Statistical Dynamics of Coastal Surface Waves: Theory, Numerical Simulation and Comparison with Experiment

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LONG-TERM GOALS

We plan to develop a well-justified consistent statistical theory of ocean waves on finite depth, supported by numerical simulations and experimental observations. This theory will be a foundation for wave prediction models of a new generation.

On water of finite depth not only four-wave interactions, but also three-wave interactions are important. We will show that on intermediate depth $0.3<kh<3$, triple interactions can be taken into account by a proper nonlinear filtration of the observed spectra. We will also develop a numerical algorithm for the filtration.

We will find analytically and numerically weak-turbulent Kolmogorov spectra on a finite depth and compare them with experimental observations. We plan to elaborate a new algorithm of high accuracy for the numerical solution of the Hasselmann equation of finite depth and apply this algorithm for the theoretical interpretation of experimental observations of wind-driven sea in a coastal area. Our recent study of fetch-limited experiments on deep water demonstrates that our theory is very much in consistency with field observations. We expect similar agreement between theory and experiment on finite depth.

We plan to develop numerical algorithms for solving the dynamic Hamiltonian equations, describing nonlinear ocean waves both on deep and shallow water. Direct numerical solution of the primitive equation can be done efficiently by the use of the Fast Fourier Transform method; it is now realistic to have the grid as fine as *1024x1024*.

OBJECTIVES

I. Derivation of Hasselmann's kinetic equation on finite depth (from first principles), and determination of conditions of its applicability; calculation of weak turbulent Kolmogorov spectra; derivation of formulae for filtration of "slave" harmonics (PI, Year 2003).

II. Development of a numerical code for filtration from slave harmonics. The goal is to turn this code to a "routine" procedure for elaboration of experimental data. (Shirshov Institute group under supervision of PI and Co--PI; Years 2003--2005)

III. Modernization of the Resio code for the numerical solution of the kinetic equation on finite depth (Shirshov Institute group under supervision of Co--PI; Years 2003--2004)

IV. Massive numerical simulation of the kinetic equation on finite depth, starting with already existing algorithms. This part will include development of duration and fetch limited codes; determination of duration dependence and fetch dependence of spectra on finite depth. (Shirshov Institute group under supervision of PI and Co--PI; Years 2003--2005).

V. Development of the phase--resolving code for solution of dynamic Hamiltonian equations in full three--dimensional geometry in approximation of weak nonlinearity. (Landau Institute group under supervision of PI; Years 2003--2005).

VI. Development of full--nonlinear code for solution of exact Euler equations for the potential fluid with a free surface in 2D geometry; numerical study of wave--breaking. (Landau Institute group under supervision of PI; Years 2003--2005).

VII. Comparison between the theory and the experiments. As our theory will be developed, we will be able to compare it with an increasing amount of experimental data; we will take this data from different experimental groups. The Shirshov Institute group under supervision of PI during 2003--2005 years will do this part of work, which is a point of our priority.

APPROACH

Creation of reliable statistical theory of ocean waves on finite-depth water is an urgent task, strongly motivated by numerous practical needs. Most of engineering constructions, including oil platforms, are placed in coastal areas; major practical problems are associated with surface coastal waves. In this proposal we plan to advance as far as possible the consistent and well-justified statistical theory of ocean waves on shallow water, which could be a foundation for wave prediction models of a new generation. This theory will be supported by massive numerical simulations and by comparison with experimental observations.

The influence of the depth h on the wave dynamics is characterized by the parameter $\delta = \tanh kh$, where k is a wave number [1, 2]. For short waves (waves in deep ocean), $kh>3$ and δ is close to unity. For long waves (waves on shallow water), $kh<0.3$ and $\delta \approx kh \ll 1$. The intermediate case, $0.3<kh<3$, is a subject of the most interest in the Coastal Dynamics [3]. At the moment, the statistical theory of waves on deep water is advanced far enough. The main nonlinear process is a four-wave interaction; energy spectra are described by Hasselmann kinetic equation [4, 5]. Since 1966 we know that this equation has Kolmogorov-type solutions, the simplest one being Zakharov-Filonenko spectrum [6]. The DIA, a simple heuristic model imitating the S_{nl} term in the kinetic equation, is now widely used in main operational models of wave predictions (WAM and SWAN). It will be replaced soon by more sophisticated models.

On shallow water, a “triple-wave” interaction could be as important (or even more important) as quadratic interactions. The theory of “triple” interaction is difficult both from physical and mathematical viewpoints. We plan to propose a practical solution of this problem for the waves of intermediate length. On deep water, a level of nonlinearity is defined by the squared wave steepness $\mu \cong (ka)^2$, where a is wave amplitude. In a typical situation $\mu \approx 10^{-2}$, an expansion in powers of μ

exploited for derivation of the Hasselmann equation is well justified. On the fluid of finite depth, the parameter of nonlinearity is $\varepsilon \approx \mu / \delta^5$ [1, 2]. If $kh \cong 1$, then $\varepsilon \cong 4 \cdot 10^{-2}$. If $kh \cong 0.5$, ε grows dramatically: $\varepsilon \cong 0.5$. The waves on the finite depth fluid are much more "nonlinear" then the waves on deep water. The "triple-wave interaction" is one of the manifestations of this increased nonlinearity. In any nonlinear system waves could be divided into "free" and "slave" waves. The slave waves are forced: their frequencies and wave vectors are not connected by a fixed relationship. The most intensive slave harmonics appear due to triple-wave resonance.

On deep water the slave harmonics are of the order of μ and are negligibly small. On shallow water they are of the order of ε and much more enhanced. Anyway, as far as $\varepsilon < 1$, the slave harmonics can be treated by perturbation theory. On very shallow water $\varepsilon \cong 1$, and slave harmonics could not be separated from free waves. In this case, the theory is essentially nonlinear. Slave harmonics are clearly seen in experimentally measured spectra. They lead to formation of "humps" at $f = nf_p$, *(n=2,3,4…)* [3, 15]. Here f_p is the spectral peak frequency.

At the moment, a lot of experimental results on spectra evolution on intermediate long waves are accumulated. In most cases $\varepsilon < 1$, and Hasselmann equation could be taken as a basic model. We propose to re-examine the derivation of Hasselmann equation for finite-depth water, determine the limits of its validity, and check carefully the analytic expression for the coupling coefficient. Formulae for the wind input and bottom friction will be re-examined as well.

The next point of our program is elaborating numerical schemes for the solution of Hasselmann equation. We propose two different algorithms: (1) based on Resio-Tracy approach [12—13]; (2) a completely new one, that uses a tuning mesh and self-controlling time step. We also exam the DIA approach and its generalizations.

Weak-turbulent Kolmogorov spectra play a central role in the statistical theory of ocean waves both on deep and on shallow water. On shallow water they are more "flat" [1, 14], becoming almost flat in the limit of very shallow water, $\delta \to 0$, where the spectrum of direct cascade of energy is $F_\omega \cong \omega^{-4/3}$ [1]. In the most interesting intermediate case the spectra are not studied yet. We propose to study the weak-turbulent Kolmogorov spectra on shallow water and intermediate zone both numerically and analytically. The weak-turbulent Kolmogorov solution can be used as a benchmark for formulating a simplified S_{nl} model for finite-depth water. We demonstrated recently that space-time evolution of the spectral peak on deep water could be described by the "conservative" Hasselmann equation, which does not include any forcing term [16]. Fetch dependence of spectra in basic fetch limited observations of wind-driven sea (JONSWAP at al.) is described by self-similar solutions of Hasselmann equation. There are no self-similar solutions on a finite depth, but the basic effect should be the same. For waves belonging to the spectral peak, the bottom friction is the most important non-conservative effect on shallow water. It should be taken into consideration first of all.

It is known since 1968 [16] that the surface waves are described by Hamiltonian system:

$$\frac{\partial \psi}{\partial t} = \frac{\delta H}{\delta \eta}, \quad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}$$

Here η is elevation of surface, ψ - potential on the surface, H – total energy serving as a Hamiltonian. For small slopes $\mu << 1$, the Hamiltonian can be expanded in powers of $\mu^{1/2}$. Keeping only the first three terms of the expansion $H = H_0 + H_1 + H_2$, one takes into account triple and quartic interactions without any assumption about their relative importance [17]. These equations can be efficiently solved numerically by use of the Fast Fourier Transform method.

In 2D geometry one can develop an economic numerical algorithm for solution of the exact hydrodynamic equations for a potential flow of a fluid with a free surface [19--20]. The algorithm is based on conformal mapping to half—plane [18], which makes possible to apply the Fast Fourier Transform [21--22]. We plan to apply this algorithm for study of strongly nonlinear effects, including wave—breaking and wave—grouping.

We are open to collaboration with any experimental groups collecting data about coastal waves; such groups have accumulated a large stockpile of experimental data to be interpreted. We will select the arrays of data collected in conditions where the wind speed was constant as long as possible. Situations, which can be interpreted as fetch limited, or duration-limited observations, are the most preferable. We will follow on the development of the NCEX experiment.

We will calculate directional spectra and perform their "filtration" from slave harmonics. After filtration, the spectra will become more smooth and free of "humps" on frequencies $f \cong nf_p$. After careful re-examination of wind input and bottom friction terms, we will perform the numerical simulation of the Hasselmann equation modeling the situation under consideration (either fetch or duration limited). The results of simulation will be compared with "filtrated" experimental spectra.

WORK COMPLETED

Our project is funded since June 2003. However we had some achievements before, declared in the objectives. The resolute breakthrough is done in the point V. We studied the long-time evolution of gravity waves on deep water excited by stochastic external force concentrated in moderately small wave numbers. We numerically implement the primitive Euler equations for potential flow of ideal fluid with a free surface written in canonical variables, using expansion of the Hamiltonian in powers of nonlinearity up to fourth order terms. We show that due to nonlinear interaction processes, the stationary energy spectrum close to $k^{-7/2}$ is formed. The observed spectrum can be interpreted as weak-turbulent Kolmogorov spectrum for direct cascade of energy.

RESULTS

We study potential flow of an ideal inviscid incompressible fluid described by the Hamiltonian expanded in powers of $\nabla\eta$. Taking into account first three terms of the expansion, one gets:

$$H = \frac{1}{2}\int\left(g\eta^2 + \psi\hat{k}\psi\right)dxdy + \frac{1}{2}\int\eta\left[\left|\nabla\psi\right|^2 - \left(\hat{k}\psi\right)^2\right]dxdy + \frac{1}{2}\int\eta\left(\hat{k}\psi\right)\left[\hat{k}\left(\eta\left(\hat{k}\psi\right)\right) + \eta\Delta\psi\right]dxdy$$

Here $\hat{k}$ is a linear operator, corresponding in k-space to multiplication of Fourier harmonics by $\sqrt{k_x^2 + k_y^2}$. For gravity waves this reduced Hamiltonian describes four-wave interaction and dynamical equations take the form:

$$\dot{\eta} = \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}\left[\eta\hat{k}\psi\right] + \hat{k}\left(\eta\hat{k}\left[\eta\hat{k}\psi\right]\right) + \frac{1}{2}\Delta\left[\eta^2\hat{k}\psi\right] + \frac{1}{2}\hat{k}\left[\eta^2\Delta\psi\right],$$

$$\dot{\psi} = -g\eta - \frac{1}{2}\left[(\nabla\psi)^2 - \left(\hat{k}\psi\right)^2\right] - \left[\hat{k}\psi\right]\hat{k}\left[\eta\hat{k}\psi\right] - \left[\eta\hat{k}\psi\right]\Delta\psi + D_\rho + F_\rho$$

Here D_ρ is an artificial damping, which simulates viscosity in small scales and F_ρ is a driving term, which simulates pumping in large scales due to wind:

$$F_k = f_k e^{iR_k(t)},$$

$$f_k = 4F_0 \frac{(k - k_{p_1})(k_{p_2} - k)}{(k_{p_2} - k_{p_1})^2}$$

$$D_k^\rho = \gamma_k \psi_k^\rho$$

$$\gamma_k = -\gamma_1, k \le k_{p_1}$$

$$\gamma_k = -\gamma_2 (k - k_d)^2, k > k_d$$

where $R_k^\rho(t)$ is uniformly distributed random number in the interval $(0,2\pi)$.

The equations (4) were numerically solved in the periodic domain $(2\pi,2\pi)$ on different grids: 128x128, 256x256, 512x512. Computations on the grid 1024x1024 are coming. Gravity acceleration was taken as $g = 1$. Parameters of the damping and forcing were taken as follows: $k_{p_1} = 5$, $k_{p_2} = 10$, $k_d = 100$. Thus, the initial interval is one decade.

According to the weak turbulent theory, the ensemble averaged surface elevation spectrum corresponding to the flux of energy from large to small scales is

$$<|\eta_k|^2> = \frac{Cg^{1/2}P^{1/3}}{k^{7/2}}$$

where P is the energy flux and C is dimensionless Kolmogorov constant. Fig.1 (Left) shows how this result improves depending on the size of the grid.

Fig.1 (Right) represents compensated waves spectra. One can see that weak-turbulent spectrum $<|\eta_k|^2> \sim k^{-7/2}$ fits obtained result better, than wave-breaking Phillips spectrum $<|\eta_k|^2> \sim k^{-4}$.

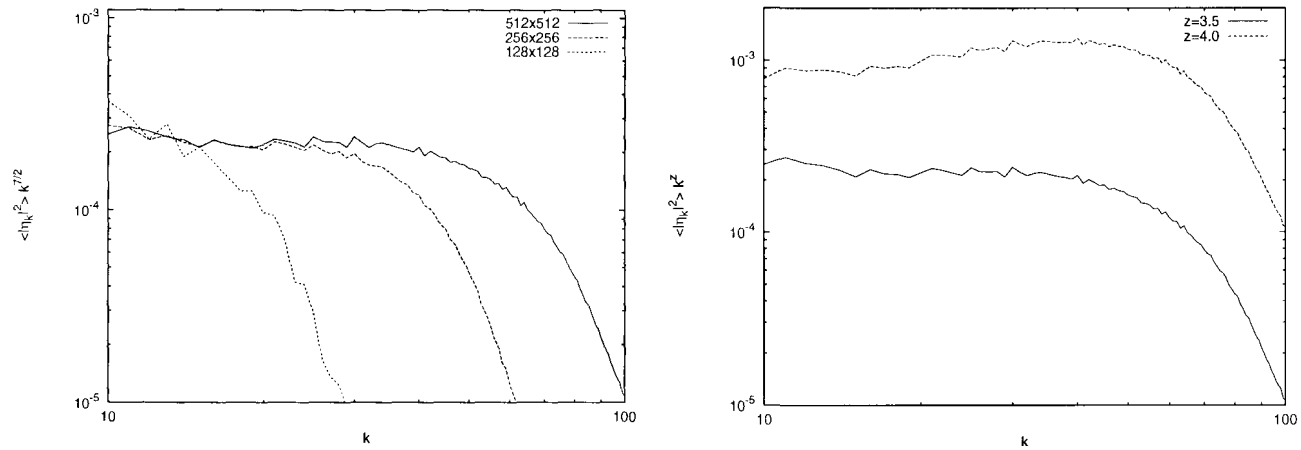

Fig.1. Left: Broadening of power-like spectrum as function of wave number for the different grids: 128x128, 256x256, and 512x512. Right: Compensated waves spectra as a function of wave number.

IMPACT/APPLICATIONS

Obtained result puts another argument in favor of weak turbulent Kolmogorov –Zakharov surface elevations spectrum $<|\eta_k|^2>\sim k^{-7/2}$ versus wave-breaking Phillips spectrum $<|\eta_k|^2>\sim k^{-4}$. It makes possible, in particular, better understanding of experimental results of ocean surface observations.

RELATED PROJECTS

US ARMY contract DACW42-03-C-0019 "Economical Models of Nonlinear Ocean Surface Evolution in Presence of the Wind", devoted to development, analysis and testing of cost-effective diffusion models of Hasselmann equation.

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PUBLICATIONS

Dyachenko A.I., Korotkevich A.O. and Zakharov V.E., Weak Turbulent Kolmogorov Spectrum for Surface Gravity Waves, submitted to *Phys.Rew.Lett.*