$ so db/dt doesn't exist!



Statistics of Solutions to the Langevin Equation

- ◆ A formal calculation leads to:

$$\langle x^2 \rangle = e^{2at} x_0^2 + \sigma^2 \frac{e^{2at} - 1}{2a}$$



Numerical Considerations

- ◆ Stochastic differential equations such as our earlier example must be modeled with special numerical techniques, e.g.:

$$x_{j+1} = (1 + a\Delta t)x_j + \sqrt{\Delta t}w_j$$

- ◆ w_j is obtained from a random number generator.



Numerical Considerations

- ◆ The variance of x after N steps is:

$$\text{var}(x_N) = \Delta t \sigma^2 \frac{(1 + a\Delta t)^{2N} - 1}{(1 + a\Delta t)^2 - 1}$$

- ◆ Which leads to the right answer as $\Delta t \rightarrow 0$
- ◆ The presence of the $\sqrt{\quad}$ indicates that special numerical techniques are required for SDE's



The Stochastic Calculus

- ◆ It turns out that white noise processes aren't even *integrable*, let alone *differentiable*.
- ◆ Treatment of stochastic differential equations requires the re-invention of ordinary calculus.



The Stochastic Calculus

◆ We want:

$$dx = f(x, t)dt + g(x, t)db$$

to mean:

$$x(t_1) - x(t_0) = \int_{t_0}^{t_1} f(x, t)dt + \int_{t_0}^{t_1} g(x, t)db(t)$$

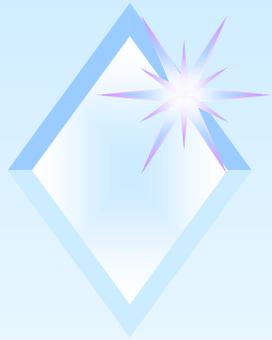


Stochastic Calculus

- ◆ How to define the second integral? One answer: evaluate the function at the beginning of the interval:

$$\int g(x, t) db = \lim \sum g(x_j, t_j)(b_{j+1} - b_j)$$

- ◆ This is the *Itô integral*



The Itô Integral

- ◆ Nice properties:

$$\left\langle \int g(x, t) db \right\rangle = 0$$

$$\left\langle \int f db \int g db \right\rangle = \sigma^2 \int \langle fg \rangle dt$$



The Itô Integral

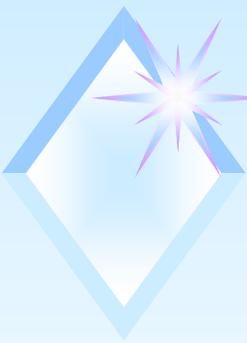
- ◆ When x is a solution to

$$dx = f(x, t)dt + g(x, t)db$$

- ◆ We find that, for $\phi(x, t)$:

$$d\phi = \phi_t dt + \phi_x dx + \phi_{xx} g^2 dt$$

- ◆ So we need an extra term in the chain rule!



A Numerical Example:

Particle Dispersion in an Idealized Jet

- ◆ Consider the following simple model for an idealized quasigeostrophic jet:

$$\psi(x, y) = -\tanh(y) + a \sec^2(y) \cos(kx) + cy$$

$$a = .01; c = \frac{1}{3} \left(1 + \sqrt{1 - \frac{3}{2} \beta} \right); k = \sqrt{2 \left(1 + \sqrt{1 - \frac{3}{2} \beta} \right)};$$

$$0 \leq \beta \leq \frac{2}{3}$$

- ◆ From Brannan et al., *Physica D* 1999



Particle Dispersion in a Jet

- ◆ We model the path of a particle moving with the jet and subject to random noise by:

$$dx = -\psi_y dt + \sigma db_1$$

$$dy = \psi_x dt + \sigma db_2$$

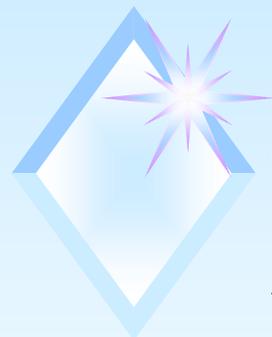
$$\sigma^2 = .005$$



Particle Dispersion in a Jet

Title:
ptrace_25.eps
Creator:
MATLAB, The Mathworks, Inc.
Preview:
This EPS picture was not saved
with a preview included in it.
Comment:
This EPS picture will print to a
PostScript printer, but not to
other types of printers.

Sample paths of 5 particles

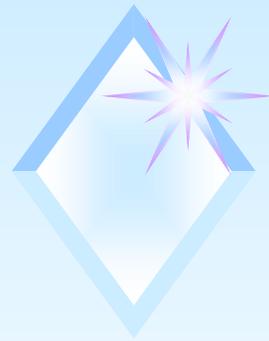


Evolution of the Probability Density Function

- ◆ The probability density function P of a solution of a stochastic differential equation evolves according to an advection diffusion equation:

$$P_t + \nabla \cdot (fP) = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} [(gg^T)_{ij} P]$$

- ◆ This is the Fokker-Planck equation



Evolution of the Probability Density Function

- ◆ The probability density function (PDF) for the particle in jet example is a PDE in 2 space dimensions, and can be solved with finite difference or finite element methods; see Brannan et al.
- ◆ The calculated PDF could be used to supply a prior for data assimilation by Bayesian estimation