Part II

Scattering of low-frequency sound from internal waves

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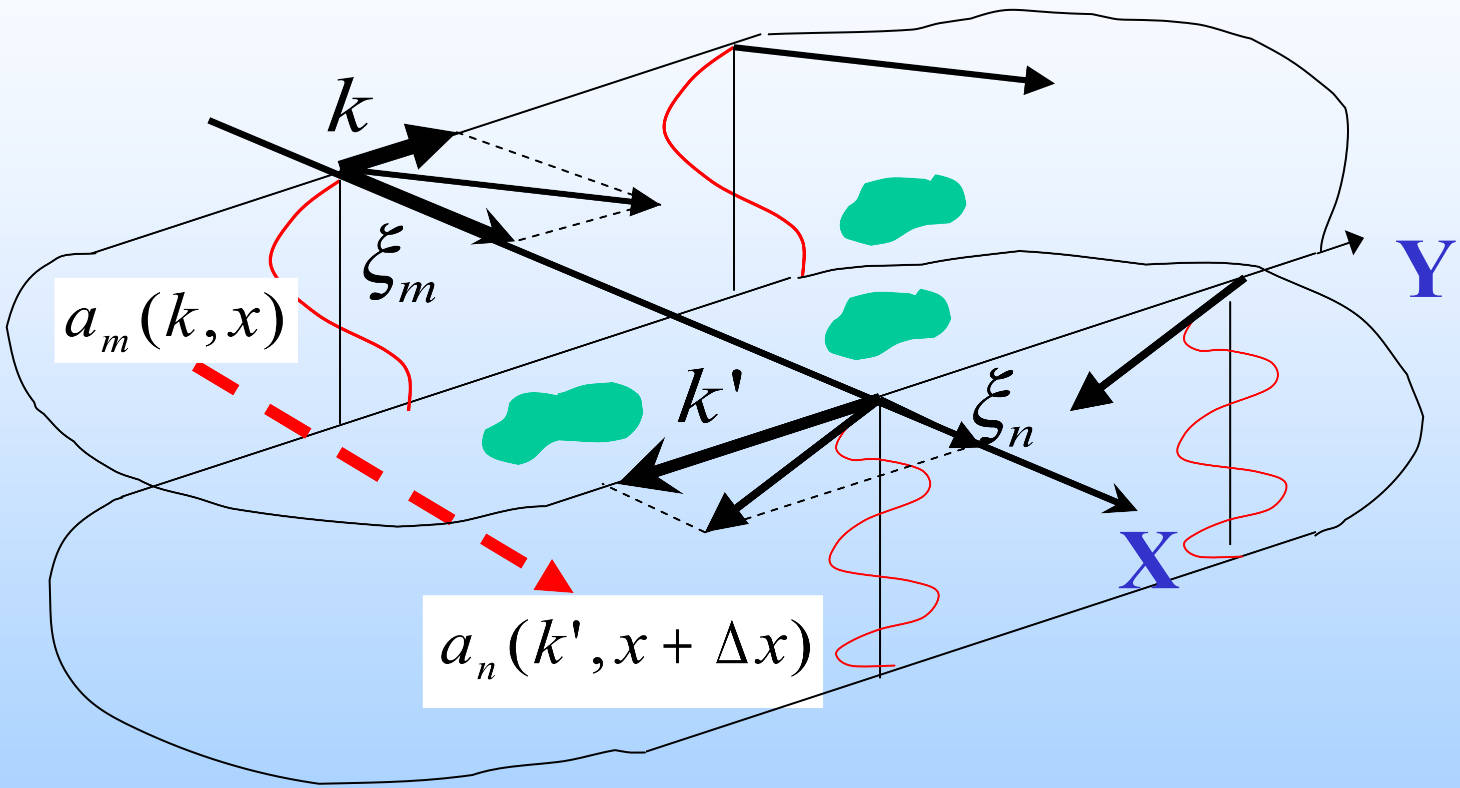
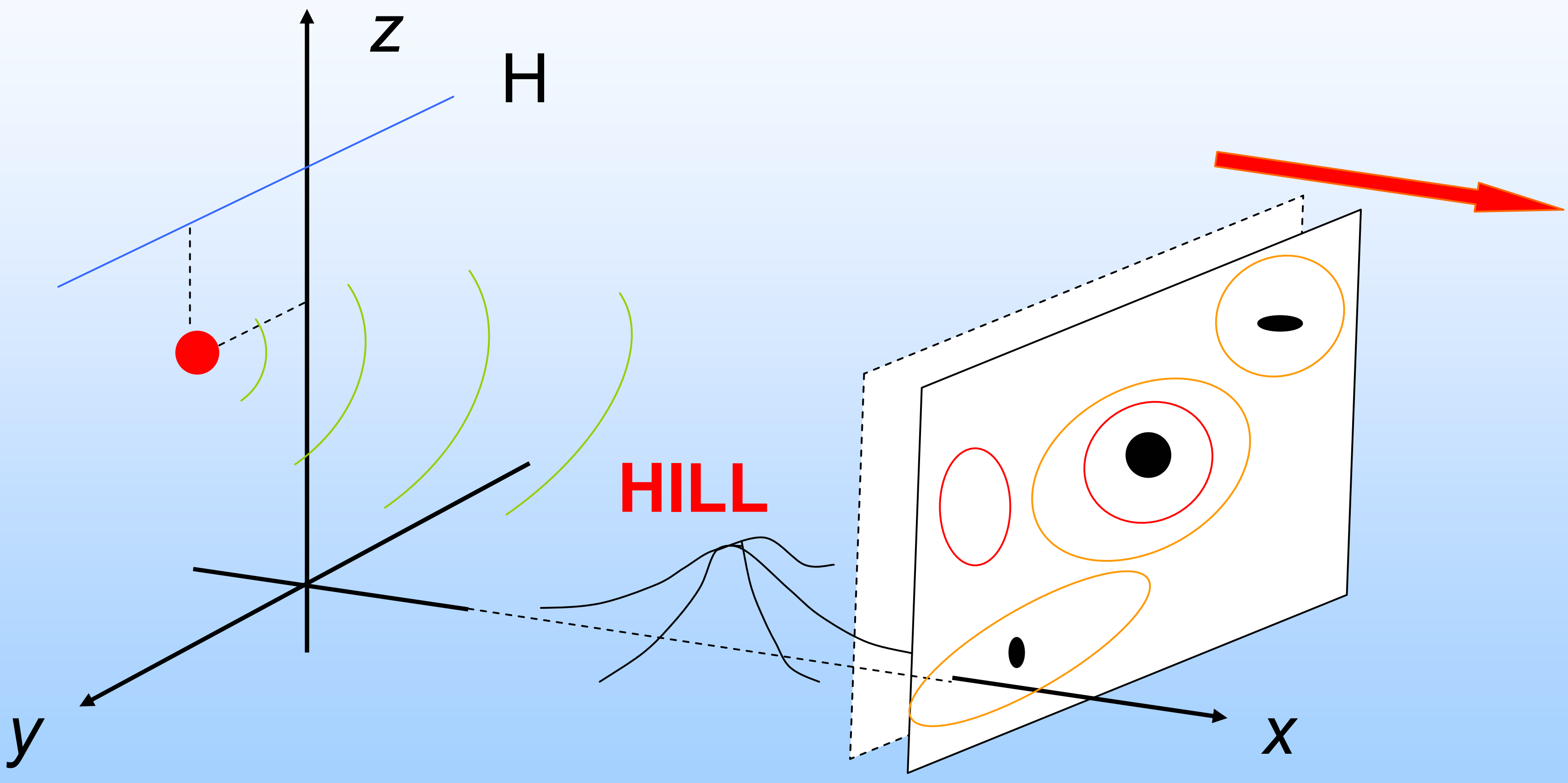
Acoustic mode scattering matrix



$$p = u_m(z)\exp(i\xi_m x + iky) + \sum_n u_n(z)\int dk' \boxed{S_{nm}(k',k)} \exp\left[i\xi'_m(x-\Delta x) + ik'y\right]$$

Source imaging: utilizing a concept of scattering matrix



Source image as a function of *X*

Parameters :

Depth = 100 m

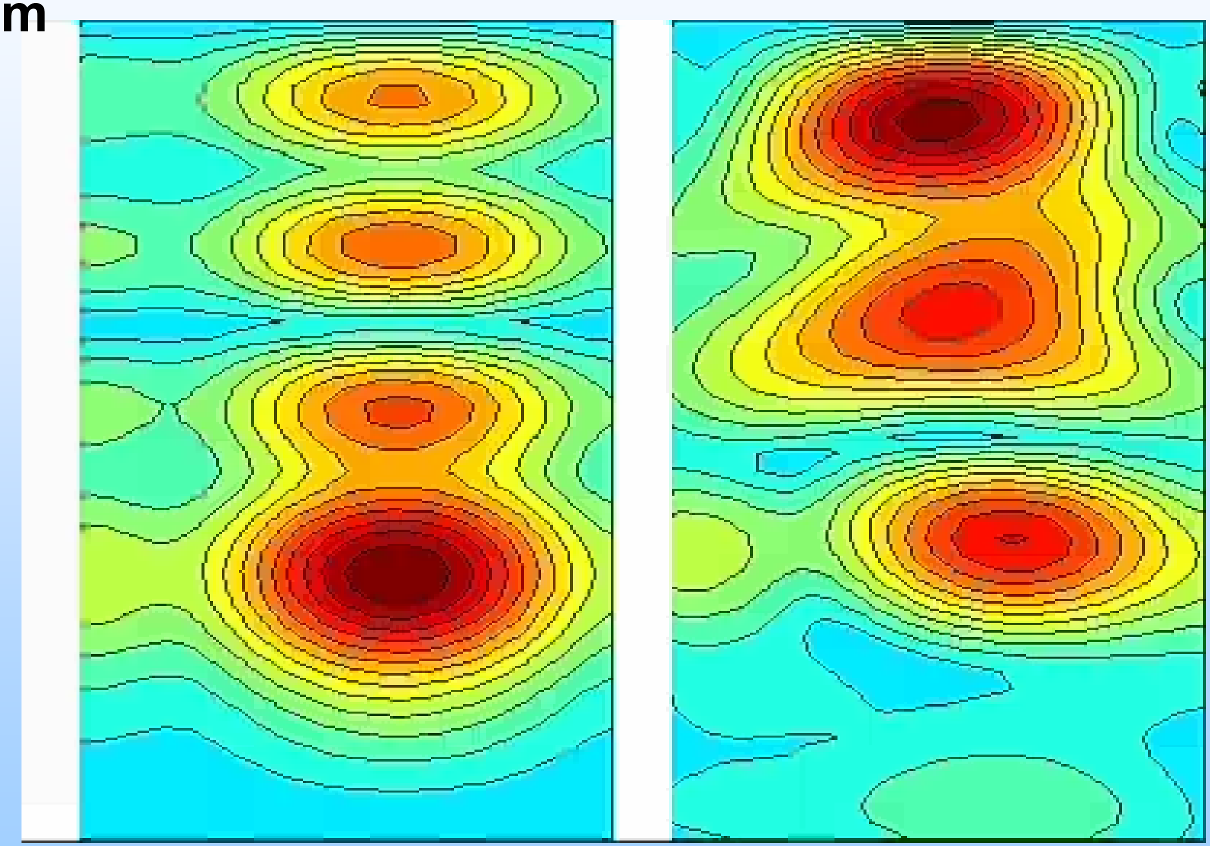
F = 100 Hz

N = 6 modes

Zs = 70 m

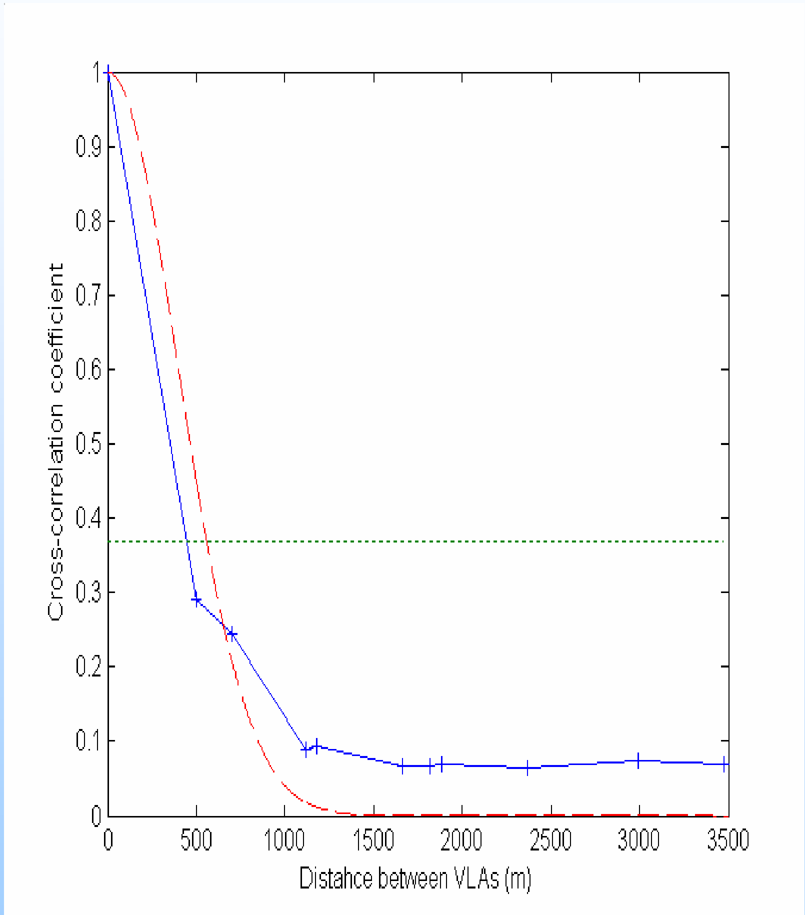
Xs = -500 m

500 m < x < 540 m

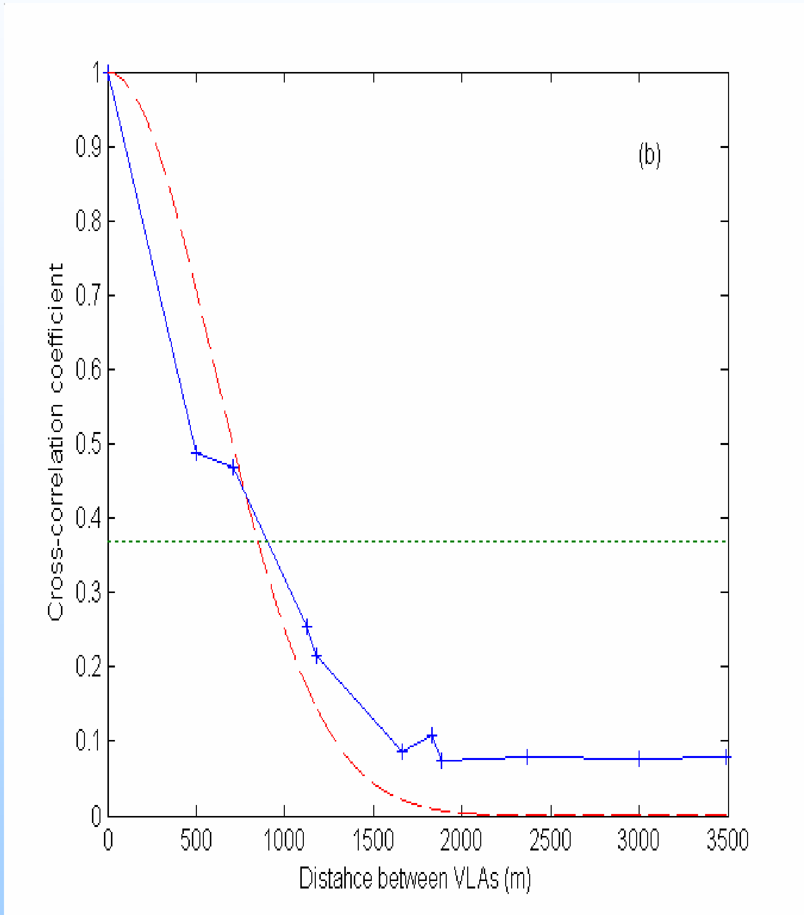


Scattering from Internal Waves

Correlation functions



$$r_c = 560\ m$$



$$r_c = 850\ m$$

Solution at large x

$$\tilde{I}_n(k,x) \to \sqrt{\frac{\pi}{k_0^4 l_* x}} \exp\left(-\frac{k^2}{4k_0^4 l_* x}\right)$$

$$l_* = \frac{\pi}{2} \sum_{n,m,a} \int k^2 P_a(b_n - b_m, k) \left[N_{nm}^a \left(\sqrt{(b_n - b_m)^2 + k^2} \right) \right]^2 dk$$

Angular spectrum: $$\frac{\delta k}{k_0} = k_0 \sqrt{l_* x}$$

Correlation function:

$$C_n(x, \Delta y) = \frac{1}{b_n} \int dk e^{ik\Delta y} \tilde{I}_n(k) = \frac{2\pi}{b_n} \exp\left[-k_0^4 l_* x (\Delta y)^2\right]$$

Modal scattering matrix (cont.)

$$a_n(k, x+\Delta x) = a_n(k,x)e^{i\xi_n \Delta x} + \sum_m \int dk' S_{nm}(k,k') a_m(k',x)$$

$$\Psi(x,y,z) = \sum_n u_n(z) \int dk\, a_n(k,x) e^{i\xi_n x + iky}$$

Some notations:

$$\frac{d^2 u_n}{dz^2} + \left(\frac{\omega^2}{c^2(z)} - b_n^2 \right) u_n = 0, \quad u_n(0) = u_n(-\infty) = 0$$

$$\xi_n = \xi_n(k) = \sqrt{b_n^2 - k^2} \qquad \int u_n u_m dz = \delta_{nm}$$

Statistical description

$$a_n(k,x) = \overline{a}_n(k,x) + \Delta a_n(k,x)$$

Characteristics of the field (to be calculated)

$$\left\langle \Delta a_n(k-\eta/2,x)\Delta a^*_m(k+\eta/2,x)\right\rangle = B(k,x;\eta)$$

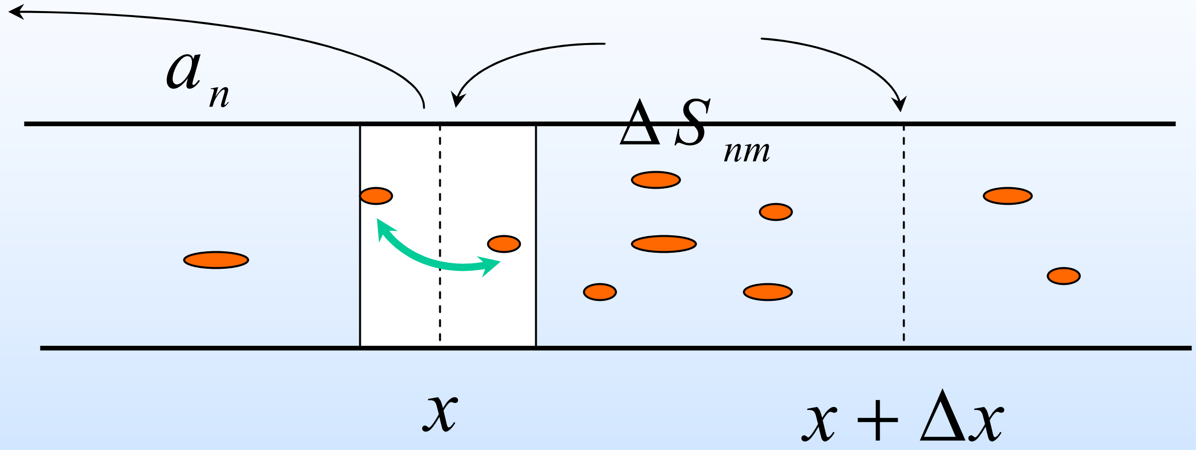
$$S_{mn}(k,k_0) = \Delta x \overline{S}_{mn}(k)\delta(k-k_0) + \Delta S_{mn}(k,k_0)$$

$$\left\langle \Delta S_{n_1m_1}(k-\eta/2,k_0-\eta/2)\Delta S^*_{n_2m_2}(k+\eta/2,k_0+\eta/2)\right\rangle =$$

$$= \Delta x \mathrm{E}_{n_1m_1,n_2m_2}(k,k_0;\eta)\delta(\eta-\eta_0)$$

Statistical parameters of the medium

Markov approximation



$$\left\langle a_n(k,x)\cdot\Delta S_{k_1m_1}\cdot\Delta S^*_{k_2m_2}\right\rangle = \left\langle a_n(k,x)\right\rangle\left\langle \Delta S_{k_1m_1}\cdot\Delta S^*_{k_2m_2}\right\rangle$$

$$\left\langle \Delta a_n\cdot\Delta a_m\cdot S_{kl}\right\rangle = \left\langle \Delta a_n\cdot\Delta a_m\right\rangle\left\langle S_{kl}\right\rangle$$

etc.

Energy conservation

Constant energy flux in the absence of losses:

$$\sum_n \int dk\,\xi_n \left[\left| a_n(k, x+\Delta x) \right|^2 + B_{nn}(k, x+\Delta x; 0) \right] =$$

$$= \sum_n \int dk\,\xi_n \left[\left| a_n(k, x) \right|^2 + B_{nn}(k, x; 0) \right]$$

Optical theorem

$$\xi_{n_1} e^{i\xi_{n_1}\Delta x}\bar{S}^*_{n_1 n_2}(k) + \xi_{n_2} e^{-i\xi_{n_2}\Delta x}\bar{S}_{n_2 n_1}(k) +$$

$$+\sum_m \int dk'\xi'_m E_{mm,n_1 n_2}(k',k;0) + \Delta x \cancel{\sum_m \xi_m \bar{S}_{mn_1}(k)\bar{S}^*_{mn_2}(k)} = 0$$

Governing equations for average field

$$\bar{a}_n(k, x+\Delta x) = \bar{a}_n(k, x) e^{i\xi_n \Delta x} + \Delta x \sum_m \boxed{\bar{S}_{nm}(k)} \bar{a}_n(k, x)$$

Characteristic of scatterers

Governing equations for correlations

$$B_{nn'}(k, x+\Delta x;\eta) = B_{nn'}(k, x;\eta)e^{i[\xi_n(k-\eta/2)-\xi_{n'}(k+\eta/2)]\Delta x} +$$

$$+\Delta x\sum_{m'} e^{i\xi_n(k-\eta/2)\Delta x}\overline{S}^{*}_{n'm'}\left(k+\eta/2\right)B_{nm'}(k, x;\eta)+$$

$$+\Delta x\sum_{m'} e^{-i\xi_{n'}(k+\eta/2)\Delta x}\overline{S}_{nm}\left(k-\eta/2\right)B_{mn'}(k, x;\eta)+$$

$$+\Delta x\sum_{m,m'}\int dk'\, E_{nn',mm'}\left(k, k';\eta\right)B_{mm'}\left(k', x;\eta\right)+$$

$$+\Delta x\sum_{m,m'}\int dk'\, E_{nn',mm'}\left(k, k';\eta\right)\overline{a}_m\left(k'-\eta/2, x\right)\overline{a}^{*}_{m'}\left(k'+\eta/2, x\right)+$$

$$+\Delta x\sum_{m,m'}\overline{S}_{nm}\left(k-\eta/2\right)\overline{S}^{*}_{n'm'}\left(k+\eta/2\right)B_{mm'}\left(k', x;\eta\right)$$

Calculation of scattering

Index of refraction fluctuations

$$\Psi(\vec{r},z)=\Psi_{in}(\vec{r},z)-\frac{\omega^2}{c_{00}^2}\int\theta(x-x')G_0(\vec{r}-\vec{r}';z,z')\Delta n^2(\vec{r}',z')\Psi(\vec{r}',z')d\vec{r}'dz'$$

Forward scattering approximation

$$G_0(\vec{r};z,z')=-\frac{i}{4\pi}\sum_m u_m(z)u_m(z')\int\frac{dk}{\xi_m(k)}\exp\left[i\xi_m(k)x+iky\right]$$

Statistics of fluctuations

$$\left\langle \Delta n^2\left(\vec{r}_1, z_1\right)\Delta n^2\left(\vec{r}_2, z_2\right)\right\rangle = N(\vec{r}_1 - \vec{r}_2; z_1, z_2)$$

$$N\left(\vec{r}_1 - \vec{r}_2; z_1, z_2\right) = \int d\vec{q}\, P_a\left(q_x, q_y\right) \sum_a \Phi_a(q, z_1)\Phi_a(q, z_2)$$

IW spectrum

IW modes

Scattering cross-sections

$$E_{nn',mm'}(k,k';\eta)=\frac{\pi}{2}\left(\frac{\omega}{c_{00}}\right)^4\frac{\exp\left[i\left(\xi_n(k-\eta/2)-\xi_{n'}(k+\eta/2)\right)\Delta x\right]}{\xi_n(k-\eta/2)\xi_{n'}(k+\eta/2)}\cdot$$

$$\cdot\frac{1-\exp(-i\kappa\Delta x)}{i\kappa\Delta x}\sum_a P_a(\Delta\xi,k-k')N^a_{nm}(q)N^a_{n'm'}(q)$$

IW spectrum

$$\Delta\xi=\frac{1}{2}\left[\xi_n(k-\eta/2)-\xi_m(k'-\eta/2)+\xi_{n'}(k+\eta/2)-\xi_{m'}(k'+\eta/2)\right]$$

$$\kappa=\xi_n(k-\eta/2)-\xi_m(k'-\eta/2)-\xi_{n'}(k+\eta/2)+\xi_{m'}(k'+\eta/2)$$

$$N^a_{nm}=\int u_n(z)u_m(z)\Phi_a(q,z)dz$$

IW modes

Simplification 1

$$B_{nn'}(k,x+\Delta x;\eta)=e^{i[\xi_n(k-\eta/2)-\xi_{n'}(k+\eta/2)]\Delta x}\Big\{B_{nn'}(k,x;\eta)+$$

$$+\frac{\Delta x}{b_{n'}}\sum_{m'}\theta^{(1)}_{m'n'}\left(k+\frac{\eta}{2}\right)s^{*}_{n'm'}B_{nm'}(k,x;\eta)+$$

$$+\frac{\Delta x}{b_{n}}\sum_{m'}\theta^{(1)}_{nm}\left(k-\frac{\eta}{2}\right)s_{nm}B_{mn'}(k,x;\eta)+$$

$$+\frac{\Delta x}{b_{n'}b_{n}}\sum_{m,m'}\theta^{(2)}_{nn',mm'}(k;\eta)\int dk'\,e_{nn',mm'}(k-k')B_{mm'}(k',x;\eta)+$$

$$+\frac{\Delta x}{b_{n'}b_{n}}\sum_{m,m'}\theta^{(2)}_{nn',mm'}(k;\eta)\int dk'\,e_{nn',mm'}(k')\bar{a}_m(k'-\eta/2,x)\bar{a}^{*}_{m'}(k'+\eta/2,x)\Big\}$$

Scattering cross-section-simplified version

IW spectrum

$$e_{nn',mm'}(k)=\frac{\pi}{2}\left(\frac{\omega}{c_{00}}\right)^4\cdot\sum_a P_a(\Delta\xi,k)N^a_{nm}(q)N^a_{n'm'}(q)$$

$$\Delta\xi=\frac{1}{2}(b_n+b_{n'}-b_m-b_{m'})$$

$$q=\sqrt{(\Delta\xi)^2+k^2}$$

$$N^a_{nm}=\int u_n(z)u_m(z)\Phi_a(q,z)dz$$

Energy conservation- simplified version

$$\sum_n b_n \int dk \left[B_{nn}(k, x; 0) + |a_n(k, x)|^2 \right] = const$$

Optical theorem – simplified version

$$s_{nm} + s^*_{mn} + \sum_l \frac{1}{b_l} \int e_{ll,nm}(k)\, dk = 0$$

Spatial averaging

$$\overline{a}_n \propto e^{i\xi_n x} \quad , \quad \overline{B}_{nn'} \propto e^{i(\xi_n - \xi_{n'})x}$$

$$e^{-i\xi_n(k)\Delta x}\overline{a}_n(k, x+\Delta x) = \overline{a}_n(k,x) + \Delta x \frac{s_{nn}}{b_n}\overline{a}_n(k,x)$$

$$e^{-i[\xi_n(k-\eta/2)-\xi_n(k+\eta/2)]\Delta x} B_{nn}(k, x+\Delta x;\eta) = B_{nn}(k,x;\eta) +$$
$$+\frac{\Delta x}{b_n}\left(s_{nn} + s^*_{nn}\right)B_{nn}(k,x;\eta) + \frac{\Delta x}{b_n^2}\sum_m \int dk'\sigma_{nm}(k-k')B_{nm}(k,x) +$$
$$+\frac{\Delta x}{b_n^2}\sum_m \int dk'\sigma_{nm}(k')\left|\overline{a}_m(k,x)\right|^2$$

Scattering cross section

$$\sigma_{nm}(k) = e_{nn,mm}(k) = \frac{\pi}{2}\left(\frac{\omega}{c_{00}}\right)^4 \sum_a P_a(b_n - b_m, k)\left[N_{nm}^a\left(\sqrt{(b_n - b_m)^2 + k^2}\right)\right]^2$$

Spectrum of fluctuations

Acoustic-IW modes interaction matrix

Optical theorem:

$$2\,\mathrm{Re}\, s_{nn} + \sum_m \frac{1}{b_m}\int dk\, \sigma_{nm}(k) = 0$$

Average field

$$\bar{a}_n(k,x) = \bar{a}_n(k,0)e^{i\xi_n(k)x-\beta_n x}$$

$$\beta_n = -\frac{2\,\mathrm{Re}\, s_{nn}}{b_n} = \frac{1}{b_n}\sum_m \frac{1}{b_m}\int dk\, \sigma_{nm}(k) > 0$$

$$\left\langle \Psi(x,y,z) \right\rangle = -\frac{i}{4}\sum_n u_n(z)u_n(z_s)H_0^{(1)}\left(b_n\sqrt{x^2+y^2}\right)e^{-\beta_n x}$$

Limit $\Delta x \to 0$

$$\frac{\partial \overline{a}_n}{\partial x} - i\xi_n \overline{a}_n = \frac{s_{nn}}{b_n}\overline{a}_n$$

$$\frac{\partial I_n}{\partial x} - \frac{i\eta k}{2b_n} I_n = \frac{2\,\mathrm{Re}\, s_{nn}}{b_n} I_n + \frac{\Delta x}{b_n^2}\sum_m \int dk' \sigma_{nm}(k-k') I_m(k') + \\ + \frac{1}{b_n^2}\sum_m \int dk' \sigma_{nm}(k') \left|\overline{a}_m\right|^2$$

$$I_n = I_n(k,x;\eta) = B_{nn}(k,x;\eta)$$

$$\sigma_{nm}(k) = e_{nn,mm}(k)$$

Diffusion approximation

$$\frac{\partial \tilde{I}_n}{\partial x} = -\beta_n \tilde{I}_n + \sum_m \sigma_{nm}^{(0)} \tilde{I}_m + \sum_m \sigma_{nm}^{(2)} \frac{\partial^2 \tilde{I}_m}{\partial k^2} + \\ + \sum_m \sigma_{nm}^{(0)} b_m \left| \overline{a}_m(0, x) \right|^2 e^{-2\beta_m x}$$

$$\tilde{I}_n = b_n I_n(k, x; \eta)$$

$$\sigma_{nm}^{(0)} = \frac{1}{b_n b_m} \int dk \sigma_{nm}(k) \quad , \quad \sigma_{nm}^{(2)} = \frac{1}{b_n b_m} \int dk \sigma_{nm}(k) k^2$$

Solution at large x

$$\tilde{I}_n(k,x) \to \sqrt{\frac{\pi}{k_0^4 l_* x}} \exp\left(-\frac{k^2}{4k_0^4 l_* x}\right)$$

Angular spectrum:

$$\frac{\delta k}{k_0} = k_0 \sqrt{l_* x}$$

Correlation function:

$$C_n(x,\Delta y) = \frac{1}{b_n} \int dk e^{ik\Delta y} \tilde{I}_n(k) = \frac{2\pi}{b_n} \exp\left[-k_0^4 l_* x (\Delta y)^2\right]$$

Numerical estimate

$$l_* = \frac{\pi}{2}\sum_{n,m,a}\int k^2 P_a\left(b_n - b_m, k\right)\left[N_{nm}^a\left(\sqrt{\left(b_n - b_m\right)^2 + k^2}\right)\right]^2 dk$$

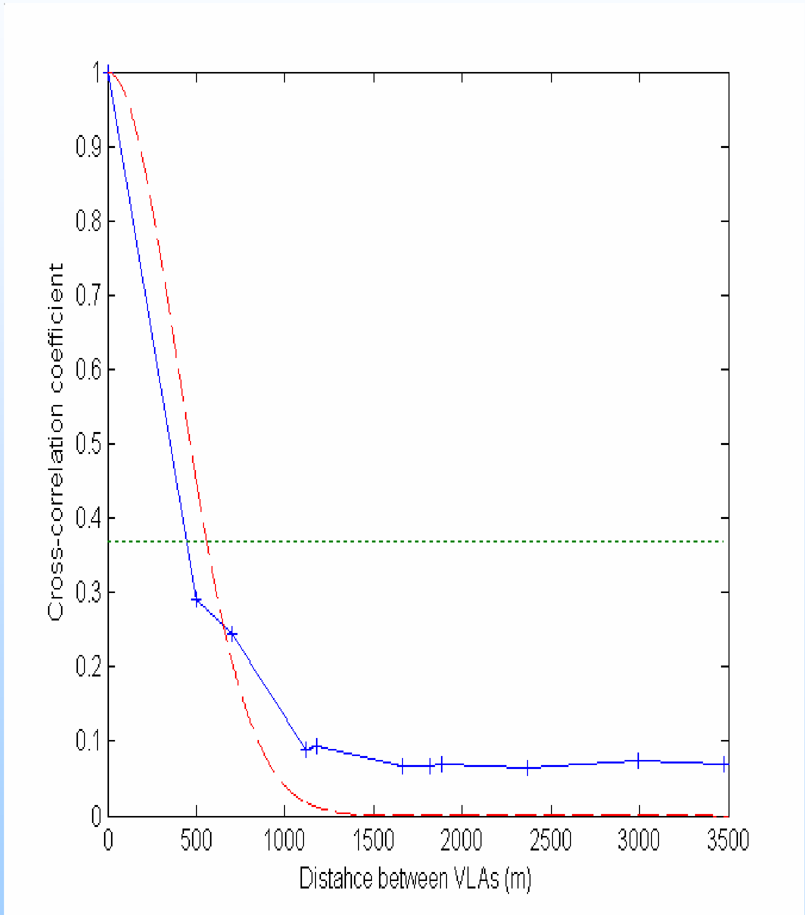
For GM spectrum:

$$l_* = 5.4 \cdot 10^{-11}\ m$$

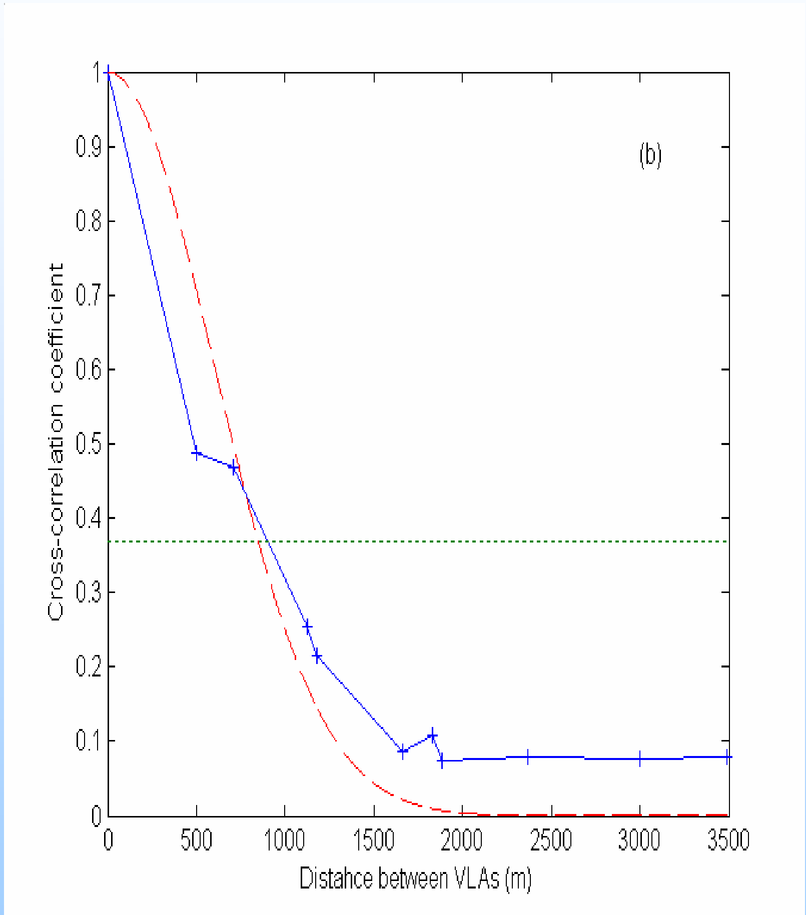
$$\sigma_\alpha = \frac{\delta k}{k_0} = k_0\sqrt{l_* x} = 4.6 \cdot 10^{-3} \approx 0.26^o$$

$$\rho_c = \frac{1}{k_0^2\sqrt{l_* x}} \approx 700\quad m$$

Correlation functions



$$r_c = 560\ m$$



$$r_c = 850\ m$$

Conclusions

- **Concept of Scattering Matrix can be useful for source imaging applications**

- **Experimental (NPAL) data indicate that correlation radius of the acoustic field in cross-propagation direction is of the order of 500 m – 1000 m. Decorrelation is explained by scattering at internal waves.**

- **A theory was developed which is applicable to the low-frequency sound propagation in the real ocean up to frequencies of the order of a few hundred Herz.**